Cyclic Homology Theory

Jean-Louis Loday

Notes taken by
Paweł Witkowski

October 2006

Contents

Chapter 1

Cyclic category

1.1 Circle and disk as a cell complexes

The circle in its simplest decomposition has one 0-cell (a point) and one 1-cell (an interval).

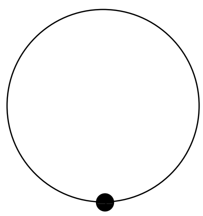
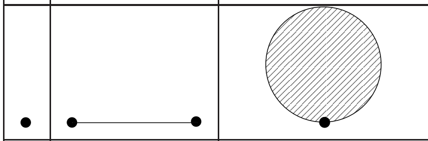
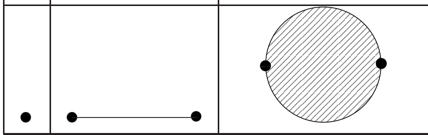
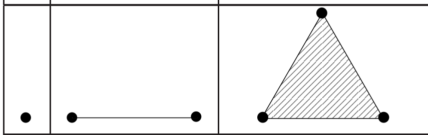
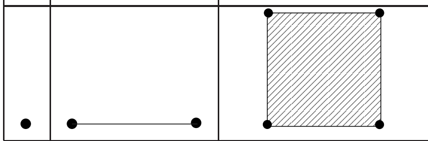
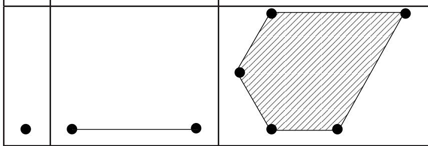
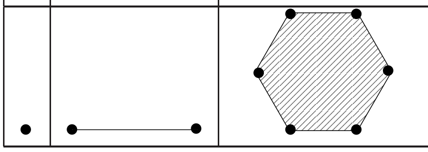
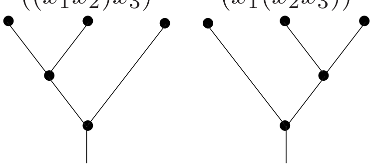


Figure 1.1: Circle

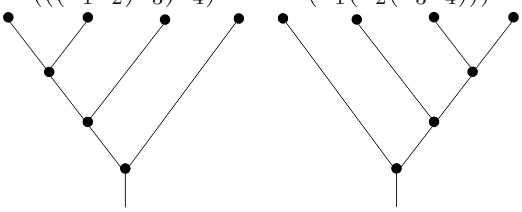
This is the only way to form a circle from an interval. If we try to decompose a disk of higher dimension, then we have choices. In the table below we give a few examples of decomposition of an n-cell.

0	1	2	...	n
				n-cell
				globular set
				n-simplex
				n-cube
				n-associahedron
				n-permutohedron

The construction of an n-associahedron can be given by the use of **Stasheff complex**. Its vertices are defined to be all ways of putting parentheses to a word of length $(n+1)$. They are in bijection with the set of planar binary rooted trees as we can see on example of words of length 3 and 4.



There is a partial order on trees in which first tree on the picture is before the second one. This can be generalized for the trees with more leaves, and is called the Tamari order.



We can associate a tree to each vertex of a 2-assciahedron and order them using the ordering on trees.

The realization of the Stasheff polytope as a subset in $\mathbb{R}^n$ is homeomorphic to a ball. To each planar binary tree t we associate a point $M(t) = (x_1, \dots, x_n)$ in $\mathbb{R}^n$ as follows. The i-th coordinate is the product of the number of leaves to the left of i-th vertex times the number of leaves to the right.

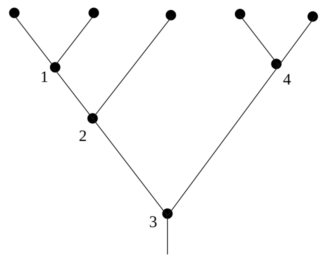


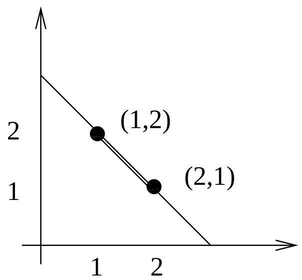
Figure 1.2: Tree t

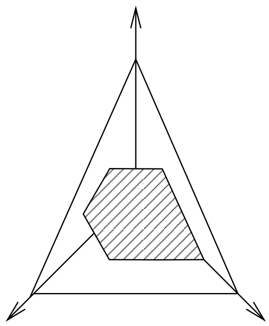
$$M(t) = (1 \cdot 1, 2 \cdot 1, 3 \cdot 2, 1 \cdot 1) = (1, 2, 6, 1) \in \mathbb{R}^4$$

The Stasheff polytope of dimension n is the convex hull of the points $M(t)$ for all planar binary trees with $(n+1)$ leaves. The sum of coordinates is

$$\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$$

so the Stasheff polytope lies in the hyperplane given by this equation. The examples of Stasheff polytopes $\mathcal{K}^1$ and $\mathcal{K}^2$ are in the following pictures.





The Stasheff polytope $\mathcal{K}^3$ has 14 vertices and 7 faces. The faces are three squares and four pentagons (2-associahedrons). In general, the Stasheff polytope $\mathcal{K}^n$ has faces of the form $\mathcal{K}^p \times \mathcal{K}^q$, where $p + q = n$.

What about the permutohedron? Take an element σ in the symmetric group S_n. Associate to it the point $M(\sigma) = (\sigma(1), \dots, \sigma(n)) \in \mathbb{R}^n$. Then we have permutohedron $\mathcal{P}^{n-1}$ as a convex hull of all points $M(\sigma)$ for all permutations. Of course $\sum_{i=1}^n \sigma(i) = \frac{n(n+1)}{2}$, so it lies in the hyperplane given by the equation $\sum_{i=1}^n x_i = \frac{n(n+1)}{2}$.

In general $\mathcal{P}^n$ has faces of the form $\mathcal{P}^p \times \mathcal{P}^q$, where $p + q = n - 1$.

Observe that we have an order on vertices of our complexes.

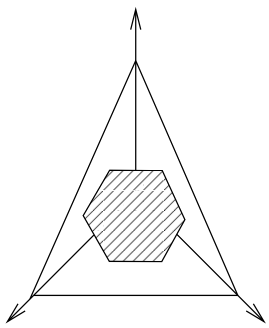
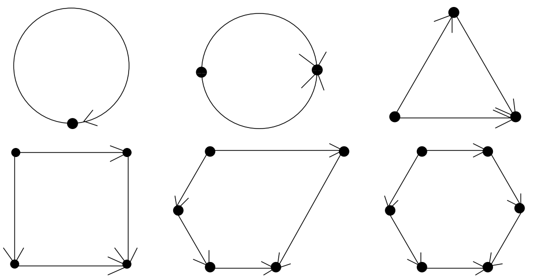


Figure 1.3: 2-permutohedron



On the set of vertices of n-simplex the order comes from the order on natural numbers, because vertices are numbered from 0 to n.

On the set of vertices n-associahedron the order is called the Tamari order.

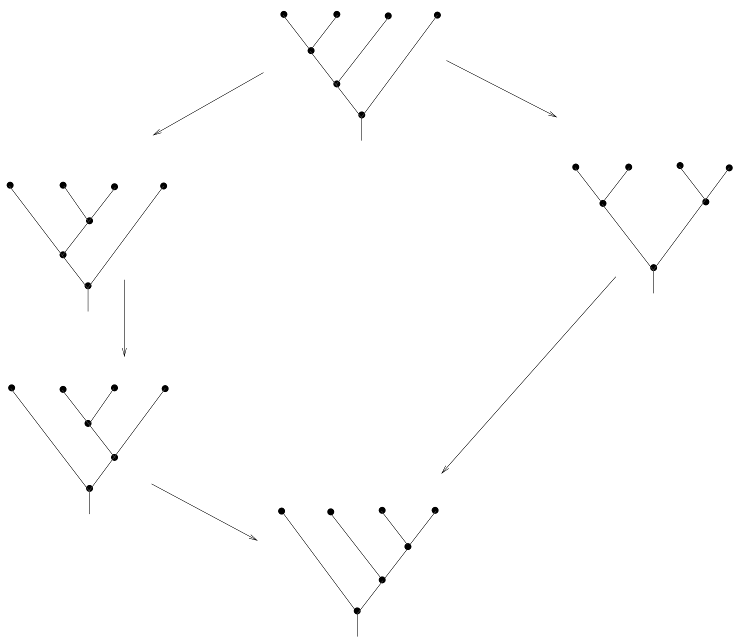


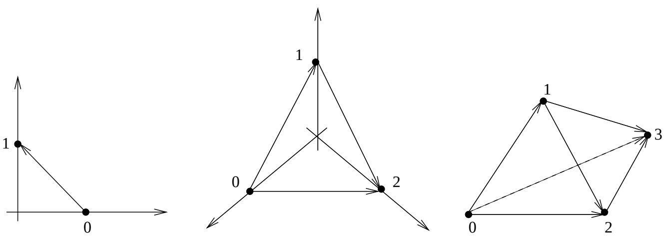
Figure 1.4: Tamari order on trees

On n-permutohedrons the order comes from the weak Bruhat order on the symmetric group S_n.

1.2 Simplicial sets

Definition 1.1. *The n-**simplex** is a ssubspace $\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_i x_i = 1, 0 \leq x_i \leq 1\}$. Denote by i the vertex on the x_i-axis.*

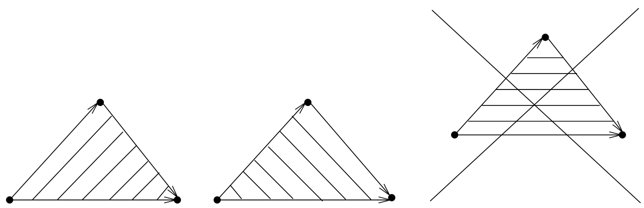
On the set of vertices of an n-simplex we have an ordering coming from the order on the set $[n] = \{0, \dots, n\}$.



Definition 1.2. *Define two kinds of order preserving maps on simplexes*

- ***Face maps*** $\delta_i\colon \Delta^{n-1} \to \Delta^n$, $i = 0, \dots, n$, *whose image is the face not containing i as image,*
$$\delta_i(x_0, \dots, x_{n-1}) = (x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$$
- ***Degeneracy maps*** $\sigma_j\colon \Delta^{n+1} \to \Delta^n$, $j = 0, \dots, n$ *which squeeze the j-th face.*
$$\sigma_j(x_0, \dots, x_{n+1}) = (x_0, \dots, x_{j-1}, x_j + x_{j+1}, x_{j+2}, \dots, x_{n+1})$$

Degeneracy map which does not preserve the ordering on vertices is not allowed. For example if $n = 2$ we have two allowed degeneracies s_0, s_1



The face and degeneracy maps satisfy the following identities

$$\delta_j \delta_i = \delta_i \delta_{j-1}, \quad i < j$$
$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \quad i \leq j$$
$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & i < j \\ \mathrm{id} & i = j,\ i = j+1 \\ \delta_{i-1} \sigma_j & i > j+1 \end{cases}$$

Definition 1.3. *A **simplicial set** is a collection of sets $\{K_n\}_{n \geq 0}$ with a collection of maps*

$$d_i\colon K_n \to K_{n-1}, \quad i = 0, \dots, n$$
$$s_j\colon K_n \to K_{n+1}, \quad j = 0, \dots, n$$

satisfying "the dual relations"

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i, \quad i < j \\
s_i s_j &= s_{j+1} s_i, \quad i \leq j \\
d_j s_i &= \begin{cases} s_{j-1} d_i & i < j \\ \mathrm{id} & i = j,\ i = j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}
\end{aligned}$$

A ***simplicial morphism*** $\varphi_\bullet \colon K_\bullet \to K'_\bullet$ *is a collection of maps* $\varphi_n \colon K_n \to K'_n$ *which commute with face and degeneracy maps*

$$\begin{array}{ccc} K_n & \xrightarrow{\varphi_n} & K'_n \\ {\scriptstyle d_i^K}\big\downarrow & & \big\downarrow{\scriptstyle d_i^{K'}} \\ K_{n-1} & \xrightarrow[\varphi_{n-1}]{} & K'_{n-1} \end{array} \qquad \begin{array}{ccc} K_{n+1} & \xrightarrow{\varphi_{n+1}} & K'_{n+1} \\ \big\uparrow{\scriptstyle s_j^K} & & {\scriptstyle s_j^{K'}}\big\uparrow \\ K_n & \xrightarrow[\varphi_n]{} & K'_n \end{array}$$

Now suppose we have a simplicial set $K_\bullet$. For all $x \in K_n$ we take a simplex Δ^n and we will build a topological space out of these data.

The **geometric realization** of a simplicial set is the following topological space

$$|X_\bullet| := \coprod_{n \geq 0} X_n \times \Delta^n / \sim,$$

where the equivalence relation $\sim$ is defined as follows. We identify $(x, \delta_i t) \in X_n \times \Delta^n$ with $(d_i x, t) \in X_{n-1} \times \Delta^{n-1}$ for any $x \in X_n$, $t \in \Delta^{n-1}$ and $(x, \sigma_j t) \in X_n \times \Delta^n$ with $(s_j x, t) \in X_{n+1} \times \Delta^{n+1}$ for any $x \in X_{n-1}$ and $t \in \Delta^{n+1}$. The topology on $|X_\bullet|$ is the quotient topology.

There exists a **simplicial category** Δ, whose objects are finite ordered sets $[n] = \{0, \dots, n\}$, and morphism $\mathrm{Mor}([n], [m])$ are nondecreasing set maps.

The category Δ can be described by generators and relations. As generators we take face and degeneracy maps

$$\begin{aligned}
\delta_i &\colon [n-1] \to [n] \\
\sigma_j &\colon [n+1] \to [n]
\end{aligned}$$

and relations are as before

$$\begin{aligned}
\delta_j \delta_i &= \delta_i \delta_{j-1}, \quad i < j \\
\sigma_j \sigma_i &= \sigma_i \sigma_{j+1}, \quad i \leq j \\
\sigma_j \delta_i &= \begin{cases} \delta_i \sigma_{j-1} & i < j \\ \mathrm{id} & i = j,\ i = j+1 \\ \delta_{i-1} \sigma_j & i > j+1 \end{cases}
\end{aligned}$$

Example 1.4. Take $X_n = \{*\}$ for all $n \geq 0$, d_i, s_j- the identity. Then $|\{*\}| = *$.

