Between the classical theorem of Young and Convergence Theorem in Set–Valued Analysis

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We obtain a version of Young's Theorem, where Young–like measures can control discontinuous functions. It determines the weak limit of $f(u^\nu)$ in the space of measures where f is (possibly) discontinuous scalar function, while $\{u^\nu\}$ is a sequence of vector–valued functions, which is weakly convergent in some L^p–space. The Representation Theorem we derive unifies the two well-known theorems in nonlinear analysis: the classical Young's theorem, where continuity of f is assumed, and Convergence Theorem known in set–valued analysis, where f may be discontinuous, but the sequence $\{u^\nu\}$ converges strongly. Such approach is dictated by the wide range of discontinuous problems arising in nonlinear PDE. The motivations and applications to various convergence problems are discussed.

1 Introduction and statement of results

The theory of Young measures originated in 1937 when L.C. Young has proved the following theorem ([68]):

Theorem 1.1 *If $I \subseteq \mathbf{R}$ is an interval, $f : \mathbf{R}^m \to \mathbf{R}$ is continuous, and $u^\nu : I \to \mathbf{R}^m$ is a sequence of measurable commonly bounded functions then there exists a subsequence $\{u^k\}$ of $\{u^\nu\}$ such that the sequence $f(u^k)$ converges weakly $*$ in $L^\infty(I)$ to the function $\overline{f}$ given by*

$$\overline{f}(x) = \int_{\mathbf{R}^m} f(\lambda)\nu_x(d\lambda), \tag{1}$$

where ν_x is a probabilistic measure on $\mathbf{R}^m$ defined for almost every $x \in \Omega$.

The discovery that the weak limit of $f(u^k)$ can be represented as an integral given by (1) turned out to be widely applicable in many disciplines of analysis such as calculus

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of variations, partial differential equations, optimal control theory, game theory, and numerical analysis, see for instance [5, 7, 17, 48, 54, 53, 51, 57], and their references.

Later there were many generalizations of the theorem of Young. They go into various directions. Let us mention a few of them.

1) The interval I in (1) is substituted by an arbitrary measurable subset of $\mathbf{R}^n$, the condition that the sequence $\{u^\nu\}$ is bounded is relaxed and one assumes that the sequence $\{f(u^\nu)\}$ is weakly $*$ convergent in the space of measures, see e.g. [2, 6, 14, 15, 31, 39, 57, 66].

2) Instead of an arbitrary sequence $\{u^\nu\}$ one takes a sequence which satisfies some additional constraints, for example $u^\nu \in M$ for every ν, where M is a submanifold of $\mathbf{R}^m$, or u^ν satisfies a given conservation law, which is expressed by the condition $Pu^\nu = 0$ where P is some differential operator, see e.g. [20, 21, 22, 36, 48, 47, 61, 62, 65, 63, 64].

3) f is a multifunction, see e. g. [1, 2].

4) Measures ν_x are defined on a completely regular topological space, see [5].

Our approach is different than each mentioned above. The following theorem is the special case of Convergence Theorem which is well known in Set–Valued Analysis, see e.g. [4, Chapter 7.2] (we use the notation given there recalling only that $f^{\#}(y)$ denotes the set of all accumulation points of f in y and $\mathrm{co}A$ is the convex hull of the set A).

Theorem 1.2 *Let $\Omega \subseteq \mathbf{R}^n$ be a compact subset equipped with a measure μ which is absolutely continuous with respect to the Lebesgue measure on Ω, $f : \mathbf{R}^m \to 2^{\mathbf{R}}$ be a nontrivial set-valued map. Assume that for every $x \in \mathrm{Dom}(f)$ there exists a neighborhood U of x such that the set $f(U)$ is bounded in $\mathbf{R}$.*
Let us consider measurable functions $u^\nu : \Omega \to \mathbf{R}^m$ and $w^\nu : \Omega \to \mathbf{R}$ such that for almost all $x \in \Omega$ we have $w^\nu(x) \in f(u^\nu(x))$, moreover
i) the sequence $\{u^\nu\}$ converges almost everywhere to a function u,
ii) $w^\nu \in L^1(\Omega, \mu)$ and the sequence $\{w^\nu\}$ is convergent weakly in $L^1(\Omega, \mu)$ to a function $w \in L^1(\Omega, \mu)$.

Then for almost all $x \in \Omega$ we have $w(x) \in \overline{\mathrm{co}} f^{\#}(u(x))$.

Our goal is to relax assumptions i) and ii) in the formulation of Convergence Theorem. Also the measure μ does not need to be absolutely continuous with respect to the Lebesque's measure. Instead we impose some additional assumptions on the multifunction f (see Theorem 8.1 and Remark 8.2 in Section 8). In particular for single–valued f we obtain a version of Young's Theorem which describes limits of sequences $f(u^\nu)$ as $\nu \to \infty$ when the sequence $\{u^\nu\}$ is weakly convergent in some L^p–type space and the sequence $\{f(u^\nu)\}$ is weakly $*$ convergent in the space of measures as $\nu \to \infty$. If we assume that f is continuous we obtain the general version of Young's Theorem proved by Alibert and Bouchitte in [2]. If f is discontinuous (and possibly set–valued) but the sequence $\{u^\nu\}$ converges strongly in $L^1(\Omega, \mu)$, the limit of $f(u^\nu)$ is described by the Convergence Theorem. Hence our approach unifies two famous theorems: Young's Theorem and Convergence Theorem.

Let us briefly describe what our main results are.

To abbreviate let us introduce the following two conditions (with the notation introduced in the next section).

Condition A.
i) The space $\mathbf{R}^m$ is compactified with the help of the finite number of disjoint Borel subsets $A_1, \ldots, A_k$, which will be called *compactification bricks.* This means that $\mathbf{R}^m = \cup_{i=1}^k A_i$, each A_i is compactified by some M_i with homeomorphism $\Phi_i : A_i \to M_i$ such that $\Phi(A_i) = M_i^0$ is dense in M_i. We assume that $M_i \subseteq \mathbf{R}^N$ for some $N \in \mathbf{N}$ and $M_i^* := M_i \setminus M_i^0$ is closed for every $i \in \{1, \ldots, k\}$.
ii) For every $i \in \{1, \ldots, k\}$ the set A_i is equipped with a *density function*: continuous function $g_i : A_i \to (0, +\infty)$ such that $g(\lambda) \geq \alpha$ for every $\lambda \in \cup_i (A_i \cap \partial A_i)$ and some $\alpha \in (0, +\infty)$, where $g := \sum_{i=1}^k \chi_{A_i} g_i$.