Example 1.5. Take a monoid M (or a group). Define $M_\bullet$ as follows.

$$M_n := M \underbrace{\times \ldots \times}_{n \text{ times}} M = M^n$$

$$d_i(m_1, \ldots, m_n) = \begin{cases} (m_2, \ldots, m_n) & i = 0 \\ (m_1, \ldots, m_i m_{i+1}, \ldots, m_n) & 0 < i < n \\ (m_1, \ldots, m_{n-1}) & i = n \end{cases}$$

$$s_j(m_1, \ldots, m_n) = (m_1, \ldots, m_j, 1, m_{j+1}, \ldots, m_n)$$

Example 1.6. Let $\mathcal{C}$ be a small category. The **nerve** of $\mathcal{C}$ is the following simplicial set

$$\mathcal{C}_n := \{C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} C_n\}$$

$$\begin{aligned} d_i(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} C_n) = &\text{ forget about } C_i \\ &= (C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \ldots \to \\ &\quad \to C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \to \ldots \xrightarrow{f_n} C_n) \\ s_j(C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} C_n) = &\text{ insert } \mathrm{id}_{C_j} \\ &= (C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \ldots \to \\ &\quad \to C_{j-1} \xrightarrow{f_j} C_j \xrightarrow{\mathrm{id}} C_j \xrightarrow{f_{j+1}} C_{j+1} \to \ldots \xrightarrow{f_n} C_n) \end{aligned}$$

The axioms of a category are exactly the conditions for $\mathcal{C}_\bullet$ to be a simplicial set.

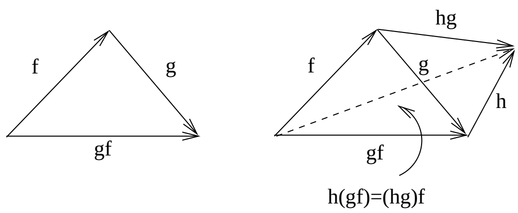


Figure 1.5: Associativity relation

To each category we associate its **classifying space**

$$\mathrm{B}\,\mathcal{C} := |\mathcal{C}_\bullet|$$

The classifying space $\mathrm{B}\,G$ of a group G is obtained from the realization of simplicial set in example (1.5). If G is discrete, then we can prove the following

$$\begin{aligned} \pi_1(\mathrm{B}\,G) &= G \\ \pi_n(\mathrm{B}\,G) &= 0, \quad n \geq 1. \end{aligned}$$

If all X_n are topological spaces, and the face and degeneracy maps are continuous, then we call $X_\bullet$ a **simplicial space**. Then the geometric realization is defined as before, but we keep track of the topology of X_n in the construction.

$$|X_\bullet| := \coprod_{n \geq 0} X_n \times \Delta^n / \sim,$$

$$(x, \delta_i t) \sim (d_i x, t)$$
$$(x, \sigma_j t) \sim (s_j x, t)$$

1.3 Fibrations

A **locally trivial fibration** is a surjective map of topological spaces $f\colon E \to B$ such that for every $b \in B$ there exists a neighbourhood U_b of b in B such that $f^{-1}(U_b) \simeq U_b \times F$, where F is a fiber.

Example 1.7. The Möbius band is a fibration over S^1. It is not a trivial fibration because it is not a product.

There is a fibration

$$G \to \mathrm{EG} \to \mathrm{BG}$$

where EG is a contractible space. For example if $G = \mathbb{Z}$, then this fibration is homotopy equivalent to

$$\mathbb{Z} \to \mathbb{R} \to S^1$$

But $\mathrm{B}\,\mathbb{Z}$ is not a space with one 0-cell and one 1-cell. The 0-cells are in bijection with $\mathbb{Z}$,

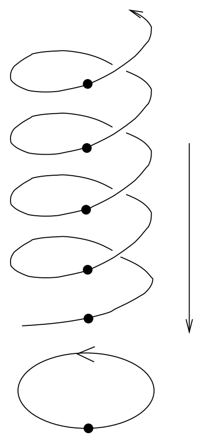


Figure 1.6: $\mathbb{Z} \hookrightarrow \mathbb{R} \to S^1$

and 1-cells are in bijection with pairs of distinct integers.

Example 1.8. The **Hopf fibration** it is a map $f\colon S^3 \to S^2$ with fiber S^1 which can be described as follows.

$$S^3 := \{(z, z') : |z|^2 + |z'|^2 = 1\} \subset \mathbb{C} \times \mathbb{C}$$
$$S^2 := \{(t, z) : t^2 + |z|^2 = 1\} \subset \mathbb{R} \times \mathbb{C}$$
$$f(z, z') = (|z|^2 - |z'|^2, 2zz') \in \mathbb{R} \times \mathbb{C}$$

The restriction of f to the north (resp. south) hemisphere is a trivial fibration.

Another description of the sphere S^3 is given by gluing two solid tori $S^1 \times D^2$ and $D^2 \times S^1$ along the boundary $S^1 \times S^1$.

If X and Y are pointed spaces, then we can perform the **join construction** $X * Y$.

$$X * Y := X \times I \times Y / \sim,$$

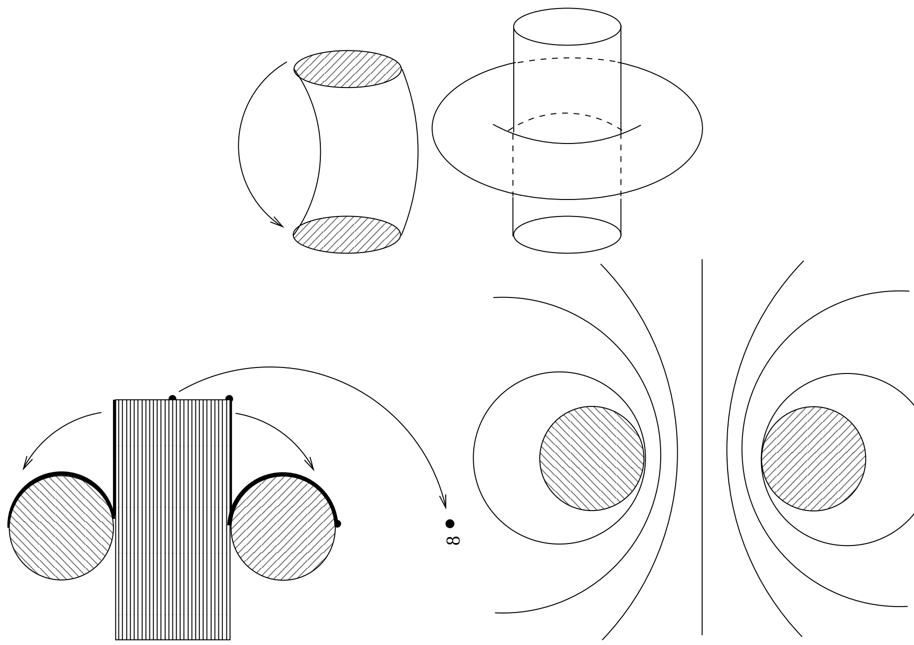


Figure 1.7: $S^3 = S^1 \times D^2 \cup_{S^1 \times S^1} D^2 \times S^1$

$$(x, 0, *) \sim (x', 0, *)$$
$$(*, 1, y) \sim (*, 1, y)$$

For example $S^1 * S^1 = S^3$.

Exercise 1.9. *Show that* $\Delta^p * \Delta^q \simeq \Delta^{p+q+1}$.

1.4 Cyclic category

We know, that $\mathrm{B}\,\mathbb{Z}$ is homotopy equivalent to S^1. Consider a question: what is the simplicial set $C_\bullet$ whose geometric realization is the circle with the cell structure consisting of one 0-cell and one 1-cell (not up to homotopy)?

The 0-cell $* \in C_0$ generates only one element, still denoted by $*$ in each C_n. Suppose we add an additional element τ to C_1. Then we get

$$\begin{aligned}
&C_0 = \{*\}\\
&C_1 = \{*, \tau\}\\
&C_2 = \{*, s_0\tau, s_1\tau\}\\
&C_3 = \{*, s_1s_0\tau, s_2s_0\tau, s_2s_1\tau\}\\
&\dots \quad \dots\\
&C_n = \{*, \dots, s_{n-1}\dots\widehat{s_i}, \dots s_0\tau, \dots\}
\end{aligned}$$

The faces are obvious to find. In particular $d_0(\tau) = * = d_1(\tau)$. Then the geometric realization $|C_\bullet|$ is a circle with its simplest cell structure. We can identify

$$C_n = \{*, \dots, s_{n-1}\dots\widehat{s_i}, \dots s_0\tau, \dots\}$$

with the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z} =: C_n$ by sending $*$ to 0, and $s_{n-1} \dots \widehat{s_i}, \dots s_0 \tau$ to $i+1$. Denote the generator of C_n by t_n.

There exists a **cyclic category** ΔC whose objects are finite ordered sets $[n] = \{0, \dots, n\}$, and morphism $\mathrm{Mor}([n],[m])$ are generated by δ_i, σ_j as in simplicial category, and additional morphisms $\tau_n \colon [n] \to [n]$ for all $n \geq 0$ satisfying the relations

$$\begin{aligned}
\tau_n^{n+1} &= \mathrm{id}_{[n]} \\
\tau_n \delta_i &= \delta_{i-1} \tau_{n-1}, \quad 1 \leq i \leq n \\
\tau_n \delta_0 &= \delta_n \\
\tau_n \sigma_j &= \sigma_{j-1} \tau_{n+1}, \quad 1 \leq j \leq n \\
\tau_n \sigma_0 &= \sigma_n \tau_{n+1}^2
\end{aligned}$$

If in this presentation we omit the relation $\tau_n^{n+1} = \mathrm{id}_{[n]}$, then we get a different category, denoted $\Delta\mathbb{Z}$.

Definition 1.10. *A cyclic set is a functor* $\Delta C^{op} \to$ **Sets**.

Proposition 1.11. $C_\bullet$ *is a cyclic set.*

Proposition 1.12.

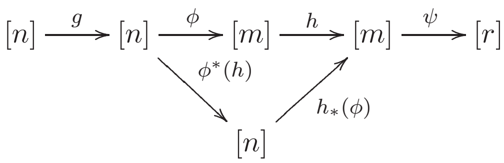
$$\mathrm{Aut}_\Delta([n]) = \{1\}$$

$$\mathrm{Aut}_{\Delta C}([n]) = C_n$$

Every morphism of ΔC can be written uniquely as $\phi \circ g$, where $\phi \in \mathrm{Mor}_\Delta([n],[m])$, $g \in C_n = \mathrm{Mor}_{\Delta C}([n],[n])$. As sets

$$\mathrm{Hom}_{\Delta C}([n],[m]) \simeq \mathrm{Hom}_\Delta([n],[m]) \times C_n$$

The compositon of two morphisms $(g \circ \phi)$ and $(h \circ \psi)$ is in ΔC, so there exist $\phi^*(h) \in C_n$ and $h_*(\phi) \in \mathrm{Mor}_\Delta([n],[m])$ such that the following diagram commutes.



Analogously, suppose we have two subgroups $A, B < G$ such that every element of G can be written uniquely as $g = ab$, $a \in A$, $b \in B$. In this situation

$$gg' = aba'b' = a \underbrace{b^*(a')}_{\in A} \underbrace{a'_*(b)}_{\in B} b'$$

The relations satisfied by ϕ^* and h_* are exactly the same as the relations satisfied by $b^* \colon A \to A$ and $a_* \colon B \to B$.

Remark 1.13. There is a way of constructing a category ΔS along the same lines, such that $\mathrm{Aut}_{\Delta S}([n]) = S_{n+1}$ - the symmetric group. Every morphism of ΔS can be written uniquely as $\phi \circ g$, where $\phi \in \mathrm{Mor}_\Delta([n],[m])$, $g \in S_n = \mathrm{Mor}_{\Delta S}([n],[n])$. As sets

$$\mathrm{Hom}_{\Delta C}([n],[m]) \simeq \mathrm{Hom}_\Delta([n],[m]) \times S_n$$

It means that for any $\phi \in \mathrm{Mor}_\Delta([m],[n])$ and $\sigma \in S_n$ there exist $\phi^*(\sigma) \in S_{m+1}$ and $\sigma_*(\phi) \in \mathrm{Mor}_\Delta([m],[n])$ such that the following diagram commutes.

$$\begin{array}{ccc} [m] & \xrightarrow{\phi} & [n] \\ {\scriptstyle \phi^*(g)\in S_{n+1}}\downarrow & & \downarrow{\scriptstyle \sigma\in S_{n+1}} \\ [n] & \xrightarrow[\sigma_*(\phi)]{} & [n] \end{array}$$

Denote by ΔB the braided category, defined along the same lines using braid groups, which

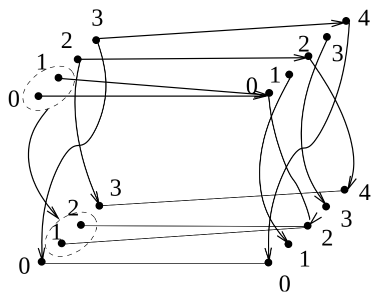


Figure 1.8: Morphisms in ΔS

contains ΔS as a quotient category. Let $H_n = (\mathbb{Z}/2)^n \rtimes S_n = \mathbb{Z}/2 \int S_n$ and denote corresponding hyperdihedral category by ΔH. Furthermore we have a dihedral category ΔD. We can arrange them in a diagram of inclusions

$$\begin{array}{ccccc} & & \Delta C & \longrightarrow & \Delta S & \longrightarrow & \Delta B \\ & & \downarrow & & \downarrow & & \\ \Delta\mathbb{Z}/2 & \longrightarrow & \Delta D & \longrightarrow & \Delta H & & \end{array}$$

There is an exact sequence of groups

$$0 \to \mathbb{Z} \xrightarrow{\cdot(n+1)} \mathbb{Z} \to \mathbb{Z}/(n+1) \to 0$$

If we treat $\mathbb{Z}$ as category, then we have following diagram of functors

$$\Delta \times \mathbb{Z} \to \Delta\mathbb{Z} \to \Delta C$$

We can ask what kind of structure on the geometric realization of the underlying simplicial set $X_\bullet$, that is $|X_\bullet|$, does the cyclic structure give? The answer is a structure of S^1-space. An open question is: can we discretize analogously $S^3 = \mathrm{SU}(2)$?