Condition B.
$\Omega \subseteq \mathbf{R}^n$ is compact, μ is a Radon measure on Ω and $u^\nu : \Omega \to \mathbf{R}^m$ is a sequence of μ–measurable functions such that
i) $\sup_\nu \int_{\mathbf{R}^m} g(u^\nu)\mu(dx) < \infty$ where g is the same as in Condition A,
ii) the sequence $\{u^\nu\}$ satisfies the tightness condition

$$\lim_{r\to\infty} (\limsup_{\nu\to\infty} \mu(\{x : |u^\nu(x)| > r\})) = 0 \tag{2}$$

and generates the classical Young measure $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$.

Suppose that Conditions A and B are satisfied. The Banach space

$$\mathcal{F} := \{f : \mathbf{R}^m \to \mathbf{R} : (f/g_i) \circ \Phi^{-1} \in C(M_i) \text{ for } i = 1, \ldots, k\} \tag{3}$$

with the norm

$$\|f\|_{\mathcal{F}} = \sum_{i=1}^k \|(f/g_i) \circ \Phi_i^{-1}\|_{\infty, M_i}$$

will be called the space of *admissible functions* associated to the compact decomposition of $\mathbf{R}^m$ into bricks $\{A_i\}_{i=1,\ldots,k}$ with densities $\{g_i\}_{i=1,\ldots,k}$. In (3) we assume that the function $f_i := (f/g_i) \circ \Phi_i^{-1}$ is the restriction of some continuous function defined on M_i to M_i^0. As this function is uniquely defined we will denote it by the same expression: f_i.

Let us state our main result.

Theorem 1.3 (Representation Theorem) *Suppose that Conditions A and B are satisfied and $f \in \mathcal{F}$. Then for every $i \in \{1, \ldots, k\}$ there exist*

i) *a subsequence of* $\{u^\nu\}$ *(denoted by the same expression),*

ii) *measures* $\overline{m}^i, m^i \in \mathcal{M}(\Omega)$*, such that* $\overline{m}^i$ *is absolutely continuous with respect to* μ *and* $\text{supp}m^i \subseteq \text{supp}\mu$*,*

iii) *families of probability measures* $\{\overline{\nu}^i_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \partial A_i \cap A_i, \mu)$ *and* $\{\nu^i_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, M^*_i, \mu)$

such that the subsequence $\{f(u^\nu(x))\}$ *converges weakly* $*$ *in the space of measures to the measure represented by*

$$\int_{\cup_{i=1}^k \text{int}A_i} f(\lambda)\mu_x(d\lambda)\mu(dx) + \sum_{i=1}^k \int_{\partial A_i \cap A_i} f(\lambda)\overline{\nu}^i_x(d\lambda)\overline{m}^i(dx) + \sum_{i=1}^k \int_{M^*_i} f_i(\lambda)\nu^i_x(d\lambda)m^i(dx). \tag{4}$$

The paper is organized as follows. After preliminary results introduced in Section 2 and some abstract results contained in Section 3 we give the proof of our Representation Theorem (in Section 4). Then in Section 5 we investigate some properties of Young–like measures from Theorem 1.3, namely we ask about its supports. In Section 6 we explain when can we expect that the sequence $\{f(u^\nu)\}$ is weakly convergent in $L^1(\Omega)$, not only in the space of measures. Some illustrations and relations with some other known results are presented in Section 7. Links between our Representation Theorem and Convergence Theorem are discussed in Section 8. Finally, in Section 9 we indicate on some possible applications of our Representation Theorem to various problems in PDE's.

The most interesting problems to apply our measures in are those which apply directly to PDE's with discontinuous constraints. Such equations appear naturally in many physical models, such as the Savage–Hutter model of the granular flow (see e.g. [24, 28, 30, 32, 33, 37], the phase flow in porus medias with discontinuous flux (flow) function (see e.g. [35, 43, 55]), hysteresis problems (that is history dependent flow properties), see e.g. [27, 45], traffic flow analysis, see e.g. [42]), debonding of adhesive joints, the delimination of multilayered plates, the ultimate strength of fiber reinforced structures or the nonstationary heat conduction equation, see e.g. [50, 46] and their references, or dislocations of cracks in geophysics, see e.g. [49], see also [23] for some other related results and applications.

The techniques used here are strongly based on those described in be paper by Alibert Bouchitte [2]. Also the idea to compactify $\mathbf{R}^n$ comes from reading of this paper.

I have a hope that the results presented here will be usefull in some discontinuous problems in nonlinear PDE's and in Optimal Control Theory.

2 Preliminaries and notation

Let A be a subset of the Euclidean space. We use the standard notation: $C(A)$, $C_b(A)$, $C_0(A)$ to denote continuous, continuous bounded, and continuous vanishing at infinity (if A is unbounded) functions on A. Open ball with center a and radius R is denoted by $B(a, R)$. If $a = 0$ then we omit a in our notation. The closure of the set $S \subseteq \mathbf{R}^K$ is denoted by $\overline{S}$.

Let $S \subseteq \mathbf{R}^K$ be the Borel subset of the Euclidean space. By $\mathcal{M}(S)$ we denote the space of Radon measures on S, while $\mathcal{P}(S)$ is its subset consisting of probability measures. If $\mu \in \mathcal{M}(S)$ and f is μ–measurable, we denote $(f, \mu) := \int_S f(\lambda)\mu(d\lambda)$.

If $\mu \in \mathcal{M}(S)$ and $K \subseteq S$ is the measurable subset of S, by $\mu \angle K$ we mean the restriction of μ to K, i. e. $(\mu \angle K)(A) = \mu(A \cap K)$.

If $C \subseteq \mathbf{R}^M$ is a Borel subset, $\phi : S \to C$ is a Borel–measurable mapping and $\mu \in \mathcal{M}(S)$, by $\phi^*(\mu)$ we denote the pushforward of the measure μ to $\mathcal{M}(C)$, that is $(\phi^*\mu)(K) = \mu(\phi^{-1}(K))$ if K is the Borel subset of C.

Arrows $\to$, $\rightharpoonup$, $\overset{*}{\rightharpoonup}$ are used to denote the strong, weak, and weak $*$ convergence respectively in the given topology.