1.5 Noncommutative sets

Let **Fin** denote the skeleton category of the category of finite sets. This means that the objects in **Fin** are the sets $[n] = \{0, 1 \dots, n\}$ and morphisms are arbitrary functions. Let **Fin**′ denote a category with the same objects, but whose morphisms satisfy $f(0) = 0$. Then there is a following diagram of categories

$$\begin{array}{ccccc} \Delta^{op} & \longrightarrow & \Delta S'^{op} & \longrightarrow & \mathbf{Fin}' \\ \downarrow & & \downarrow & & \downarrow \\ \Delta C = \Delta C^{op} & \longrightarrow & \Delta S & \longrightarrow & \mathbf{Fin} \end{array}$$

For a set $[n]$ we have

$$\mathrm{Aut}_{\Delta S}([n]) = S_{n+1},$$
$$\mathrm{Aut}_{\Delta S'}([n]) = S_n.$$

The top row of this diagram will correspond to Hochschild homology, and the bottom row to cyclic homology, which we will define in the next chapter.

If A is an algebra, then $[n] \mapsto A^{\otimes(n+1)}$ is a well defined functor $\Delta S \to \mathbf{Mod}$.

$$A^{\otimes 2} \rightrightarrows A, \ a \otimes b \mapsto ab, \ a \otimes b \mapsto ba.$$

The two maps $d_1, d_0 \colon [1] \to [0]$ become the same in **Fin**. If A is commutative, then $[n] \to A^{\otimes(n+1)}$ factors through **Fin**.

Thus ΔS can be viewed as a category of noncommutative sets. It has a following description

$$\mathrm{Ob}(\Delta S) = \{[n]\}$$
$$\mathrm{Mor}_{\Delta S}([n],[m]) = \text{ set of maps with an order on the fibers } f^{-1}(i) \text{ for } i \in [m].$$

1.6 Adjoint functors

Suppose we have two categories $\mathcal{A}$ and $\mathcal{B}$ and a pair of functors $F \colon \mathcal{A} \to \mathcal{B}$, $G \colon \mathcal{B} \to \mathcal{A}$. We say that F is **right adjoint** to G and G is **left adjoint** to F if there is an isomorphism of sets

$$\mathrm{Hom}_{\mathcal{A}}(G(B), A) \simeq \mathrm{Hom}_{\mathcal{B}}(B, F(A))$$

for every $A \in \mathrm{Ob}(\mathcal{A})$, $B \in \mathrm{Ob}(\mathcal{B})$, and the isomorphism is functorial in A and B.

Example 1.14. Let $\mathcal{A}, \mathcal{B} = \mathbf{Sets}$. Take a set X and define

$$G(B) = B \times X, \qquad F(A) = \mathrm{Hom}_{\mathbf{Sets}}(X, A)$$

Then

$$\mathrm{Hom}(B \times X, A) \simeq \mathrm{Hom}(B, \mathrm{Hom}(X, A))$$
$$\varphi \colon B \times X \to A \mapsto (B \to \mathrm{Hom}(X, A))$$

Many examples follow the pattern in (1.14), but with additional structure.

Example 1.15. Let $\mathcal{A}, \mathcal{B} = \mathbf{Vect}$, V vector space over a field k. Define

$$G(B) = B \otimes_k V, \qquad F(A) = \mathrm{Hom}_k(V, A)$$

Then

$$\mathrm{Hom}_k(B \otimes_k V, A) = \mathrm{Hom}_k(B, \mathrm{Hom}_k(V, A))$$

Example 1.16. Let R be a ring, $\mathcal{A}$ be the category of left R-modules, and $\mathcal{B}$ the category or right R-modules. Take a left R-module V and define

$$G(B) = B \otimes_R V, \qquad F(A) = \mathrm{Hom}_R(V, A)$$
$$\mathrm{Hom}_{\mathbb{Z}}(B \otimes_R V, A) = \mathrm{Hom}_{\mathbb{Z}}(B, \mathrm{Hom}_R(V, A))$$

Example 1.17. Define the loop space and the suspension of a topological space X with base point as follows.

$$\Omega X = \{f \colon S^1 \to X : f(*) = *\}$$
$$SX = S^1 \wedge X / S^1 \vee X$$

Then

$$\mathrm{Hom}_{\mathbf{Top}_*}(SX, Y) \simeq \mathrm{Hom}_{\mathbf{Top}_*}(X, \Omega Y)$$

where $\mathbf{Top}_*$ is the category of topological spaces with base point.

1.7 Generic example of a simplicial set

Let X be a topological space. Define

$$\mathcal{S}_n(X) := \{f\colon \Delta \to X,\ \text{continuous}\}$$

We claim that $\mathcal{S}_\bullet(X)$ is a simplicial set with the following face and degeneracy maps:

$$d_i\colon \mathcal{S}_n(X) \to \mathcal{S}_{n-1}(X), \quad d_i(f) := f \circ \delta_i$$
$$s_j\colon \mathcal{S}_n(X) \to \mathcal{S}_{n+1}(X), \quad s_j(f) := f \circ \sigma_j$$

It is called the **singular functor**. It goes from the category of topological spaces to the category of simplicial sets.

$$\mathcal{S}_\bullet(-)\colon \mathbf{Top} \to \mathbf{SSets}$$

Recall the functor of geometric realization of a simplicial set,

$$K_\bullet \mapsto |K_\bullet|, \quad |-|\colon \mathbf{SSets} \to \mathbf{Top}$$

Proposition 1.18. *The functors $\mathcal{S}_\bullet(-)$ and $|-|$ are adjoint, that is*

$$\mathrm{Hom}_{\mathbf{Top}}(|K_\bullet|, X) \simeq \mathrm{Hom}_{\mathbf{SSets}}(K_\bullet, \mathcal{S}_\bullet(X)).$$

In the example (1.16) R-modules can be replaced by functors. Left modules correspond to covariant functors, and right modules correspond to contravariant functors. Then the geometric realization functor can be seen as a tensor product over the simplicial category

$$|K_\bullet| = K_\bullet \otimes_\Delta \Delta^\bullet$$

In an analogous way we can present the singular functor as

$$\mathcal{S}_\bullet(X) = \mathrm{Hom}_{\mathbf{Top}}(\Delta^\bullet, X)$$

Hence we can derive adjointness

$$\mathrm{Hom}_{\mathbf{Top}}(K_\bullet \otimes_\Delta \Delta^\bullet, X) \simeq \mathrm{Hom}_\Delta(K_\bullet, \mathrm{Hom}_{\mathbf{Top}}(\Delta^\bullet, X))$$

Now the question arises: how to compare X and $|\mathcal{S}_\bullet(X)|$? Take $\mathrm{id} \in \mathrm{Hom}_{\mathbf{SSets}}(\mathcal{S}_\bullet(X), \mathcal{S}_\bullet(X))$. This identity goes to a map

$$\varepsilon\colon |\mathcal{S}_\bullet(X)| \to X$$

which is called a **unit**. Also $\mathrm{id} \in \mathrm{Hom}_{\mathbf{Top}}(|K_\bullet|, |K_\bullet|)$ goes to a map

$$\eta\colon K_\bullet \to \mathcal{S}_\bullet(|K_\bullet|)$$

which is called a **counit**. If X is a CW-complex, then this map is a homotopy equivalence.

Now we will prove the following theorem.

Theorem 1.19. *If $X_\bullet$ is a cyclic set, then the geometric realization $|X_\bullet|$ is an S^1-space.*

Before the proof, we will give some necessary propositions.

Lemma 1.20. *The functor $\Delta \to$ **Top** given by $[n] \mapsto \Delta^n$ is in fact a functor on ΔC (it is a **cocyclic space**).*

Proof. It is enough to define the image of τ_n

$$\tau_n \mapsto \{\Delta^n \to \Delta^n\}$$

$$\text{vertex } i \mapsto \text{vertex } i-1$$

$$\text{vertex } 0 \mapsto \text{vertex } n$$

□

Let $C_\bullet$ be the cyclic set, whose geometric realization is the circle. A naive way to define an S^1-action would be to use

$$C_\bullet \times X_\bullet \to X_\bullet$$

$$(g, x) \mapsto g_*(x)$$

But it does not work, since it gives a trivial action of S^1 for $X_\bullet = C_\bullet$.

There is a forgetful functor from the category of cyclic sets to the category of simplicial sets.

$$G\colon \mathbf{CSets} \to \mathbf{SSets}$$

We will define its left adjoint

$$F\colon \mathbf{SSets} \to \mathbf{CSets}$$

If $Y_\bullet$ is a simplicial set, then put

$$F(Y_\bullet)_n := C_n \times Y_n, \quad C_n = \mathbb{Z}/(n+1)\mathbb{Z}$$

If f is a morphism in Δ^{op}, then we define

$$f_*(g, y) := (f_*(g), (g^*(f))_*(y))$$

$$\begin{array}{ccc} [n] & \xrightarrow{f} & [m] \\ {\scriptstyle f_*(g)}\downarrow & & \downarrow{\scriptstyle g} \\ [n] & \xrightarrow[g^*(f)]{} & [m] \end{array}$$

If h is a morphism in C_m, then we define

$$h^*(g, y) := (h(g), y)$$

Proposition 1.21. *The set $F(Y_\bullet)$ equipped with the simplicial structure given by f_* and the cyclic structure given by h^* is a cyclic set.*

Proposition 1.22. *If $X_\bullet$, $Y_\bullet$ are simplicial sets, and if $|X_\bullet| \times |Y_\bullet|$ is a CW-complex, then the map*

$$|X_\bullet \times Y_\bullet| \to |X_\bullet| \times |Y_\bullet|$$

is a homeomorphism.

Proposition 1.23. *If $X_\bullet$ is a cyclic set, then we have a homeomorphism*

$$|F(X_\bullet)| \simeq |C_\bullet| \times |X_\bullet| = S^1 \times |X_\bullet|$$

Observe that the composite

$$|F(X_\bullet)| \to |C_\bullet| \times |X_\bullet| \xrightarrow{\simeq} |C_\bullet \times X_\bullet|$$

is not the geometric realization of a simplicial map.

Proof. It is induced by the two projections

$$|F(X_\bullet)| \xrightarrow{p_1 \times p_2} |C_\bullet| \times |X_\bullet|$$

The map p_1 is induced by $(g, y) \mapsto g$, and p_2 is induced by $(g, y) \mapsto y$.

Next we define

$$C_n \times X_n \times \Delta^n \to X_n \times \Delta^n$$
$$(g, y, t) \mapsto (y, g^*(t))$$

and show that it is compatible with the equivalence relation. It induces a cyclic map called the **evaluation**

$$F(X_\bullet) \xrightarrow{ev} X_\bullet$$

which gives a map

$$|F(X_\bullet)| \xrightarrow{|ev|} |X_\bullet|$$

□

Proof. (of theorem (1.19)) Define a map

$$S^1 \times |X_\bullet| \xrightarrow{\simeq} |C_\bullet| \times |X_\bullet| \xrightarrow{(p_1,p_2)^{-1}} |F(X_\bullet)| \xrightarrow{ev} |X_\bullet|$$

If we want it to be an S^1-action on $|X_\bullet|$, then the following diagram has to commute

$$\begin{array}{ccc} S^1 \times S^1 \times |X_\bullet| & \longrightarrow & S^1 \times |X_\bullet| \\ \downarrow & & \downarrow \\ S^1 \times |X_\bullet| & \longrightarrow & |X_\bullet| \end{array}$$

Let $X_\bullet = C_\bullet$. We will show that, the action $S^1 \times S^1 \to S^1$ is the classical multiplication of units in $\mathbb{C}$.

$$\begin{aligned} F(C_\bullet)_0 &= (*, *), \quad 1 \in C_0 \\ F(C_\bullet)_1 &= \underbrace{(*, t_1),\ (t_1, *),\ (t_1, t_1),}_{\text{nondegenerate simplices}}\ (*, *) \\ F(C_\bullet)_2 &= (t_2, t_2),\ (t_2^2, t_2^2),\ \text{all other simplices are degenerate} \end{aligned}$$

The higher rank simplices are degenerate.

We will examine the evaluation map

$$S^1 \times S^1 = |F(C_\bullet)| \to |C_\bullet| = S^1$$

Take $(u, v) \in |F(C_\bullet)|$. Then

$$(u, v) \in \begin{cases} \{(t_2^2, t_2^2) \times \Delta^2\} & \text{if } u + v \leq 1 \\ \{(t_2, t_2) \times \Delta^2\} & \text{if } u + v \geq 1 \end{cases}$$

The formulas

$$d_0(t_2, t_2) = (*, t_1)$$
$$d_2(t_2^2, t_2^2) = (t_1, *)$$

show that the 0-th face of the triangle (t_2, t_2) has to be identified with the 2-nd face of the triangle (t_2^2, t_2^2).

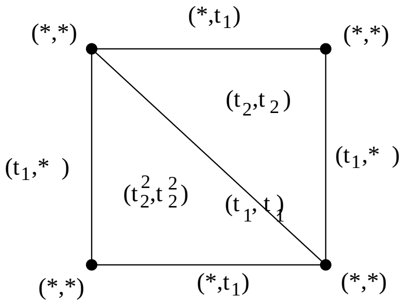


Figure 1.9: 0,1, and 2-faces

$$F(C_\bullet) \xrightarrow{ev} C_\bullet, \quad (t_2, t_2) \mapsto t_2$$

$$ev(t_2^2, t_2^2) = t_2^4 = t_2 = s_1(t_1), \text{ because } t_2^3 = 1$$
$$ev(t_2, t_2) = t_2^2 = s_0(t_1)$$

$$\begin{array}{ccc} & F(C_\bullet) & \\ {}^{p_1 \times p_2}\swarrow & & \searrow^{|ev|} \\ |C_\bullet| \times |C_\bullet| & \longrightarrow & |C_\bullet| \end{array}$$

$$S^1 \times |C_\bullet| \to |C_\bullet|$$

$$C_0 = \{1\}$$
$$C_1 = \{1, t_1\}$$
$$C_2 = \{1, t_2, t_2^2\}$$

Degenerate simplices will be identified with the interval. There are two ways to do that.

$$\begin{array}{ccc} \bigcup_{n\geq 0} F(C_\bullet) \times \Delta^n & \longrightarrow & \bigcup_{n\geq 0} C_n \times \Delta^n \\ \downarrow & & \downarrow \\ |F(C_\bullet)| & \longrightarrow & |C_\bullet| \end{array}$$

$$ev\colon (t_2, t_2) \times \Delta^2 \mapsto (s_0 t_1, \Delta^2)$$
$$ev\colon (t_2^2, t_2^2) \times \Delta^2 \mapsto (s_1 t_1, \Delta^2)$$

Therefore the map $|ev|\colon S^1 \times S^1 \to S^1$ is the multiplication of complex units (under the exponential map $\exp(2\pi i -)\colon \mathbb{R}/\mathbb{Z} \to \mathrm{SO}(2)$).

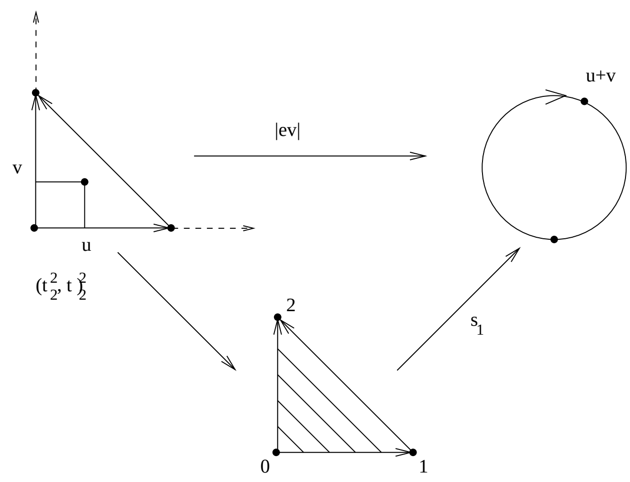


Figure 1.10:

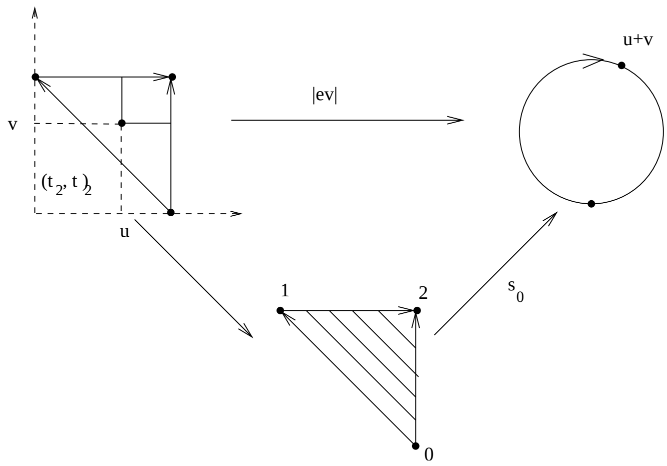
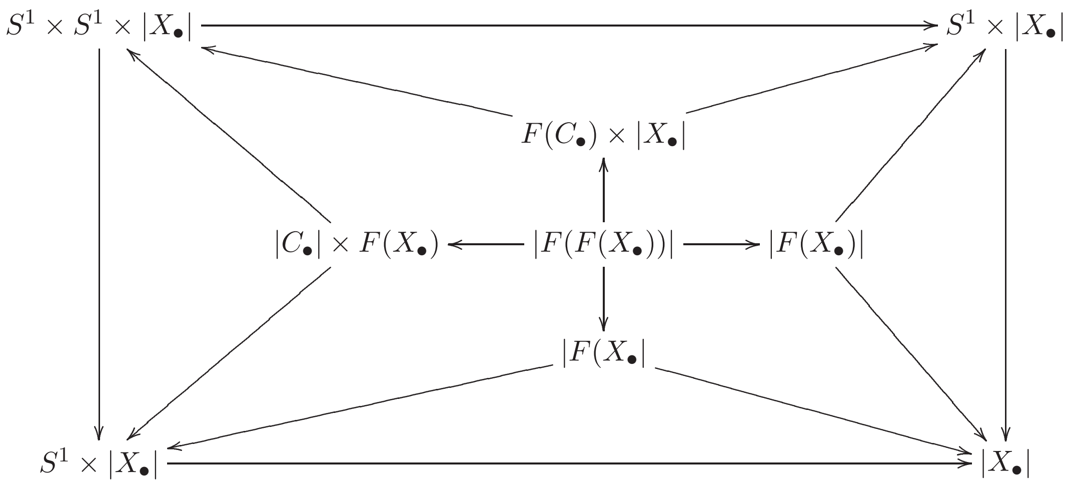


Figure 1.11:

At the end we get a commutative diagram:



As a consequence $|X_\bullet|$ is an S^1-space. □

1.8 Simplicial modules

Definition 1.24. *A* ***simplicial module*** *is a functor*

$$\Delta^{op} \to \mathbf{Mod}_k, \quad [n] \mapsto M_n$$

There is a chain complex associated to a simplicial module

$$M_\bullet : \quad \ldots \to M_n \xrightarrow{b_n} M_{n-1} \xrightarrow{b_{n-1}} M_{n-1} \to \ldots$$

where $b = b_n = \sum_{i=0}^{n} (-1)^i d_i$. We have $b^2 = 0$ as an immediate consequence of $d_i d_j = d_{j-1} d_i$, $i < j$, for example:

$$\ldots M_2 \xrightarrow{d_0 - d_1 + d_2} M_1 \xrightarrow{d_0 - d_1} M_0$$

$$(d_0 - d_1)(d_0 - d_1 + d_2) = \underbrace{d_0 d_0 - d_0 d_1}_{0} + \underbrace{d_0 d_2 - d_1 d_0}_{0} + \underbrace{d_1 d_1 - d_1 d_2}_{0} = 0$$

We define the homology of a simplicial module as

$$\mathrm{H}_n(M_\bullet) := \ker(b_n) / \operatorname{im}(b_{n-1})$$

It is well defined for presimplicial module, that is using only face maps.

Lemma 1.25. *The submodule M'_n of M_n spanned be the degeneracy elements gives a subcomplex $M'_\bullet$ of $M_\bullet$.*

Proof. This is a consequence of the relations between s_j, d_i. □

Define the **normalized** complex $\overline{M}_\bullet$ as a quotient

$$0 \to M'_\bullet \to M_\bullet \to \overline{M}_\bullet \to 0$$

Theorem 1.26. *The quotient map $M_\bullet \to \overline{M}_\bullet$ is an quasi-isomorphism, i.e. it induces an isomorphism in homology.*

Proof. From the long exact sequence in homology

$$\ldots \to \mathrm{H}_n(M'_\bullet) \to \mathrm{H}_n(M_\bullet) \to \mathrm{H}_n(\overline{M}_\bullet) \xrightarrow{\delta} \mathrm{H}_{n-1}(M'_\bullet) \to \ldots$$

it is enough to prove that $\mathrm{H}_n(M'_\bullet) = 0$.

If one wants to prove that some complex $C_\bullet$ is acyclic, then it is enough to construct a homotopy from id to 0 (contraction), that it a sequence of maps $h_n : C_n \to C_{n+1}$ such that $h_{n-1} d_{n-1} + d_n h_n = \mathrm{id}$. Unfortunately it is hard to find a contracting homotopy for $M'_\bullet$ to prove that it is acyclic. But one can define a filtration on $M'_\bullet$

$$F_k \hookrightarrow F_{k+1} \to G_k$$

with F_k spanned by the first k degeneracies, and quotient $G_\bullet$ for which we can construct a contracting homotopy. Then we can proceed by induction. □

Let A be a k-algebra and M an A-module. There is a simplicial module

$$C_\bullet(A, M) := M \otimes A^{\otimes n}$$

$$
\begin{aligned}
d_i(a_0, a_1, \ldots, a_n) &= (a_0, \ldots, a_i a_{i+1}, \ldots, a_n), \quad i = 0, \ldots, n-1 \\
d_n(a_0, a_1, \ldots, a_n) &= (a_n a_0, \ldots, a_{n-1}) \\
s_j(a_0, a_1, \ldots, a_n) &= (a_0, \ldots, a_j, 1, a_{j+1}, \ldots, a_n)
\end{aligned}
$$

Define

$$b := \sum_{i=0}^{n} (-1)^i d_i$$

Then $(C_\bullet(A, M), b)$ is called the **Hochschild chain complex**, and its homology $\mathrm{H}_*(A; M)$ the **Hochschild homology** of A with coefficients in M. If $M = A$, then we denote

$$\mathrm{H}_*(A; A) =: \mathrm{HH}_*(A)$$

Suppose that A is augmented and let $\overline{A}$ be its augmentation ideal in A, that is $A = \overline{A} \oplus k1$. Define the **reduced Hochschild complex** as

$$\overline{C}_n(A, M) := M \otimes \overline{A}^{\otimes n}$$

If $M = A = \bar{A} \oplus k1$, then $C_\bullet(A, A)$ has extra degeneracy

$$s_{-1}(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n).$$

We have

$$
\begin{aligned}
d_0(1, a_1, \ldots, a_n) &= (a_1, \ldots, a_n) \\
d_n(1, a_1, \ldots, a_n) &= (a_n, \ldots, a_1)
\end{aligned}
$$

Define also two maps on $\overline{A}^{\otimes n}$

$$
\begin{aligned}
t(a_1, \ldots, a_n) &:= (-1)^n (a_n, a_1, \ldots, a_{n-1}) \\
b' &:= \sum_{i=0}^{n-1} (-1)^i d_i, \quad (b = b' + (-1)^n d_n)
\end{aligned}
$$

1.9 Bicomplexes

Assume we have an array of k-modules

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
C_{02} & \xleftarrow{d^h} & C_{12} & \xleftarrow{d^h} & C_{22} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
C_{01} & \xleftarrow{d^h} & C_{11} & \xleftarrow{d^h} & C_{21} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \downarrow{\scriptstyle d^v} & & \\
C_{00} & \xleftarrow{d^h} & C_{10} & \xleftarrow{d^h} & C_{20} & \longleftarrow & \cdots
\end{array}
$$

We call it a **bicomplex** of k-modules if the maps d^v and d^h, called vertical and horizontal differential, satisfy

$$\begin{aligned} d^v \circ d^v &= 0 \\ d^h \circ d^h &= 0 \\ d^h \circ d^v + d^v \circ d^h &= 0 \end{aligned}$$

For a bicomplex $C_{\bullet\bullet}$ we define a **total complex** as

$$\mathrm{Tot}(C_{\bullet\bullet})_n := \bigoplus_{p+q=n} C_{pq}, \quad d := d^h + d^v$$

After taking homology with respect to the vertical differential we obtain a complex

$$\ldots \leftarrow \mathrm{H}^v_{(p-1),\bullet} \leftarrow \mathrm{H}^v_{p,\bullet} \leftarrow \mathrm{H}^v_{(p+1),\bullet} \leftarrow \ldots$$

with the differential induced on homology by horizontal differential in the bicomplex. Now we can take homology of this complex and obtain

$$E^2_{pq} := \mathrm{H}^h_q(\mathrm{H}^v_{p,\bullet})$$

There is a decomposition of the reduced Hochschild complex

$$\overline{C}_n(A,A) = A \otimes \overline{A}^{\otimes n} = (\overline{A} \oplus k1) \otimes \overline{A}^{\otimes n} = \overline{A}^{\otimes(n+1)} \oplus \overline{A}^{\otimes n}$$

and a map

$$\begin{pmatrix} b & 1-t \\ 0 & -b' \end{pmatrix} : \overline{A}^{\otimes(n+1)} \oplus \overline{A}^{\otimes n} \to \overline{A}^{\otimes n} \oplus \overline{A}^{\otimes(n-1)}$$

which fits in the diagram

$$\begin{array}{ccc} \overline{C}_n(A,A) & \xrightarrow{\simeq} & \overline{A}^{\otimes(n+1)} \oplus \overline{A}^{\otimes n} \\ {\scriptstyle b}\downarrow & & \downarrow \begin{pmatrix} b & 1-t \\ 0 & -b' \end{pmatrix} \\ \overline{C}_{n-1}(A,A) & \xrightarrow{\simeq} & \overline{A}^{\otimes n} \oplus \overline{A}^{\otimes(n-1)} \end{array}$$

This complex can be thought of as the total complex of a bicomplex

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ \overline{A}^{\otimes 3} & \xleftarrow{1-t} & \overline{A}^{\otimes 3} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle -b'} \\ \overline{A}^{\otimes 2} & \xleftarrow{1-t} & \overline{A}^{\otimes 2} \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle -b'} \\ \overline{A} & \xleftarrow{1-t} & \overline{A} \end{array}$$

Here we see the beginning of the complex computing the homology of the cyclic group with coefficients in a module. This will lead to the cyclic bicomplex.