Recall that the compact topological space M is the compactification of the topological space A if there is the homeomorphism $\Phi : A \to M^0 \subseteq M$ such that $\Phi(A) = M^0$ and M^0 is dense in M (see e.g. [41]).

We will consider such compactifications only that $M^* := M \setminus M^0$ is closed in $\mathbf{R}^N$. Then we will say that A is associated to the triple (Φ, M^0, M).

By $C(\hat{A})$ we denote the space of those continuous functions $f : A \to \mathbf{R}$ that $f \circ \Phi^{-1} : M^0 \to \mathbf{R}$ is the restriction of some $F \in C(M)$ to M^0. Since M^0 is dense in M, it follows that F is uniquely determined by f. Moreover, every $f \in C(\hat{A})$ is bounded.

If $S \subseteq \mathbf{R}^k$ is closed, by $L^\infty_{w*}(\Omega, \mathcal{M}(S), \mu)$ we denote the set of families $\{\mu_x\}_{x\in\Omega}$ of Radon measures on S which are weakly μ–measurable in the sense of Pettis i.e. for every $f \in C(S)$ the mapping $x \mapsto \int_S f(\lambda)\mu_x(d\lambda)$ is μ–measurable (see e.g. Definition 1 of Section V.4 in [67]).

In general the symbol $\mathcal{P}(\Omega, S, \mu)$ stands for the set of families $\{\mu_x\}_{x\in\Omega}$ of Radon probability measures on the closed subset S of some Euclidean space which are weakly μ–measurable in the sense of Pettis.

The same expression will be used for the set of families of Radon probability measures defined on Borel, but not necessarily closed subsets $S = A_i \cap \partial A_i$ of $\mathbf{R}^m$, where A_i is one of bricks of compactification of $\mathbf{R}^m$ in Condition A. Then by weak measurability of $\{\overline{\nu}_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, S, \mu)$ we will mean that the mapping $x \mapsto \int_S f(\lambda)\overline{\nu}_x(d\lambda)$ is μ–measurable for every $f \in C(\hat{A}_i)$.

At the end of this section we recall one version of the classical theorem of Young. Such formulation can be easily deduced from statements given in [6, 31], see also [5, Lemma 4.11 and Corollary 5.4] and [2], and is known to the specialists. For completeness of our arguments we include the proof in Section 3.

Theorem 2.1 *Let $u^\nu : \Omega \to \mathbf{R}^m$ be a sequence of μ–measurable functions.*

1) *There exists a subsequence of $\{u^\nu\}_{\nu\in\mathbf{N}}$ still denoted by the same expression and a family of positive measures $\{\mu_x\}_{x\in\Omega} \in L^\infty_{w*}(\Omega, \mathbf{R}^m, \mu)$ such that*

- **i)** $\|\mu_x\|_{\mathcal{M}(\mathbf{R}^m)} := \int_{\mathbf{R}^m} \mu_x(d\lambda) \le 1$ *for μ almost all $x \in \Omega$,*
- **ii)** $f(u^\nu) \stackrel{*}{\rightharpoonup} \langle f, \mu_x\rangle$ *as $\nu \to \infty$, in $L^\infty(\Omega, \mu)$, for every $f \in C_0(\mathbf{R}^m)$.*

2) *The following conditions are equivalent:*

- **i)** $\|\mu_x\|_{\mathcal{M}(\mathbf{R}^m)} = 1$ *for μ almost all $x \in \Omega$.*
- **ii)** *The sequence $\{u^\nu\}_{\nu\in\mathbf{N}}$ satisfies the tightness condition:* $\limsup_{\nu\in\mathbf{N}} \mu(\{x \in \Omega : |u^\nu(x)| \ge r\}) \stackrel{r\to\infty}{\to} 0$.
- **iii)** *For every measurable subset $\Omega' \subseteq \Omega$ and every continuous function $f : \mathbf{R}^m \to \mathbf{R}$ such that the sequence $\{f(u^\nu)\}$ is sequentially weakly relatively compact in $L^1(\Omega', \mu)$, we have $f(u^\nu) \stackrel{\nu\to\infty}{\rightharpoonup} (f, \mu_x)$ in $L^1(\Omega', \mu)$.*

Recall that the sequence $\{u^\nu\}$ generates the Young measure $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$ if for every $f \in C_0(\mathbf{R}^m)$ the sequence $\{f(u^\nu)\}$ converges weakly $*$ in $L^\infty(\Omega, \mu)$ to the function $\overline{f}(x) = \int_{\mathbf{R}^m} f(\lambda)\mu_x(d\lambda)$.

3 Four abstract lemmas

Let us assume that the following conditions are satisfied.

1) $A \subseteq \mathbf{R}^m$ is the Borel subset associated to the triple (Φ, M^0, M) where $M, M^0 \subseteq \mathbf{R}^N$, and $M^* = M \setminus M^0$ is a closed subset of $\mathbf{R}^N$.

2) $u^\nu : \Omega \to \mathbf{R}^m$ is the given sequence of μ–measurable functions, and $g : A \to \mathbf{R}_+$ is the continuous function such that $\sup_\nu \int_{\{x: u^\nu(x)\in A\}} g(u^\nu(x))\,\mu(dx) < \infty$.

Our goal now is to prove the following four lemmas. Three of them (Lemmas 3.1 3.2 and 3.4) will be crucial in the proof of Theorem 1.3. Lemma 3.3 will play a role later. Their proofs are strongly based on arguments of Alibert and Bouchitte used in [2], where a variant of Young's theorem is obtained.

Lemma 3.1 *Define the sequence of measures* $\{L^\nu\}_{\nu\in\mathbf{N}}$ *on* $\Omega \times M$ *by the expression*

$$(F, L^\nu) := \int_{\{x:u^\nu(x)\in A\}} F(x, \Phi(u^\nu(x)))g(u^\nu(x))\mu(dx), \text{ where } F \in C(\Omega \times M). \tag{5}$$