1.10 Spectral sequences

Having computed $E^2_{pq} = \mathrm{H}^h_q(\mathrm{H}^v_{p,\bullet})$ of a bicomplex $C_{\bullet\bullet}$ it seems that we have used all data, that is vertical and horizontal differentials in the bicomplex. However, there is a piece of information which we can extract in addition to E^2_{pq}. We can define a homomorphism

$$d^2\colon E^2_{pq} \to E^2_{p-2,q+1}$$

as follows.

$$\begin{array}{ccccc} C_{p-2,q+1} & \xleftarrow{d_h} & C_{p-1,q+1} & & \\ & & \downarrow{\scriptstyle d^v} & & \\ & & C_{p-1,q} & \xleftarrow{d^h} & C_{pq} \end{array}$$

Using a horizontal cycle $x \in \mathrm{Z}_p(C_{\bullet,q})$ we want to define an element in $C_{p-2,q-1}$ which represents an element in horizontal cycles of vertical homology complex, that is in $\mathrm{Z}^h_p(\mathrm{H}^v_q(C_{\bullet\bullet}))$. Our x gives $[x] \in \mathrm{H}^v_q(C_{\bullet\bullet})$. Using the induced map

$$d^h_*\colon\ \mathrm{H}^v_q(C_{p,\bullet}) \to \mathrm{H}^v_q(C_{p-1,\bullet})$$

we have $d^h_*([x]) = 0 = [d^h(x)]$. Saying that the homology class is zero means that the cycle is in fact a boundary. Therefore there exists an $y \in C_{p-1,q+1}$ such that $d^v(y) = d^h(x)$. Now we define our cycle as $d^h(y) \in C_{p-2,q+1}$.

$$\begin{array}{ccccc} d^h(y) & \xleftarrow{d_h} & y & & \\ & & \downarrow{\scriptstyle d^v} & & \\ & & d^v(y) = d^h(x) & \xleftarrow{d^h} & x \end{array}$$

We claim that this element defines an element in $E^2_{p-2,q+1}$ which does not depend on the choice of y nor on the choice of the representative of $[x]$. Thus we have defined

$$d^2\colon E^2_{pq} \to E^2_{p-2,q+1}, \quad [x] \mapsto [d^h(y)].$$

Furthermore $d^2 \circ d^2 = 0$, so now we can take homology to obtain E^3_{pq} and

$$d^3\colon E^2_{pq} \to E^3_{p-3,q+2}.$$

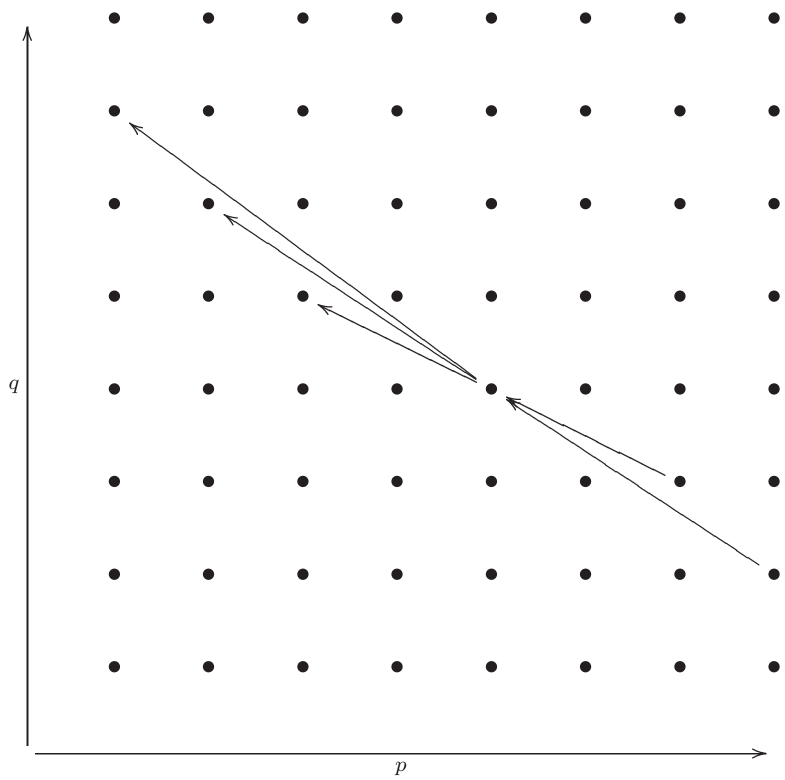
This procedure can be continued and as a result we get a sequence of arrays E^r_{pq} for any $r \geq 2$ and maps

$$d^r\colon E^r_{pq} \to E^r_{p-r,q+r-1}$$

such that E^r_{pq} is the homology of the complex (E^{r-1}, d^{r-1}) at the place (p,q). Furthermore there are subspaces B^r_{pq}, Z^r_{pq} of C_{pq}

$$B^2_{pq} \subseteq B^3_{pq} \subseteq \ldots \subseteq B^\infty_{pq} \subseteq Z^\infty_{pq} \subseteq \ldots \subseteq Z^2_{pq} \subseteq Z^2_{pq} \subseteq C_{pq}$$

such that $E^r_{pq} = Z^r_{pq}/B^r_{pq}$.



When both differentials (leaving and entering) for E^r_{pq} are zero, this component does not change furthermore and we have $E^r_{pq} = E^{r+1}_{pq} = \ldots$. We denote this stable component by E^{∞}_{pq}.

There is a filtration on the total complex

$$\mathrm{F}_p \operatorname{Tot} C_{\bullet\bullet} := \operatorname{Tot} \bigoplus_{k \le p} C_{k\bullet},$$

$$0 \subseteq F_0 \subseteq F_1 \subseteq \ldots \subseteq F_{p-1} \subseteq F_p \subseteq \ldots \subseteq \operatorname{Tot} C_{\bullet\bullet}.$$

This filtration induces a filtration on $\mathrm{H}_*(\operatorname{Tot} C_{\bullet\bullet})$

$$\mathrm{F}_p := \mathrm{F}_p \,\mathrm{H}_*(\operatorname{Tot} C_{\bullet\bullet}) := \operatorname{im}(\mathrm{H}_*(\mathrm{F}_p \operatorname{Tot} C_{\bullet\bullet}) \to \mathrm{H}_*(\operatorname{Tot} C_{\bullet\bullet}))$$

Denote the quotient

$$F_p/F_{p-1} =: \operatorname{gr}_p(\mathrm{H}_{p+q}(\operatorname{Tot} C_{\bullet\bullet})).$$

All data defined above, that is $\{E^r_{pq}, d^r\}_{p,q,r}$ and a filtration $\{F_p\}_p$ define a **spectral sequence** of a bicomplex $C_{\bullet\bullet}$. We say that the spectral sequence abuts to $\mathrm{H}_*(\operatorname{Tot} C_{\bullet\bullet})$, which means that there is an isomorphism

$$E^{\infty}_{pq} \simeq \operatorname{gr}_p(\mathrm{H}_{p+q}(\operatorname{Tot} C_{\bullet\bullet}))$$

We write

$$E^2_{pq} = \mathrm{H}^h_p(\mathrm{H}^v_q(C_{\bullet\bullet})) \implies \mathrm{H}_{p+q}(\operatorname{Tot} C_{\bullet\bullet}).$$

which is to be read as: there is a spectral sequence starting at E^2_{pq} and converging to $\mathrm{H}_{p+q}(\operatorname{Tot} C_{\bullet\bullet})$

Example 1.27. The typical theorem using spectral sequences in algebraic topology looks as follows

Theorem 1.28. *Let $F \to E \to B$ be a fibration of connected spaces, with B simply connected. Then there is a spectral sequence*

$$E^2_{pq} = \mathrm{H}_p(B; \mathrm{H}_q(F)) \implies \mathrm{H}_{p+q}(E).$$

The implicit data in this theorem are $E^3_{pq}, E^4_{pq}, \ldots$, the filtration F_p on $\mathrm{H}_*(E)$. The sign " $\implies$ " means that there is an isomorphism

$$E^\infty_{pq} \simeq \mathrm{gr}_p(\mathrm{H}_{p+q}(E)).$$

In many cases we do not need to look at E^r_{pq} for $r \geq 3$ and at the filtration. That is why these data are often omitted in the theorems.

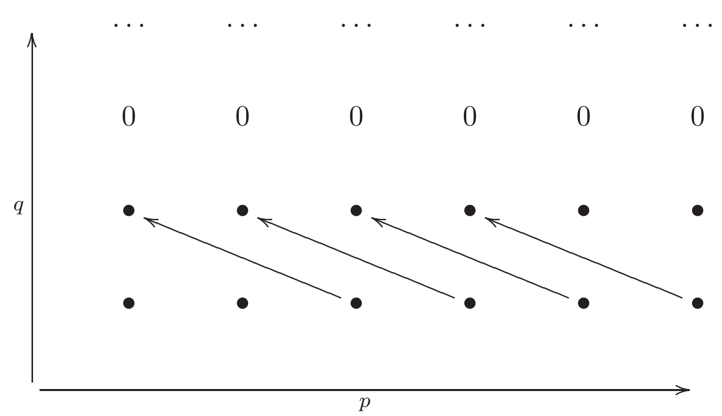
Example 1.29. Let X be an S^1-space, $\mathrm{E}\,S^1$ contractible space of paths on S^1. Consider the Borel space $\mathrm{E}\,S^1 \times_{S^1} X$ an S^1-fibration

$$S^1 \hookrightarrow \mathrm{E}\,S^1 \times_{S^1} X \to X.$$

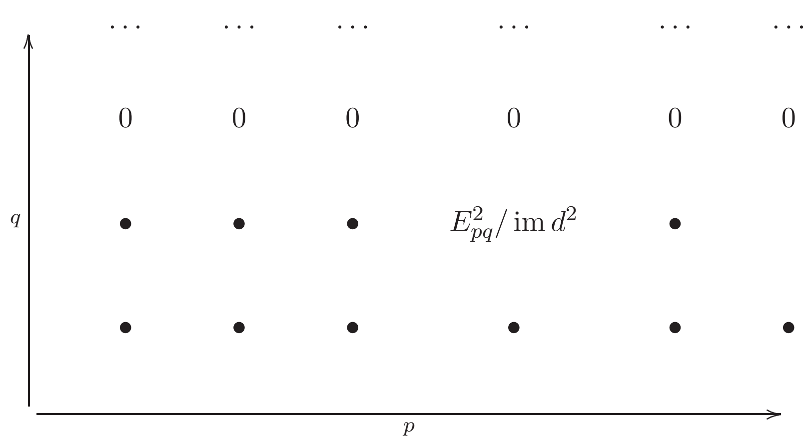
The homology of the fiber is

$$\begin{aligned} &\mathrm{H}_0(S^1) = \mathbb{Z}, \\ &\mathrm{H}_1(S^1) = \mathbb{Z}, \\ &\mathrm{H}_q(S^1) = 0, \quad q \geq 2. \end{aligned}$$

E^2 :



$E^3 = \ldots = E^\infty$:



For any S^1-fibration $S^1 \hookrightarrow E \xrightarrow{f} B$ of pointed spaces we obtain a Gysin sequence

$$\ldots \to \mathrm{H}_n(E) \xrightarrow{f_*} \mathrm{H}_n(B) \xrightarrow{d^2} \mathrm{H}_{n-2}(B) \to \mathrm{H}_{n-1}(E) \to \ldots.$$

Recall that for the bicomplex we took the vertical homology and then horizontal homology. We could have done it the other way. Any bicomplex gives a rise to two spectral sequences

$$E'^2_{pq} = \mathrm{H}^h_p(\mathrm{H}^v_q(C_{\bullet\bullet})) \implies \mathrm{H}_{p+q}(\mathrm{Tot}(C_{\bullet\bullet}))$$
$$E''^2_{pq} = \mathrm{H}^v_p(\mathrm{H}^h_q(C_{\bullet\bullet})) \implies \mathrm{H}_{p+q}(\mathrm{Tot}(C_{\bullet\bullet}))$$

But remark that the filtrations are different on $\mathrm{Tot}(C_{\bullet\bullet})$.

Chapter 2

Cyclic homology

2.1 The cyclic bicomplex

Let $C_\bullet$ be a cyclic module with

$$\begin{aligned} d_i \colon C_n &\to C_{n-1}, \\ t_n \colon C_n &\to C_n. \end{aligned}$$

Consider the following two-column bicomplex

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
C_2 & \xleftarrow{1-t} & C_2 \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle -b'} \\
C_1 & \xleftarrow{1-t} & C_1 \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle -b'} \\
C_0 & \xleftarrow{1-t} & C_0
\end{array}$$

One checks that it has anticommuting squares, so it is indeed a bicomplex. It can be extended to the right using the map $N := 1 + t + \dots t^n \colon C_n \to C_n$.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_2 & \xleftarrow{1-t} & C_2 & \xleftarrow{N} & C_2 & \xleftarrow{1-t} & C_2 & \xleftarrow{N} & \cdots \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle -b'} & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle -b'} & & \\
C_1 & \xleftarrow{1-t} & C_1 & \xleftarrow{N} & C_1 & \xleftarrow{1-t} & C_1 & \xleftarrow{N} & \cdots \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle -b'} & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle -b'} & & \\
C_0 & \xleftarrow{1-t} & C_0 & \xleftarrow{N} & C_0 & \xleftarrow{1-t} & C_0 & \xleftarrow{N} & \cdots
\end{array}$$

For example if $C_n = A \otimes A^{\otimes n}$ we have a cyclic bicomplex $C_{\bullet\bullet}(A)$ with t being the cyclic operator, and $N = 1 + t + \dots t^n$.

Definition 2.1. *The* ***cyclic homology*** *of a cyclic module* $C_\bullet$ *is defined as*

$$\mathrm{HC}_n(C_\bullet) := \mathrm{H}_n(\mathrm{Tot}(C_{\bullet\bullet})).$$

When $C_n = A \otimes A^{\otimes n}$ *then the cyclic homology of an algebra* A *is denoted by* $\mathrm{HC}_n(A)$.

Proposition 2.2. *The complex* $(C_\bullet, b')$ *is acyclic.*

Proof. Use extra degeneracy

$$(a_0, \ldots, a_n) \mapsto (1, a_0, \ldots, a_n)$$

to construct a homotopy of the identity and the zero map. □

Whenever we have a sequence of complexes

$$K'_\bullet \rightarrowtail K_\bullet \twoheadrightarrow K''_\bullet$$

and we know that $K'_\bullet$ is acyclic, then the complexes $K_\bullet$ and $K''_\bullet$ are quasi-isomorphic. This allows us to quotient out the acyclic subcomplexes of a given complex when computing homology. But $(C_\bullet, -b')$ is not a subcomplex. We will get rid of one column at a time using

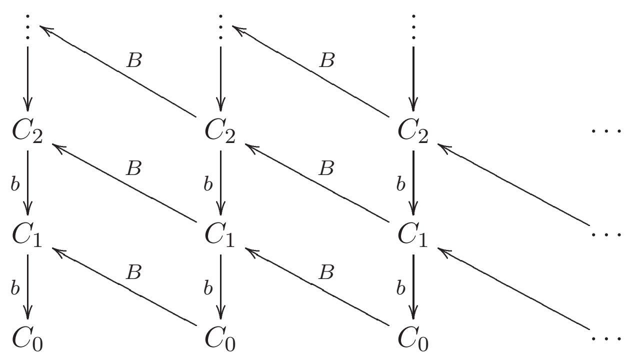
Lemma 2.3 (Killing contractible complexes)**.** *Suppose we have o complex*

$$\ldots \to A_n \oplus A'_n \xrightarrow{d=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}} A_{n-1} \oplus A'_{n-1} \to \ldots$$

and $(A'_\bullet, \delta)$ *has a homotopy* h *between* id *and 0. Then the following inclusion is a quasi-isomorphism*

$$(A_\bullet, \alpha - \beta h \gamma) \xrightarrow{(\mathrm{id}, -h\gamma)} (A_\bullet \oplus A'_\bullet, d).$$

The cokernel of $(\mathrm{id}, -h\gamma)$ is $(A'_\bullet, \delta)$. Applied infinitely many times to the cyclic bicomplex we end up with the total complex of the bicomplex $B_\bullet C_\bullet$



This is the normalized version of the bicomplex $C_{\bullet\bullet}$ used to define cyclic homology. Because of the quasi-isomorphism in the lemma (2.3) we have

$$\mathrm{H}_*(C_\bullet) = \mathrm{H}_*(\mathrm{Tot}(B_\bullet C_\bullet)).$$

We can rearrange the bicomplex $B_\bullet C_\bullet$ to obtain

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C_2 & \xleftarrow{B} & C_1 & \xleftarrow{B} & C_0 & & \\
{\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & & & \\
C_1 & \xleftarrow{B} & C_0 & & & & \\
{\scriptstyle b}\downarrow & & & & & & \\
C_0 & & & & & &
\end{array}$$

It is indeed a bicomplex, that is we have the identities

$$b^2 = 0, \quad B^2 = 0, \quad bB + Bb = 0.$$

The morphism B on the normalized complex $B_\bullet C_\bullet(A)$ is given explicitly by

$$B = (1-t)sN \colon A \otimes \overline{A}^{\otimes n} \to A \otimes \overline{A}^{\otimes (n+1)},$$

$$(a_0, \dots, a_n) \mapsto \sum_{i=0}^{n} (-1)^{in} (1, a_i, \dots, a_n, a_0, \dots, a_{n-1}).$$

In the non-normalized complex there are more terms, but they are trivial in the normalized complex.