Then there exists a subsequence of $\{L^\nu\}$*, still denoted by the same expression, a measure* $L \in \mathcal{M}(\Omega \times M)$*, a family of probability measures* $\{\tilde{\nu}_x\} \in \mathcal{P}(\Omega, M, \mu)$*, and* $\tilde{m} \in \mathcal{M}(\Omega)$ *such that*

$$L^\nu \overset{*}{\rightharpoonup} L \text{ in } \mathcal{M}(\Omega \times M), \tag{6}$$

$$(F, L) = \int_\Omega \int_M F(x, \lambda)\tilde{\nu}_x(d\lambda))\tilde{m}(dx), \text{ where } F \in C(\Omega \times M), \tag{7}$$

$$\operatorname{supp} \tilde{m} \subseteq \operatorname{supp} \mu. \tag{8}$$

Lemma 3.2 *Let* $\tilde{m}$ *and* $\tilde{\nu}_x$ *be as in Lemma 3.1, and let* $\tilde{m} = p(x)\mu + \tilde{m}_s$ *be the Lebesgue's–Nikodym decomposition of* $\tilde{m}$ *with respect to* μ*. Then* $\tilde{\nu}_x(M^*) = 1$ *for* $\tilde{m}_s$ *almost all* $x \in \Omega$*.*

Lemma 3.3 *Let* $\tilde{m}$ *and* $\tilde{\nu}_x$ *be as in Lemma 3.1, and let* $K \subseteq M$ *be a closed subset. Let* $f \in C(A)$ *be such that* $f \geq 0$ *and* $(f/g) \circ \Phi^{-1} : M^0 \to \mathbf{R}$ *extends to the continuous function on* M*. The following conditions are equivalent*

i) *There exists a sequence* $\{U^r\}_{r\in\mathbf{N}}$ *of open subsets in* M *such that* $\bigcap_r U^r = K$, $\overline{U}^{r+1} \subseteq U^r$ *for every* $r \in \mathbf{N}$ *and*

$$A_r = \limsup_{\nu\to\infty} \int_{\{x:u^\nu(x)\in A, \Phi(u^\nu(x))\in U^r\}} f(u^\nu(x))\mu(dx) \overset{r\to\infty}{\to} 0 \tag{9}$$

ii) $\int_K (f/g) \circ \Phi^{-1}(\lambda)\tilde{\nu}_x(d\lambda) = 0$ *for* $\tilde{m}$ *almost all* $x \in \Omega$*.*

Lemma 3.4 *Let* $\tilde{m}$ *and* $\tilde{\nu}_x$ *be as in Lemma 3.1, and let* $K \subseteq M$ *be a closed subset. The following conditions are equivalent*

i) *There exists a sequence* $\{U^r\}_{r\in\mathbf{N}}$ *of open subsets in* M *such that* $\bigcap_r U^r = K$, $\overline{U}^{r+1} \subseteq U^r$ *for every* $r \in \mathbf{N}$ *and*

$$A_r = \limsup_{\nu\to\infty} \int_{\{x:u^\nu(x)\in A, \Phi(u^\nu(x))\in U^r\}} g(u^\nu(x))\mu(dx) \overset{r\to\infty}{\to} 0 \tag{10}$$

ii) $\tilde{\nu}_x(K) = 0$ *for* $\tilde{m}$*–almost all* $x \in \Omega$*.*

Proofs of our lemmas will be preceded by the sequence of remarks and one corollary.

Remark 3.1 It is easy to verify that if $u : \Omega \to \mathbf{R}^m$ is μ–measurable, A is a Borel subset of $\mathbf{R}^m$, and $f \in C(A)$, then the function $f(u(x))\chi_{u(x)\in A}$ is μ–measurable.

Remark 3.2 Note that the function f in Lemma 3.3 satisfies $f(\lambda) \le g(\lambda)\|f/g\|_{L^\infty(A)}$. Hence condition (10) is stronger than (9).

Corollary 3.1 *Let C be the closed subset of M, $\epsilon > 0$, $C_\epsilon = \{\lambda \in \mathbf{R}^N : \mathrm{dist}(\lambda, C) < \epsilon\}$, and assume that*

$$\phi(\epsilon) = \lim_{\nu\to\infty} \int_{\{x: u^\nu(x)\in A, \Phi(u^\nu(x))\notin C_\epsilon\}} g(u^\nu(x))\mu(dx) \overset{\epsilon\to 0}{\to} 0. \tag{11}$$

Then $\mathrm{supp}\, \tilde{\nu}_x \subseteq C$ *for μ almost all $x \in \Omega$.*

Proof. Since the function $\phi(\epsilon)$ is nonnegative and decreasing with respect to $\epsilon \in [0, \infty)$, it follows that $\phi(\epsilon) \equiv 0$. Take an arbitrary $\epsilon_0 > 0$, $K = M \setminus C_{\epsilon_0}$ and $U_\epsilon = M \setminus \overline{C}_\epsilon$ where $\epsilon < \epsilon_0$, and let $\epsilon \to \epsilon_0$. Then Lemma 3.4 implies $\tilde{\nu}_x(M \setminus C_{\epsilon_0}) = 0$, hence $\mathrm{supp}\, \tilde{\nu}_x \subseteq C_{\epsilon_0}$. Since instead of ϵ_0 we can take an arbitrary smaller number, we have $\mathrm{supp}\, \tilde{\nu}_x \subseteq \cap_{\epsilon<\epsilon_0} C_\epsilon = C$. □

Remark 3.3 Take $A = \mathbf{R}^m$, $g \equiv 1$, let K be a closed subset of $\mathbf{R}^m$, and $C = \Phi(K)$. Then a condition (11) is equivalent to the fact that $u^\nu \to K$ in the measure μ: $\lim_{\nu\to\infty} \mu(\{x \in \Omega : u^\nu(x) \notin U\}) = 0$, where U is an arbitrary neighbourhood of K in $\mathbf{R}^m$.

Remark 3.4 In general it may happen that the set C in Corollary 3.1 lies in the boundary of the compactification of A. Let for example $A = \mathbf{R}^m$, $g \equiv 1$, let $\Phi : \mathbf{R}^m \to S^m$ be the stereographic projection, $C = (0, \ldots, 0, 1)$ be the pole identified with the point at infinity. Then condition (11) reads as: $\lim_{\nu\to\infty} \mu\{x \in \Omega : |u^\nu(x)| < K\} = 0$, and expresses the fact that the sequence $\{u^\nu\}$ concentrates at infinity.

Remark 3.5 Take $A = \mathbf{R}^m$, $g \equiv 1$ and $K = M^*$ in Lemma 3.4. Then the condition (10) is equivalent to the classical tightness condition:

$$\limsup_{\nu\to\infty} \mu(\{x : |u^\nu(x)| > r\}) \overset{r\to\infty}{\to} 0, \tag{12}$$

no matter what kind of compactification of $\mathbf{R}^m$ we use.

Remark 3.6 If the sequence $\{g(u^\nu)\chi_{u^\nu \in A}\}_{\nu\in\mathbf{N}}$ is equintegrable with respect to the measure μ, C is the same as in Corollary 3.1 and for an arbitrary $\epsilon > 0$, we have

$$\lim_{\epsilon\to 0}(\lim_{\nu\to\infty} \mu(\{x \in \Omega : u^\nu(x) \in A, \Phi(u^\nu(x)) \notin C_\epsilon\})) = 0, \tag{13}$$

then condition (11) is satisfied.

Proof of Lemma 3.1. The existence of the measure L satisfying (6) follows from Banach–Alaoglu's theorem (see e.g. [59], page 131), as the space of measures on $\Omega \times M$ is dual to the separable space $C(\Omega \times M)$ and the sequence of measures L^ν is bounded.

Let $\tilde{m}$ be the projection of L onto $\mathcal{M}(\Omega)$, that is $(\Phi, \tilde{m}) = \int_\Omega \Phi(x)L(dx, d\lambda)$ for $\Phi \in C(\Omega)$, By the slicing measure argument ([56]) there exists the family of positive measures $\{\tilde{\nu}_x\}_{x\in\Omega} \in L^\infty_{w*}(\Omega, \mathcal{M}(M), \mu)$ such that the representation formula (7) holds. We will show that $\tilde{\nu}_x$ are probability measures $\tilde{m}$ almost everywhere. Take $F(x, \lambda) = h(x)f(x, \lambda)$ where $h \in C(\Omega)$, substitute it to (5), and set

$$f^\nu(x) = f(x, \Phi(u^\nu(x)))g(u^\nu(x))\chi_{\{u^\nu(x)\in A\}}, \ \overline{f}(x) = \int_M f(x, \lambda)\tilde{\nu}_x(d\lambda).$$

Since $(F, L^\nu) = \int_\Omega h(x)f^\nu(x)\mu(dx)$, $(F, L) = \int_\Omega h(x)\overline{f}(x)\mu(dx)$ and h was taken arbitrary, we deduce from (6) that

$$f^\nu(x)\mu(dx) \overset{*}{\rightharpoonup} \overline{f}\tilde{m}(dx) \tag{14}$$

in $\mathcal{M}(\Omega)$. In particular for $f \equiv 1$, we get

$$g(u^\nu(x))\chi_{\{u^\nu(x)\in A\}}\mu(dx) \overset{*}{\rightharpoonup} (\int_M 1\tilde{\nu}_x(d\lambda))\tilde{m}(dx). \tag{15}$$

On the other hand, since $\tilde{m}$ is the projection, if $h \in C(\Omega)$, we have

$$\int_\Omega h(x)\tilde{m}(dx) = \lim_{\nu\to\infty} \int_{\Omega\cap\{u^\nu\in A\}} h(x)g(u^\nu(x))\mu(dx),$$

which by (15) is equal to $\int_\Omega h(x)(\int_M \lambda\tilde{\nu}_x(d\lambda))\tilde{m}(dx)$. This implies that $\tilde{\nu}_x(M) = 1$, for $\tilde{m}$ almost all $x \in \Omega$.

To verify that (8) holds it suffices to show that if K is an open subset of Ω and $\mu(K) = 0$ then $\tilde{m}(K) = 0$. Let π^ν be the projection of L^ν onto $\mathcal{M}(\Omega)$. Then

$$\pi^\nu(K) = \int_{\{u^\nu\in A\}} \chi_K g(u^\nu(x))\mu(dx) = 0,$$

and $\pi^\nu \overset{*}{\rightharpoonup} \tilde{m}$ in $\mathcal{M}(\Omega)$. Thus $0 = \liminf_{\nu\to\infty} \pi^\nu(K) \geq \tilde{m}(K)$ (see e. g. [18, Theorem 1, Section 1.9]) and $\tilde{m}(K) = 0$. That ends the proof of the lemma. □

Proof of the Lemma 3.2. Let $F(\lambda) := \mathrm{dist}(\lambda, M^*) \in C(M)$.

At first we note that the sequence $\{h^\nu\}$ defined by $h^\nu(x) = F(\Phi(u^\nu(x)))g(u^\nu(x))$ if $u^\nu(x) \in A$ and $h^\nu(x) = 0$ otherwise is uniformly integrable in $L^1(\Omega, \mu)$.

Essentially, let $A^*_\epsilon = \{\lambda \in A : \mathrm{dist}(\Phi(\lambda), M^*) < \epsilon\}$. Then for every $K \in R_+$ we have $\int_{\{|h^\nu(x)|>K\}} |h^\nu(x)|\mu(dx) =$

$$\int_{\{|h^\nu(x)|>K\}\cap\{u^\nu(x)\in A^*_\epsilon\}} |h^\nu(x)|\mu(dx) + \int_{\{|h^\nu(x)|>K\}\cap\{u^\nu(x)\in A\setminus A^*_\epsilon\}} |h^\nu(x)|\mu(dx).$$

The first term is not larger than $\epsilon \sup_\nu \int_{u^\nu\in A} g(u^\nu(x))\mu(dx)$. The second one is zero if we take $K > \mathrm{diam} M \sup_{\lambda\in A\setminus A^*_\epsilon} |g(\lambda)|$ (this supremum is finite, because the set $A \setminus A^*_\epsilon$ is compact in $\mathbf{R}^m$).

Hence there exists $h \in L^1(\Omega, \mu)$ such that

$$h^\nu(x)\mu(dx) \overset{*}{\rightharpoonup} h\mu(dx) \text{ in } \mathcal{M}(\Omega). \tag{16}$$

On the other hand, according to Lemma 3.1, we have for arbitrary $\phi \in C(\Omega)$,

$$\begin{aligned} \int_\Omega \phi(x)h^\nu(x)\mu(dx) &= (\phi F, L^\nu) \to (\phi F, L) = \int_\Omega \phi(x)(\int_M F(\lambda)\tilde{\nu}_x(d\lambda))\tilde{m}(dx) = \\ &= \int_\Omega \phi(x)\overline{F}(x)p(x)\mu(dx) + \int_\Omega \phi(x)\overline{F}(x)\tilde{m}_s(dx), \end{aligned}$$

where $\overline{F}(x) = \int_M F(\lambda)\tilde{\nu}_x(d\lambda)$. Combining this with (16) we observe that the second term above vanishes, so $\overline{F}(x) = 0$ for $\tilde{m}_s$ almost all x. Since $F > 0$ on M^0, we get $\tilde{\nu}_x(M^0) = 0$ for $\tilde{m}_s$ almost all x, and we are done. □