Theorem 2.4. *For a cyclic module $C_\bullet$ there exits a* ***periodicity exact sequence***

$$\dots \to \mathrm{H}_n(C_\bullet) \xrightarrow{I} \mathrm{HC}_n(C_\bullet) \xrightarrow{S} \mathrm{HC}_{n-2}(C_\bullet) \xrightarrow{B} \mathrm{H}_{n-1}(C_\bullet) \to \dots, \tag{2.1}$$

where the map I is induced by the inclusion of the simplicial complex for $C_\bullet$ into bicomplex $C_{\bullet\bullet}$.

If $C_n = A^{\otimes n}$ the sequence takes the form

$$\dots \to \mathrm{HH}_n(A) \xrightarrow{I} \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \xrightarrow{B} \mathrm{HH}_{n-1}(A) \to \dots. \tag{2.2}$$

Proof. It follows from the bicomplex $(B_\bullet C_\bullet, b, B)$ and the sequence of complexes

$$C_\bullet \rightarrowtail \mathrm{Tot}(B_\bullet C_\bullet) \twoheadrightarrow \mathrm{Tot}\, B_\bullet C_\bullet[-2].$$

Prove that the boundary map is given by B. Find an explicit formula for S. □

2.2 Characteristic 0 case

Recall the computation of the homology of the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Let M be a module over $\mathbb{Z}/n\mathbb{Z}$, that is a module over the group ring $k[\mathbb{Z}/n\mathbb{Z}]$ for some ring k. to compute $\mathrm{H}_i(\mathbb{Z}/n\mathbb{Z}; M)$ one uses the complex

$$M \xleftarrow{1-t} M \xleftarrow{N} M \xleftarrow{1-t} M \xleftarrow{N} \dots$$

When the ring k is a field of characteristic 0, there is a homotopy from id to 0,

$$M \xrightarrow{h} M \xrightarrow{h'} M \xrightarrow{h} M \xrightarrow{h'} \dots,$$

$$
\begin{aligned}
h &:= -\frac{1}{n}\sum_{i=1}^{n-1} i t^i, \\
h' &:= \frac{1}{n}\mathrm{id}, \\
& h(1-t) + Nh' = t^n = \mathrm{id}.
\end{aligned}
$$

It proves that

$$
\begin{aligned}
\mathrm{H}_0(\mathbb{Z}/n\mathbb{Z}; M) &= M/1-t, \\
\mathrm{H}_n(\mathbb{Z}/n\mathbb{Z}; M) &= 0, \quad n \geq 1.
\end{aligned}
$$

Now instead of considering the bicomplex $C_{\bullet\bullet}$ we can take the reduced complex $C^\lambda_\bullet$ which is defined as a cokernel of the map $(1-t)$ between first and zeroth column of $C_{\bullet\bullet}$

$$
\begin{array}{cc}
\vdots & \vdots \\
\downarrow & \\
C_3/(1-t) & 0 \\
\downarrow b & \\
C_2/(1-t) & 0 \\
\downarrow b & \\
C_1/(1-t) & 0 \\
\downarrow b & \\
C_0/(1-t) & 0
\end{array}
$$

If $C_n = A^{\otimes(n+1)}$, then $C^\lambda_n(A) = A^{\otimes(n+1)}/(1-t)$ and we denote

$$\mathrm{H}^\lambda_n(A) := \mathrm{H}_n(C^\lambda_\bullet)$$

As a corollary we have that if $k \supset \mathbb{Q}$, then $\mathrm{H}^\lambda(A) \simeq \mathrm{HC}_n(A)$ and there exists an exact sequence

$$\ldots \to \mathrm{HH}_n(A) \xrightarrow{I} \mathrm{H}^\lambda_n(A) \xrightarrow{S} \mathrm{H}^\lambda_{n-2}(A) \xrightarrow{B} \mathrm{HH}_{n-1}(A) \to \ldots.$$

In the case of characteristic not equal 0 the maps are still defined, but the sequence is not exact.

2.3 Computations

Let $A = k$, the ground ring. Then

$$
\begin{aligned}
\mathrm{HH}_0(k) &= k, \\
\mathrm{HH}_n(k) &= 0, \quad n \geq 1.
\end{aligned}
$$

The periodicity exact sequence (2.2) implies that

$$
\begin{aligned}
\mathrm{HC}_{2n}(k) &= k, \\
\mathrm{HC}_{2n+1}(k) &= 0,
\end{aligned}
$$

so also

$$\begin{aligned} \mathrm{H}^\lambda_{2n}(k) &= k, \\ \mathrm{H}^\lambda_{2n+1}(k) &= 0. \end{aligned}$$

Let $A = T(V)$ be the tensor algebra over V, that is

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}, \quad (v_1, \dots, v_n)(v_{n+1}, \dots, v_{n+m}) = (v_1, \dots, v_{n+m}) \in V^{\otimes(n+m)}$$

Then

$$\begin{aligned} \mathrm{HH}_0(T(V)) &= \bigoplus_{m\geq 0} V^{\otimes m}/(1-\tau) = \bigoplus_{m\geq 0} (V^{\otimes m})_{\mathbb{Z}/m\mathbb{Z}}, \\ \mathrm{HH}_1(T(V)) &= \bigoplus_{m\geq 0} (V^{\otimes m})^{\mathbb{Z}/m\mathbb{Z}}, \\ \mathrm{HH}_1(T(V)) &= 0, \end{aligned}$$

where τ is the cyclic operator without sign.

$$\mathrm{HC}_n(T(V)) = \mathrm{HC}_n(k) \oplus \underbrace{\bigoplus_{m>0} \mathrm{H}_n(\mathbb{Z}/m\mathbb{Z}; V^{\otimes m})}_{\text{This is zero in the characteristic 0 case.}}$$

Consider now the matrix algebra $M_n(A)$ for a unital associative algebra A over a field k. There are isomorphisms

$$\begin{aligned} \mathrm{HH}_*(M_r(A)) &\simeq \mathrm{HH}_*(A), \\ \mathrm{HC}_*(M_r(A)) &\simeq \mathrm{HC}_*(A). \end{aligned}$$

The map $A \to M_r(A)$ is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

In the opposite way $\mathrm{Tr}\colon M_r(A) \to A$ we have the trace map

$$\alpha = [\alpha_{ij}] \mapsto \sum_i \alpha_{ii}.$$

There is also a trace map $\mathrm{Tr}\colon M_r(A)^{\otimes(n+1)} \to A^{\otimes(n+1)}$

$$\mathrm{Tr}(\alpha^0, \dots, \alpha^n) := \sum_{(i_0,\dots,i_n)} \alpha^0_{i_0 i_1} \otimes \alpha^1_{i_1 i_2} \otimes \dots \otimes \alpha^n_{i_n i_0}$$

called the **Dennis trace map**. We claim that this map commutes with the faces and with the cyclic operator.

Let k be a field and A a commutative k-algebra. Define the space of 1-forms on A, denoted by $\Omega^1_{A/k} = \Omega^1_A$, as an A-module generated by elements da for every $a \in A$ satisfying following relations

$$\begin{aligned} d(\lambda a + \mu b) &= \lambda da + \mu db \text{ (linearity)}, \\ d(ab) &= adb + bda \text{ (Leibniz rule)}. \end{aligned}$$

Define the space of n-forms as an n-th exterior power of Ω^1_A

$$\Omega^n_A := \Lambda^n_A \Omega^1_A.$$

Elements of Ω^n_A can be written as $a_0 da_1 \dots da_n$, $a_i \in A$, $i = 0, \dots, n$, with the relation

$$da da' = -da' da.$$

Define a differential of an n-form as

$$d(a_0 da_1 \dots da_n) := 1 da_0 da_1 \dots da_n.$$

$$d \colon \Omega^n_A \to \Omega^{n+1}_A, \quad d \circ d = 0.$$

Now $\Omega^\bullet_A$ is a cochain complex and its homology is called deRham cohomology of the algebra A

$$\mathrm{H}_{\mathrm{dR}}(A) := \mathrm{H}_n(\Omega^\bullet_A, d).$$

If A is commutative, M an A-module, then

$$\mathrm{H}_1(A; M) = M \otimes_A \Omega^1_A.$$

There is a map

$$\begin{aligned} \pi \colon C_n(A) = A^{\otimes(n+1)} &\to \Omega^n_A \\ (a_0, \dots, a_n) &\mapsto a_0 da_1 \dots da_n \end{aligned} \tag{2.3}$$

There is a map also in the opposite way

$$\Omega^n_A \xrightarrow{\varepsilon_n} \mathrm{HH}_n(A)$$

$$\varepsilon_n(a_0 da_1 \dots da_n) := \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)(a_0, a_{\sigma(1)}, \dots, a_{\sigma(n)}). \tag{2.4}$$

Passing to Hochschild homology it gives a well defined map $\Omega^n_A \to \mathrm{HH}_n(A)$. In charecteristic 0 case the composition of the maps in (2.4) and (2.3) gives an isomorphism

$$\Omega^n_A \to \mathrm{HH}_n(A) \to \Omega^n_A.$$

Proposition 2.5. *The following diagram is commutative*

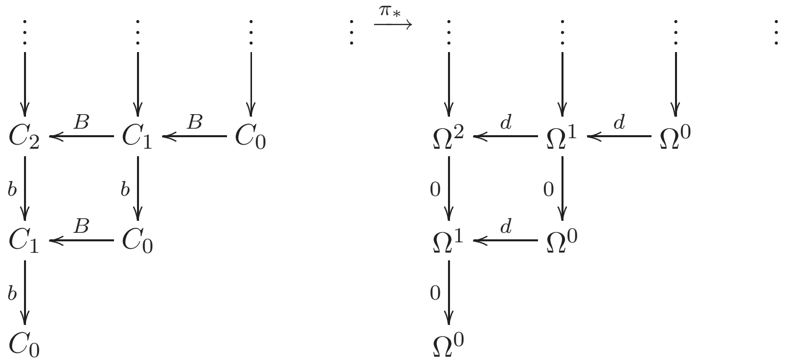
$$\begin{array}{ccc} \Omega^n_A & \xrightarrow{\varepsilon_n} & \mathrm{HH}_n(A) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle B} \\ \Omega^{n+1}_A & \xrightarrow{\varepsilon_{n+1}} & \mathrm{HH}_{n+1}(A) \end{array}$$

Proof. There is a bijection of sets $S_{n+1} \simeq S_n \times \mathbb{Z}/(n+1)\mathbb{Z}$. First one proves the commutativity of the following diagram

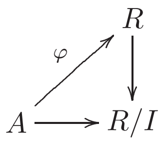
$$\begin{array}{ccc} A \otimes \Lambda^n A & \xrightarrow{\varepsilon_n} & C_n(A) \\ \downarrow & & \downarrow \\ A \otimes \Lambda^{n+1} A & \xrightarrow{\varepsilon_{n+1}} & C_{n+1}(A) \end{array}$$

and then passes to the quotient. □

Now we can form a map of bicomplexes



Definition 2.6. *A commutative algebra A is **formally smooth** if for any commutative algebra R and two sided ideal $R \supset I$ such that $I^2 = 0$ and a map $A \to R/I$, there is a lifting $\varphi\colon A \to R$.*



Theorem 2.7 (Hochschild-Kostant-Rosenberg)**.** *If A is formally smooth, then*

$$\varepsilon_*\colon M \otimes_A \Omega^n_A \to \mathrm{H}_*(A; M)$$

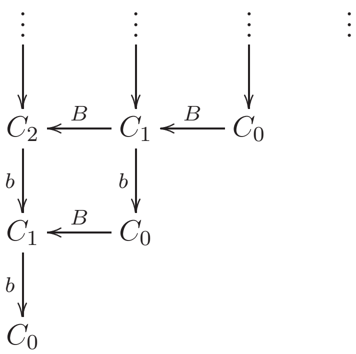
is an isomorphism.

As a corollary we have that for a formally smooth algebra A over characteristic 0 field k

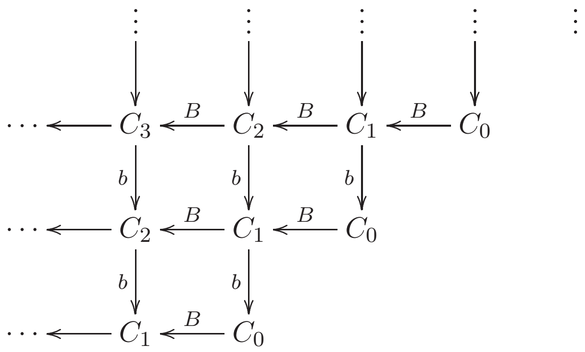
$$\mathrm{HC}_n(A) \simeq \Omega^n_A / d\Omega^{n-1}_A \oplus {\mathrm{H}_{\mathrm{dR}}}^{n-2}(A) \oplus {\mathrm{H}_{\mathrm{dR}}}^{n-4}(A) \oplus \ldots \oplus {\mathrm{H}_{\mathrm{dR}}}^{0}(A) \text{ or } {\mathrm{H}_{\mathrm{dR}}}^{1}(A).$$

2.4 Periodic and negative cyclic homology

Recall the cyclic bicomplex

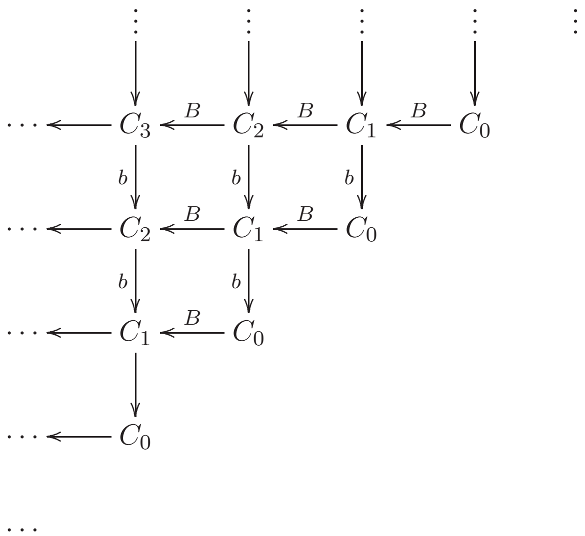


which after passing to total complex gives a complex computing cyclic homology of an algebra. There is an obvious way to extend this bicomplex to the left using the same differentials



Furthermore we can repeat each row going down continuing the same pattern.

(2.5)



This is called the **periodic bicomplex**. If the columns of the cyclic bicomplex we started with were indexed by natural numbers starting from 0, then in the periodic bicomplex (2.5) we have columns indexed by integers.

To work with the total complex of the periodic bicomplex one should use the product instead of the sum. Otherwise one would get zero in the homology.

Definition 2.8. *The cohomology of the total complex of bicomplex (2.5) is called* ***periodic cyclic homology****. If* $C_n = A^{\otimes(n+1)}$*, then we denote this homology by* $\mathrm{HP}_*(A)$ *or* $\mathrm{HC}_*^{per}(A)$*.*

The homology of the total complex consisting of columns with nonpositive indices is called ***negative cyclic homology****. If* $C_n = A^{\otimes(n+1)}$*, then we denote this homology by* $\mathrm{HN}_*(A)$ *or* $\mathrm{HC}_*^-(A)$*.*

2.5 Harrison homology

Recall that when A is an algebra over characteristic 0 field k, then

$$\mathrm{HH}_*(A) \xrightarrow{\simeq} \Omega_A^*.$$

In general there is a decomposition into direct sum

$$\mathrm{HH}_n(A) = \underbrace{\square \oplus \ldots \oplus \square}_{n \text{ terms}} \oplus \Omega^n_A$$
$$\ldots$$
$$\mathrm{HH}_2(A) = \square \oplus \Omega^2_A$$
$$\mathrm{HH}_1(A) = \square$$

When one considers the first summands in each gradation then what one obtains is called **Harrison homology** of the algebra A. When M is an A-bimodule, then $C_n(A, M) = M \otimes_A A^{\otimes n}$ gives a complex computing Hochschild homology of the algebra A with coefficients in M. The complex for Harrison homology we obtain by taking a quotient by the shuffles in $C_n(A, M)$.

2.6 Derived functors

The Hochschild homology of an algebra A over a field k with coefficients in an A-bimodule M can be interpreted as a derived functor

Proposition 2.9. *There is an isomorphism*

$$\mathrm{H}_n(A; M) \simeq \mathrm{Tor}^{A^e}_n(M, A),$$

where $A^e = A \otimes A^{op}$ (so M is a right A^e-module).

The definition of the derived functor $\mathrm{Tor}^{A^e}_n$ goes as follows. Having an exact sequence of right A^e-modules

$$0 \to M' \to M \to M'' \to 0$$

we tensor it with A over A^e to get a sequence which is exact on the right

$$M' \otimes_{A^e} A \to M \otimes_{A^e} A \to M'' \otimes_{A^e} A \to 0,$$

but the map $M' \otimes_{A^e} A \to M \otimes_{A^e}$ can have a nontrivial kernel, which we define as $\mathrm{Tor}^{A^e}_1(M'', A)$. Next we can define in an analogous way $\mathrm{Tor}^{A^e}_1(M, A)$ and $\mathrm{Tor}^{A^e}_1(M', A)$ which fit into an exact sequence

$$\mathrm{Tor}^{A^e}_1(M', A) \to \mathrm{Tor}^{A^e}_1(M, A) \to \mathrm{Tor}^{A^e}_1(M', A) \to M' \otimes_{A^e} A \to M \otimes_{A^e} A \to M'' \otimes_{A^e} A \to 0.$$

General construction uses a resolution of A by free left A^e-modules, $C_\bullet \twoheadrightarrow A \to 0$,

$$\begin{array}{ccccccc} \cdots \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\ &&&&& \downarrow \\ &&&&& A \\ &&&&& \downarrow \\ &&&&& 0 \end{array}$$

Then we define

$$\mathrm{Tor}^{A^e}_n(M, A) := \mathrm{H}_n(M \otimes_{A^e} C_\bullet).$$

As a resolution we can take $C_n := A^e \otimes A^{\otimes n}$ and obtain the isomorphism $\mathrm{H}_n(A, M) \simeq \mathrm{Tor}_n^{A^e}(M, A)$.

Recall that the simplicial module $C_\bullet$ is a functor $\Delta^{op} \to \mathbf{Mod}$, for example $[n] \mapsto M \otimes_{A^e} A^n$. The homology of $C_\bullet$ with respect to $b = \sum_i (-1)^i d_i$ can be written as a derived functor

$$\mathrm{H}_n(C_\bullet) \simeq \mathrm{Tor}_n^{\Delta^{op}}(k, C_\bullet),$$

where $C_\bullet$ is a left module over Δ^{op}, and k is a right module over Δ^{op}, that is a functor $\Delta \to \mathbf{Mod}$, $[n] \mapsto k$. The resolution for k can be given by

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & k[\mathrm{Hom}_\Delta([n], -)] & \longrightarrow \cdots \longrightarrow & k[\mathrm{Hom}_\Delta([1], -)] & \longrightarrow & k[\mathrm{Hom}_\Delta([0], -)] \\ & & & & & & \downarrow \\ & & & & & & k \end{array}$$

In general for a category $\mathcal{C}$ we have the following correspondence

Category $\mathcal{C}$	Algebra A
Functor $F \colon \mathcal{C} \to \mathbf{Mod}$	Left A-module M
Functor $G \colon \mathcal{C}^{op} \to \mathbf{Mod}$	Right A-module N
Tensor product over a category $G \otimes_{\mathcal{C}} F$	Tensor product over algebra $N \otimes_A M$

The tensor product over a category is defined as

$$G \otimes_{\mathcal{C}} F := \bigoplus_{C \in \mathrm{Ob}(\mathcal{C})} G(C) \otimes F(C) / \sim,$$

where the equivalence relation $\sim$ is given by

$$y \otimes f_*(x) \sim f^*(y) \otimes x, \quad C \xrightarrow{f} D, \quad x \in F(C),\ y \in G(D),$$
$$F(C) \xrightarrow{f_*} F(D), \quad G(C) \xleftarrow{f^*} G(D).$$

Using cyclic category ΔC we can present cyclic homology of a cyclic module $C_\bullet$ as a derived functor.

Proposition 2.10. *There is an isomorphism*

$$\mathrm{HC}_n(C_\bullet) \simeq \mathrm{Tor}_n^{\Delta C^{op}}(k, C_\bullet).$$

We can write $\mathrm{Tor}_0^{\mathcal{C}}(G, F)$ simply as the tensor product $G \otimes_{\mathcal{C}} F$. To define higher derived functors $\mathrm{Tor}_n^{\mathcal{C}}(G, F)$ we need a notion of a free module over a category. Let $\mathcal{C}^{triv}$ be the category with the same objects as $\mathcal{C}$, but with only the identity morphisms. For a functor $F \colon \mathcal{C} \to \mathbf{Mod}$ there is a corresponding forgetful functor $\mathrm{forget(F)} \colon \mathcal{C}^{triv} \to \mathbf{Mod}$. Suppose we have an adjoint pair

$$\mathrm{Funct}(\mathcal{C}, \mathbf{Mod}) \underset{\text{left adjoint}}{\overset{\text{forgetful}}{\rightleftarrows}} \mathrm{Funct}(\mathcal{C}^{triv}, \mathbf{Mod})$$

Then we say that a functor $F \colon \mathcal{C} \to \mathbf{Mod}$ is free if it is an image of this left adjoint functor to a forgetful functor. For example

$$A - \mathbf{Mod} \to k - \mathbf{Mod}$$

has a left adjoint

$$k^n \mapsto A^n.$$

Chapter 3

Relation with K-theory

We will define invariant of rings, called algebraic K-theory and denoted by $\mathrm{K}_*(A)$ for a ring A. Next we will describe its relation with cyclic homology by defining a map

$$\mathrm{K}_*(A) \to \mathrm{HC}_*(A).$$

3.1 K-theory

First we will define **K-theory** of a ring A in gradation 0, that is $\mathrm{K}_0(A)$. We say that a finitely generated module over A is **free** if it is isomorphic to the product A^n for some n. A finitely generated A-module P is **projective** if it is a direct summand in a free A-module, that is there exists an A-module Q such that $P \oplus Q \simeq A^n$ for some n. Such projective module P corresponds to idempotent in the matrix algebra $M_n(A)$. The set of isomorphism classes of finitely generated projective modules over A is a monoid with respect to direct sum of classes defined by

$$[P] + [Q] =: [P \oplus Q].$$

There is an universal abelian group for this monoid (called the Grothendieck group), and we take it as the definition of the K-theory of A, denoted by $\mathrm{K}_0(A)$.

Let A be a commutative algebra over k. There exists a map

$$\mathrm{ch}\colon\ \mathrm{K}_0(A) \to {\mathrm{H}_{\mathrm{dR}}}^2(A)$$

that we will construct later.

First consider an example of a map from a tori $S^1 \times S^1$ to a sphere S^2 given by contracting the boundary of a square with opposite edges identified. This map has degree 1 and induces

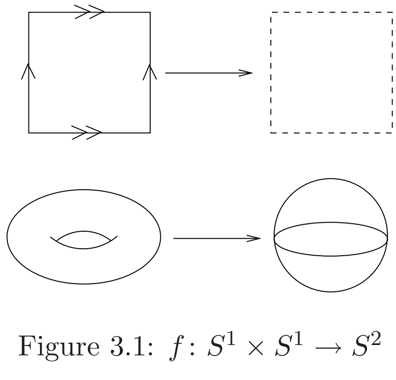


Figure 3.1: $f\colon S^1 \times S^1 \to S^2$

an isomorphism

$$\mathrm{H_{dR}}^2(S^2) \xrightarrow{\deg(f)} \mathrm{H_{dR}}^2(S^1 \times S^1).$$

If we want to find an algebraic map of corresponding coordinate rings

$$S^2_a := \mathbb{C}[X,Y,Z]/(X^2+Y^2+Z^2-1) \to \mathbb{C}[U,U^{-1},V,V^{-1}] =: (S^1 \times S^1)_a$$

then we will not succeed, because any algebraic map $S^1 \times S^1 \to S^2$ is homotopic to the constant map. The situation is very different now than it was in case of maps $S^3 \to S^2$. Indeed, assume we have the map

$$f^*\colon S^2_a \to S^1 \times S^1_a.$$

Then it induces a map on K-theory

$$\mathrm{K}_0(S^2_a) \to \mathrm{K}_0((S^1 \times S^1)_a),$$

and we would have a commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z} & = & \widetilde{\mathrm{K}}_0(S^2_a) & \longrightarrow & \widetilde{\mathrm{K}_0}((S^1 \times S^1)_a) & = & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{C} & = & \mathrm{H_{dR}}^2(S^2_a) & \xrightarrow{\deg(f)} & \mathrm{H_{dR}}^2((S^1 \times S^1)_a) & = & \mathbb{C}
\end{array}$$

which gives a contradiction, because a generator of $\mathbb{Z} = \widetilde{\mathrm{K}}_0(S^2_a)$ goes to generator of $\mathbb{C} = \mathrm{H_{dR}}^2(S^2_a)$.

Define a projector p and idempotent e in $M_2(S^2_a)$ by the formulas

$$p := \begin{pmatrix} X & Y+iZ \\ Y-iZ & -X \end{pmatrix}, \quad p^2 = 1, \quad e := \frac{p+1}{2}, \quad e^2 = e.$$

Fact 3.1. *The class of an image of e, $[\operatorname{im} e]$, generates $\mathrm{K}_0(S^2_a)$.*

Fact 3.2. *For any ring A there is an isomorphism*

$$\widetilde{\mathrm{K}}_0(A[X,X^{-1}]) \simeq \mathrm{K}_0(A).$$

3.2 Trace map

There is a trace map defined as

$$\mathrm{Tr}\colon M_r(A) \to A, \quad [a_{ij}]_{i,j=1}^r \mapsto \sum_{i=1}^r a_{ii}.$$

We can extend it to a map

$$\mathrm{Tr}\colon M_r(A)^{\otimes(n+1)} \to A^{\otimes(n+1)},$$

$$[a_{i_0 j_0}] \otimes \ldots \otimes [a_{i_n j_n}] \mapsto \sum_{k_0,k_1,\ldots,k_n} a_{k_0 k_1} \otimes a_{k_1 k_2} \otimes \ldots \otimes a_{k_n k_0}$$

for any $r \geq 1$, $n \geq 0$. It induces a maps on Hochschild, cyclic, periodic cyclic and negative cyclic homology.

$$\mathrm{HH}_n(M_r(A)) \to \mathrm{HH}_n(A), \quad \mathrm{HC}_n(M_r(A)) \to \mathrm{HC}_n(A), \text{ etc.}$$

Let us take an idempotent $e^2 = e$ in $M_r(A)$. Under the map b in Hochschild complex for $M_r(A)$ we have

$$e^{\otimes(n+1)} \mapsto \begin{cases} 0 & n \text{ even} \\ e^{\otimes n} & n \text{ odd} \end{cases}$$

In $C_n^\lambda(M_r(A))$ we have $e^{\otimes(n+1)} = (-1)^n e^{\otimes(n+1)}$. If n is odd, then $[e^{\otimes(n+1)}] = 0$. If $n = 2m$ is even, then $b[e^{\otimes(n+1)}] = 0$, so $[e^{\otimes(n+1)}]$ is a cycle, and we can define a map $[e] \mapsto [\mathrm{Tr}(e^{\otimes(n+1)})]$,

$$\mathrm{K}_0(A) \to \mathrm{H}_{2m}^\lambda(M(A)) \xrightarrow{\mathrm{Tr}} \mathrm{H}_{2m}^\lambda(A),$$

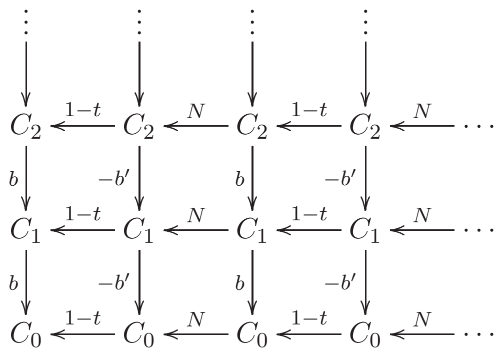
$$M(A) = \bigcup_r M_r(A), \quad M_r(A) \hookrightarrow M_{r+1}(A), \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}.$$

We have to show that the element $[\mathrm{Tr}(e^{\otimes(n+1)})] \in \mathrm{H}_{2m}^\lambda(A)$ depends only on the isomorphism class.