Proof of Lemma 3.3. Let $h^n : M \to [0, 1]$ be such that $h^n(\lambda) \equiv 1$ if $\lambda \in \overline{U}^{n+1}$, $h^n(\lambda) \equiv 0$ outside U^n, and $0 < h^n < 1$ on the remaining set. Define $H^{\nu,n}(x) = h^n(\Phi(u^\nu(x)))$ if $u^\nu(x) \in A$ and $H^{\nu,n}(x) = 0$ otherwise. To abbreviate let us denote $F = f/g \circ \Phi^{-1}$. Using the diagonal procedure, after extracting a subsequence from $\{u^\nu\}$ we can assume that there exists a subsequence of Radon measures $\mu_n \in \mathcal{M}(\Omega)$ such that

$$H^{n,\nu}(x)f(u^\nu(x))\chi_{u^\nu(x)\in A}\mu(dx) \overset{*}{\rightharpoonup} \mu_n \text{ as } \nu \to \infty$$

in the space of measures. According to Lemma 3.1, we have

$$\mu_n(dx) = \int_M h^n(\lambda)F(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}(dx),$$

and as $1 \geq h^n$ and $h^n \equiv 1$ on K, we see that

$$A_n = \limsup_{\nu\to\infty} \int_{\{u^\nu\in A, \Phi(u^\nu)\in U^n\}} f(u^\nu(x))\mu(dx) \geq \|\mu_n\| \geq \int_\Omega \int_K F(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}(dx).$$

Therefore, if the condition i) is satisfied, we have $\int_K F(\lambda)\tilde{\nu}_x(d\lambda) = 0$ for $\tilde{m}$ almost all $x \in \Omega$.

On the other hand, as $h^n(\lambda) = 1$ on $\overline{U}^{n+1}$, and $1 \geq h^n \geq 0$, we see that

$$\int_\Omega \int_M h^{n-1}(\lambda) F(\lambda) \tilde{\nu}_x(d\lambda) \tilde{m}(dx) = \limsup_{\nu\to\infty} \int_{\{u^\nu \in A\}} h^{n-1}(\Phi(u^\nu(x)) f(u^\nu(x)) \mu(dx) \geq$$
$$\geq A_n \geq \int_\Omega \int_K F(\lambda) \tilde{\nu}_x(d\lambda) \tilde{m}(dx).$$

Applying the Lebesgue's Dominated Convergence Theorem (as $h^n \to \chi_K$ pointwise), we see that condition ii) implies

$$\lim_{n\to\infty} A_n = \int_\Omega \int_K F(\lambda) \tilde{\nu}_x(d\lambda) \tilde{m}(dx) = 0.$$

The lemma is proved. □

Proof of Lemma 3.4 We take $f = g$ in Lemma 3.3. □

At the end of this section, for completeness of our arguments, we give the proof of Theorem 2.1. The proof we present is a slight modification of that given in [6, 31], and [2] and uses our three abstract lemmas. This is only one of the possible ways of proving Theorem 2.1. One could also use more direct methods based on the papers of Ball [6] and Hungerbühler [31].

Proof of Theorem 2.1.

"1):" We use similar arguments as in [2]. Let us compactify $\mathbf{R}^m$ to the sphere S^m by adding one point at infinity with the help of the stereographic projection. We take $g \equiv 1$ and $A = \mathbf{R}^m$.

Let $f \in C_0(\mathbf{R}^m)$. According to Lemma 3.1 the measure $f(u^\nu(x))\mu(dx)$ weakly $*$ converges in the space of measures to the measure $\overline{f}(x)\tilde{m}(dx)$ where

$$\overline{f}(x) = \int_{S^m} f(\Phi^{-1}(\lambda)) \tilde{\nu}_x(d\lambda).$$

Since $f \circ \Phi^{-1}$ vanishes at the north pole $\{*\}$ of the sphere, after changing variables we obtain

$$\overline{f}(x) = \int_{\mathbf{R}^m} f(\lambda) \nu_x(d\lambda),$$

where $\nu_x = (\Phi^{-1})^*(\tilde{\nu}_x \angle (S^m \setminus \{*\}))$ is the pushforward of the measure $\tilde{\nu}_x$ restricted to $S^m \setminus \{*\}$.

At the same time the sequence $\{f(u^\nu)\}$ is weakly convergent in $L^1(\Omega, \mu)$ (even weakly $*$ in $L^\infty(\Omega, \mu)$) to some function F. This implies that in Lemma 3.2 $(f, \nu_x) = 0$ m_s almost everywhere and $(f, \nu_x)p(x) = F(x)$ for μ almost all x. Set $\mu_x = p(x)\nu_x$. As (f, μ_x) is the weak $*$ limit of $f(u^\nu)$ in $L^\infty(\Omega, \mu)$, we have

$$\limsup_{\nu\to\infty} \|f(u^\nu)\|_{L^\infty(\Omega,\mu)} \geq \|(f, \mu_x)\|_{L^\infty(\Omega,\mu)}.$$

In particular $(f, \mu_x) \le 1$ if $f \le 1$ μ almost everywhere, and $\|\mu_x\|_{\mathcal{M}(\mathbf{R}^m)} \le 1$ for μ almost all x.

"2):"; "i)$\Longrightarrow$ ii)": We forward arguments from [31] and suppose by contradiction that after passing to a subsequence we have

$$\mu(\{|u^i| \ge i\}) > \epsilon$$

for some $\epsilon > 0$ and every $i \in \mathbf{N}$. Thus

$$\mu(\Omega) - \epsilon > \mu(\Omega) - \mu(\{|u^i| \ge i\}) = \int_{\{\Omega \cap \{|u^i| < i\}\}} \mu(dx) \ge \int_\Omega f^r(u^i) \mu(dx),$$

where $r \in \mathbf{N}, r \le i$, and f^r is the continuous function such that $f^r(\lambda) \equiv 1$ for $|\lambda| < r/2$, $f^r(\lambda) \equiv 0$ for $|\lambda| \ge r$ and $0 \le f^r \le 1$ elsewhere.