Lemma 3.3. *An interior automorphism (conjugation) induces an identity for Hochschild, cyclic, periodic cyclic, negative cyclic homology.*

We have constructed a functorial map $\mathrm{K}_0(A) \to \mathrm{H}_{2m}^\lambda(A)$. Now we ask if we can construct a map $\mathrm{K}_0(A) \to \mathrm{HC}_{2m}(A)$?

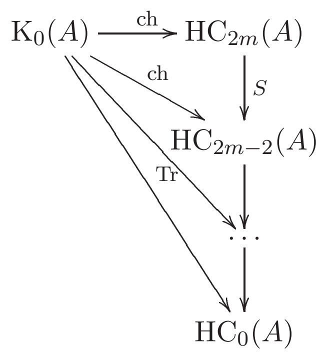
Recall the cyclic bicomplex $C_{\bullet\bullet}(A)$



Define

$$y_i := (-1)^i \frac{(2i)!}{i!} \mathrm{Tr}(e^{\otimes(2i+1)}),$$
$$z_i := (-1)^{i-1} \frac{(2i)!}{2(i!)} \mathrm{Tr}(e^{\otimes(2i)}).$$

Proposition 3.4. *The element* $\mathrm{ch}([e]) := (y_m, z_m, y_{m-1}, z_{m-1}, \ldots, y_0, z_0) \in (\mathrm{Tot}(C_{\bullet\bullet}(A)))_n$, $n = 2m+1$ *is a cycle. Furthermore the following diagram is commutative*



For the bicomplex $B_\bullet C_\bullet$ we have to use $\mathrm{ch}([e]) := (y_n, y_{n-1}, \dots, y_0) \in (\mathrm{Tot}(B_\bullet C_\bullet(A)))_n$. We can define a map

$$\mathrm{ch}\colon \mathrm{K}_0(A) \to \mathrm{H_{dR}}^{ev}(A), \quad \mathrm{ch}([e]) := \mathrm{Tr}(edede \dots de).$$

3.3 Algebraic K-theory

Let A be a ring with unit. Define a discrete group $\mathrm{GL}(A)$ as a direct limit of the groups $\mathrm{GL}_r(A)$ with respect to the maps

$$\mathrm{GL}_r(A) \hookrightarrow \mathrm{GL}_{r+1}(A), \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

There is a classifying space $\mathrm{B\,GL}(A)$ with

$$\pi_1(\mathrm{B\,GL}(A)) = \mathrm{GL}(A),$$
$$\pi_n(\mathrm{B\,GL}(A)) = 0, \quad n \neq 1.$$

We can apply the Quillen's plus construction to obtain a space $\mathrm{B\,GL}(A)^+$ with the following three properties

1. the fundamental group is an abelianization of $\mathrm{GL}(A)$,

$$\pi_1(\mathrm{B\,GL}(A)^+) = \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)],$$

2. there is an isomorphism on homology $\mathrm{H}_i(\mathrm{B\,GL}(A)) \simeq \mathrm{H}_i(\mathrm{B\,GL}(A)^+)$,

3. there is an H-space structure on $\mathrm{B\,GL}(A)^+$.

Thus $\mathrm{H}_*(\mathrm{B\,GL}(A)^+)$ is a commutative, cocommutative (and connected) Hopf algebra.

Definition 3.5. *Higher K-theory groups of A are defined as*

$$\mathrm{K}_n(A) := \pi_n(\mathrm{B\,GL}(A)^+), \quad n \geq 1.$$

Prior to this definition there were defined K_1, K_2, K_3. We will describe these earlier definitions.

The K_1 group of a ring A was defined as an abelianization of $\mathrm{GL}(A)$,

$$\mathrm{K}_1(A) = \mathrm{GL}(A)/[\mathrm{GL}(A), \mathrm{GL}(A)].$$

For example if $A = F$ is a field, then $\mathrm{K}_1(F) = F^\times$, the group of invertible elements in F. The determinant map $\det\colon \mathrm{GL}(F) \to F^\times$ can be generalized to noncommutative rings by the map $\mathrm{GL}(A) \to \mathrm{K}_1(A)$.

Denote by $\mathrm{E}(A)$ the group generated by elementary matrices e_{ij}^a, where each e_{ij}^a is an identity matrix plus the matrix with only one nonzero entry equal a in i-th row and j-th column. Then

$$[\mathrm{GL}(A), \mathrm{GL}(A)] = \mathrm{E}(A).$$

The elementary matrices e_{ij}^a satisfy the following relations

$$\begin{cases} e_{ij}^a e_{ij}^b &= e_{ij}^{a+b}, \\ e_{ij}^a e_{kl}^b &= e_{kl}^b e_{ij}^a, \text{ for } j \neq k,\ i \neq l, \\ e_{ij}^a e_{jk}^b &= e_{jk}^b e_{ik}^{ab} e_{ij}^a. \end{cases} \tag{3.1}$$

The group $\mathrm{E}(A)$ can be presented using generators e^a_{ij} which satisfy the relations (3.1) above plus some relations which depend on A. Define the Steinberg group $\mathrm{St}(A)$ of A as the group with the set of generators $\{x^a_{ij}\}$ with the relations (3.1). There is an epimorphism $\mathrm{St}(A) \twoheadrightarrow \mathrm{E}(A)$ and we define $\mathrm{K}_2(A)$ as the kernel of this map. Then $\mathrm{K}_2(A)$ is abelian, and the sequence

$$\mathrm{K}_2(A) \rightarrowtail \mathrm{St}(A) \twoheadrightarrow \mathrm{E}(A)$$

can be shown to be a central extension.

Theorem 3.6 (Whitehead-Kervaire)**.** *The group* $\mathrm{E}(A)$ *is perfect, that is*

$$\mathrm{H}_1(E(A)) = 0,$$

and

$$\mathrm{H}_2(E(A)) \simeq \mathrm{K}_2(A).$$

Proof. The proof relies on the spectral sequence of the fibration

$$\mathrm{B}\,\mathrm{K}_2(A) \to \mathrm{B}\,\mathrm{St}(A) \to \mathrm{B}\,\mathrm{E}(A)$$

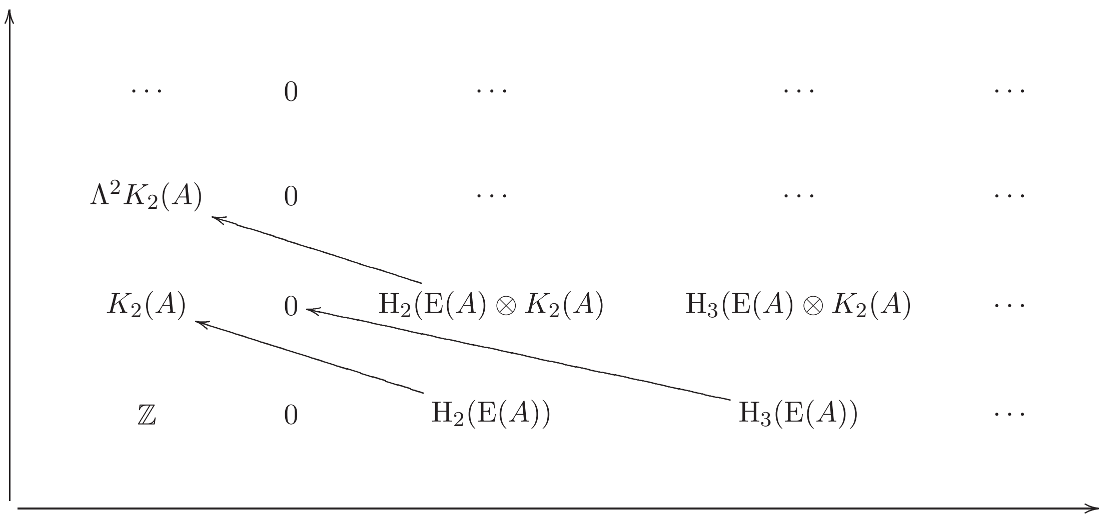
On the second table we have

$$E^2_{pq} = \mathrm{H}_p(\mathrm{B}\,\mathrm{E}(A); \mathrm{H}_q(\mathrm{B}\,\mathrm{K}_2(A)))$$

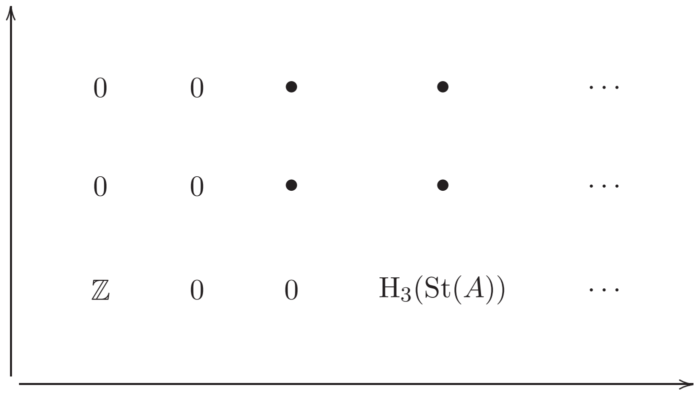
and the sequence converges to $\mathrm{H}_{p+q}(\mathrm{St}(A))$. We have

$$\mathrm{H}_p(\mathrm{B}\,\mathrm{E}(A); \mathrm{H}_q(\mathrm{B}\,\mathrm{K}_2(A))) \simeq \mathrm{H}_p(\mathrm{E}(A); \mathrm{H}_q(\mathrm{K}_2(A))) \simeq \mathrm{H}_p(\mathrm{E}(A)) \otimes \mathrm{H}_q(\mathrm{K}_2(A))$$

The second table looks like follows.



One needs to prove that $\mathrm{H}_2(\mathrm{St}(A)) = 0$, and that E^∞_{pq} looks like



□

Theorem 3.7 (Gersten)**.** *There is an isomorphism*

$$\mathrm{H}_3(\mathrm{St}(A)) \simeq \mathrm{K}_3(A).$$

Proof. One has to prove that there is a fibration

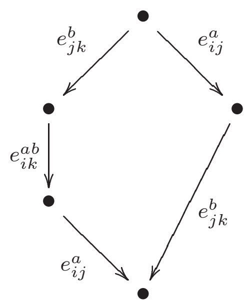
$$\begin{array}{ccccc} \mathrm{B}\,K_2(A) & \longrightarrow & \mathrm{B}\,\mathrm{St}(A)^+ & \longrightarrow & \mathrm{B}\,\mathrm{E}(A)^+ \\ \| & & & & \\ \mathrm{B}\,K_2(A)^+ & & & & \end{array}$$

and then use the above spectral sequence. □

Summarizing earlier results we have

$$\begin{aligned} \mathrm{H}_1(\mathrm{GL}(A)) &= \mathrm{K}_1(A), \\ \mathrm{H}_2(\mathrm{E}(A)) &= \mathrm{K}_2(A), \\ \mathrm{H}_3(\mathrm{St}(A)) &= \mathrm{K}_3(A). \end{aligned}$$

Let us look once more at the relations for the Steinberg group (3.1). We can label the edges of the Stasheff polytope of dimension 2 as follows



to encode the relation $e^a_{ij}e^b_{jk} = e^b_{jk}e^{ab}_{ik}e^a_{ij}$. There is a way to put labels on the Stasheff polytope of dimension 3 in the coherent way. It can be generalized to higher dimension.

Proposition 3.8 (Cartan)**.** *Let X be an H-space. Then*

$$\operatorname{Prim} \mathrm{H}_*(X;\mathbb{Q}) \simeq \pi_*(X) \otimes \mathbb{Q}$$

where the primitive elements of the Hopf algebra $\mathcal{H}$ are

$$\operatorname{Prim}(\mathcal{H}) := \{x \in \mathcal{H} \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

Corollary 3.9.

$$\operatorname{Prim} \mathrm{H}_*(\mathrm{GL}(A);\mathbb{Q}) \simeq \mathrm{K}_*(A) \otimes \mathbb{Q}.$$

Let $G = \mathrm{GL}_r(A)$. The map $f\colon k[\mathrm{GL}_r(A)] \to M_r(A)$ is the unique k-algebra map which extends the inclusion of invertible matrices to matrices.

For any n there is defined a map of cyclic modules

$$k[G^n] \xrightarrow{\iota} k[G^{n+1}] \simeq k[G]^{\otimes(n+1)} \xrightarrow{f^{\otimes(n+1)}} M_r(A)^{\otimes(n+1)} \xrightarrow{\mathrm{Tr}} A^{\otimes(n+2)},$$

where $\iota(g_1, \ldots, g_n) := ((g_1 \ldots g_n)^{-1}, g_1, \ldots, g_n)$.

We can apply any of the cyclic theories HH, HC, HC^-, HC^{per} to this sequence to get, for instance

$$\mathrm{H}_*(\mathrm{GL}(A)) \to \mathrm{HC}^-_*(A).$$

Working over $\mathbb{Q}$ and using collorary (3.9) we get the **Chern character**

$$\mathrm{ch}\colon \mathrm{K}_*(A) \to \mathrm{HC}^-_*(A).$$