After letting $i \to \infty$ and using part 1) we deduce that

$$\mu(\Omega) - \epsilon > \int_\Omega (f^r, \mu_x) \mu(dx).$$

Next we let $r \to \infty$, apply the Lebegue's Dominated Convergence Theorem twice and verify that the right hand side of the above inequality tends to $\int_\Omega \|\mu_x\|_{\mathcal{M}(\mathbf{R}^m)} \mu(dx)$. This implies

$$\mu(\Omega) > \int_\Omega \|\mu_x\|_{\mathcal{M}(\mathbf{R}^m)} \mu(dx),$$

what contradicts the assumption i).

"ii)$\Longrightarrow$ iii)": We may assume that $\Omega' = \Omega$. Let f satisfy the assumption iii). Take $f^r \in C_0(\mathbf{R}^m)$ such that $f^r \equiv f$ on $B(r) \subseteq \mathbf{R}^m$ and $|f^r| \le |f|$. Let us decompose

$$f(u^\nu)\mu = f^r(u^\nu)\mu + (f - f^r)(u^\nu)\mu.$$

According to part 1), after extracting the subsequence $\{u^\nu\}$ the first term converges weakly $*$ in the space of measures to $\mu^r = (f^r, \mu_x)\mu$. At the same time the second one converges to some measure ν^r. Since $\{f(u^\nu)\}$ is weakly compact in $L^1(\Omega, \mu)$ and the tightness condition ii) is satisfied, we easily check that $\|\nu^r\|_{\mathcal{M}(\mathbf{R}^m)} \to 0$, as $r \to \infty$. On the other hand, by the Lebesque's Dominated Convergence Theorem we get $\mu^r \to (f, \mu_x)\mu$ as $r \to \infty$. Thus

$$\overline{\mu} = * \lim_{\nu \to \infty} f(u^\nu)\mu = \mu^r + \nu^r \stackrel{r \to \infty}{\to} (f, \mu_x)\mu.$$

Since by assumption the sequence $\{f(u^\nu)\}$ is weakly convergent in $L^1(\Omega, \mu)$, its limit must be equal to (f, μ_x).

"iii)$\Longrightarrow$ i)" We substitute $f \equiv 1$ in iii). Then $\{f(u^\nu)\}$ converges weakly in $L^1(\Omega, \mu)$ to $(1, \mu_x) = \|\mu_x\|_{\mathcal{M}(\mathbf{R}^m)}$, and $f(u^\nu) \equiv 1$. Hence $\|\mu_x\|_{\mathcal{M}(\mathbf{R}^m)} = 1$ for μ almost all x.

That ends the proof of the theorem. $\square$

4 Proof of the Representation Theorem

Our goal now is to give the proof of Theorem 1.3. The proof will be based on the following lemma.

Lemma 4.1 *Assume that*

1) $A \subseteq \mathbf{R}^m$ *is the Borel subset associated to the triple* (Φ, M^0, M) *where* $M, M^0 \subseteq \mathbf{R}^N$*, and* $M^* = M \setminus M^0$ *is closed subset of* $\mathbf{R}^N$*.*

2) $u^\nu : \Omega \to \mathbf{R}^m$ *is a given sequence of* μ*–measurable functions, which satisfy the tightness condition (12) and generate the Young measure* $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$*.*

3) $g : A \to \mathbf{R}_+$ *is the continuous function such that* $\sup_\nu \int_{\{x:u^\nu(x)\in A\}} g(u^\nu(x))\, \mu(dx) < \infty$*.*

Then the following statements hold true.

i) *If* $\tilde{M}^0 = \Phi(\mathrm{int}A)$*, and* $f \in C(M)$ *is such that* $F \in C_0(\mathbf{R}^m)$*, where* $F(\lambda) = f(\Phi(\lambda))g(\lambda)$ *if* $\lambda \in A$ *and* $F(\lambda) = 0$ *otherwise, we have*

$$\int_{\mathrm{int}A} f(\Phi(\lambda))g(\lambda)\mu_x(d\lambda) = p(x) \int_{\tilde{M}^0} f(\lambda)\tilde{\nu}_x(d\lambda), \tag{17}$$

for μ *almost all* $x \in \Omega$*, where* $p(x)$ *is the same as in assertion ii) of Lemma 3.1.*

ii) *Let* $\tilde{\nu}_x^0 = \tilde{\nu}_x \angle \tilde{M}^0 \in \mathcal{M}(\tilde{M}^0)$*,* $\mu_{A,x} = g(\lambda)\chi_{\{\lambda\in\mathrm{int}A\}}\mu_x(d\lambda) \in \mathcal{M}(\mathrm{int}A)$*. Then for* μ *almost all* $x \in \Omega$*, we have*

$$\Phi^*(\mu_{A,x}) = p(x)\tilde{\nu}_x^0. \tag{18}$$

Proof. "i):" Take $F(\lambda) = f(\Phi(\lambda))g(\lambda)$ if $\lambda \in A$ and $F(\lambda) = 0$ otherwise, and assume that $F \in C_0(\mathbf{R}^m)$. Then F is supported in A, and vanishes at infinity and on ∂A. Applying the theorem of Young we get

$$F(u^\nu(x)) \rightharpoonup \overline{F}(x) = \int_{\mathbf{R}^m} F(\lambda)\mu_x(d\lambda) = \int_{\mathrm{int}A} f(\Phi(\lambda))g(\lambda)\mu_x(d\lambda) \text{ in } L^1(\Omega, \mu).$$

Hence $F(u^\nu(x))\mu(dx) \stackrel{*}{\rightharpoonup} \overline{F}(x)\mu(dx)$ in $\mathcal{M}(\Omega)$. According to Lemma 3.2,

$$\overline{F}(x)\mu = (f, \tilde{\nu}_x)\tilde{m} = (f, \tilde{\nu}_x)p(x)\mu + (f, \tilde{\nu}_x)\tilde{m}_s.$$

Since f vanishes on M^*, which is the support of $\tilde{\nu}_x$ for $\tilde{m}_s$ almost all x, it follows that the second term of the above decomposition vanishes. Note also that $f \equiv 0$ on $M_1 = M \setminus \Phi(\mathrm{int}A)$. Hence $(f, \tilde{\nu}_x) = \int_{\tilde{M}^0} f(\lambda)\tilde{\nu}_x(d\lambda)$.

"ii):" It suffices to note that the left hand side of (17) reads as

$$\int_{\mathrm{int}A} f\circ\Phi(\lambda)\mu_{A,x}(d\lambda) = (f\circ\Phi, \mu_{A,x}) = (f, \Phi^*(\mu_{A,x})),$$

while the right hand side of (17) reads as $(f, p(x)\tilde{\nu}_x^0)$. Now the assertion follows from the fact that if $U \subseteq \mathbf{R}^N$ is an open set, $\nu_1, \nu_2 \in \mathcal{M}(U)$ and $(f,\nu_1) = (f,\nu_2)$ for every $f \in C(\overline{U})$ such that $f \equiv 0$ on ∂U then $\nu_1 \equiv \nu_2$. In our case $U = \tilde{M}^0$. □

Now we are in the position to prove Theorem 1.3.

Proof of Theorem 1.3. We may assume that f does not vanish on one of the bricks: $A = A_i$ of compactification of $\mathbf{R}^m$ only. To abbreviate we will omit the index "i" in this part of the proof.

Let $u^\nu(x) \in A$. We have

$$f(u^\nu(x)) = \frac{f}{g}(u^\nu(x))g(u^\nu(x)) = F(\Phi(u^\nu(x))g(u^\nu(x)),$$

where $F = (f/g)\circ\Phi^{-1}$. According to Lemma 3.1, we have (as $f = f\chi_{\{\lambda\in A\}}$)

$$f(u^\nu(x))\mu(dx) \overset{*}{\rightharpoonup} \int_M F(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}(dx) = \mathcal{A}, \tag{19}$$

where $\tilde{m}$ and $\{\tilde{\nu}_x\}_{x\in\Omega}$ are the same as in Lemma 3.1. Using the Lebesgue–Nikodym's decomposition of $\tilde{m}$ with respect to μ as in Lemma 3.2 we verify that

$$\mathcal{A} = \int_M F(\lambda)\tilde{\nu}_x(d\lambda)p(x)\mu(dx) + \int_{M^*} F(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}_s(dx).$$

Since the first integral is the sum of two: the one over $\tilde{M}^0 = \Phi(\mathrm{int}A)$ and the second over $M_1 = M \setminus \Phi(\mathrm{int}A)$, we derive from Lemma 4.1 that

$$\begin{aligned}\mathcal{A} &= \int_{\mathrm{int}A} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{M_1} F(\lambda)\tilde{\nu}_x(d\lambda)p(x)\mu(dx) + \\ &+ \int_{M^*} F(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}_s(dx) = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3.\end{aligned}$$

Decomposing further $M_1 = M^* \cup (M_1 \setminus M^*)$, and recalling that $\tilde{m} = p(x)\mu + \tilde{m}_s$, we see that the last two integrands sum up to

$$\int_{M_1\setminus M^*} F(\lambda)\tilde{\nu}_x(d\lambda)p(x)\mu(dx) + \int_{M^*} F(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}(dx) = \mathcal{B}_1 + \mathcal{B}_2. \tag{20}$$

Let $h(x) = \tilde{\nu}_x(M^*)$, $\Omega_1 = \{x \in \Omega : h(x) \neq 0\}$, and choose $y \in M^*$. Then the second term in (20) can be modified to

$$\mathcal{B}_2 = \int_{M^*} F(\lambda)\nu_x(d\lambda)m(dx), \tag{21}$$

where ν_x is such that $(F, \nu_x) = \frac{1}{\tilde{\nu}_x(M^*)} \int_{M^*} F(\lambda)\tilde{\nu}_x(d\lambda)$ for $x \in \Omega_1$ and $(F, \nu_x) = F(y)$ otherwise, and $m(dx) = h(x)\tilde{m}(dx)$.

Since $M_1 \setminus M^* = \Phi(\partial A \cap A)$, after the change of variables in $\mathcal{B}_1$ we derive

$$\mathcal{B}_1 = \int_{\partial A \cap A} \frac{f}{g}(\lambda)(\Phi^{-1})^*(\tilde{\nu}_x(d\lambda))p(x)\mu.$$

Let

$$w(x) = \int_{\partial A \cap A} \frac{1}{g(\lambda)}(\Phi^{-1})^*(\tilde{\nu}_x(d\lambda)) = \int_{M_1 \setminus M^*} \frac{1}{g(\Phi^{-1}(\lambda))}\tilde{\nu}_x(d\lambda),$$

and $q(x) = w(x)p(x)$. Since we assume that $g(\lambda) > \alpha$ if $\lambda \in \partial A \cap A$, it follows that $w(x)$ cannot be infinite. Choose an arbitrary $a \in \partial A \cap A$, set $\Omega_2 = \{x \in \Omega : \tilde{\nu}_x(\partial M \setminus M^*) > 0\}$, and define the measure $\overline{\nu}_x$ by

$$\overline{\nu}_x = \frac{1}{w(x)} \frac{1}{g(\lambda)}(\Phi^{-1})^*((\tilde{\nu}_x \angle (M_1 \setminus M^*))(d\lambda)), \text{ if } x \in \Omega_2, \tag{22}$$

and $\overline{\nu}_x = \delta_a$ otherwise. Then

$$\mathcal{B}_1 = \int_{\partial A \cap A} f(\lambda)\overline{\nu}_x(d\lambda)q(x)\mu.$$

That ends the proof of the theorem. □

5 On supports of Young–like measures

In this section we will prove the following theorem which deals with supports of measures appearing in Theorem 1.3.

Theorem 5.1 *Suppose that Conditions A and B are satisfied. Let $i \in \{1, \ldots, k\}$ and assume further that $K \subseteq M_i$ is a closed subset such that there exists a sequence of open subsets in M_i: $\{U^r\}_{r \in \mathbf{N}}$ such that $\cap_r U^r = K$, $\overline{U}^{r+1} \subseteq U^r$ for every $m \in \mathbf{N}$ and*

$$A_r := \limsup_{\nu \to \infty} \int_{\{x : u^\nu(x) \in A_i, \Phi_i(u^\nu(x)) \in U^r\}} g_i(u^\nu(x))\mu(dx) \overset{r \to \infty}{\to} 0. \tag{23}$$

Let $\overline{m}^i$, m^i, $\{\overline{\nu}^i_x\}_{x \in \Omega}$, $\{\nu^i_x\}_{x \in \Omega}$ be as in Theorem 1.3.

Then we have.

i) $\operatorname{supp} \overline{\nu}^i_x, \operatorname{supp} \mu_x \subseteq \Phi_i^{-1}(M_i^0 \setminus K)$ *for μ–almost every x, and* $\operatorname{supp} \nu^i_x \subseteq M_i^* \setminus K$ *for m^i–almost every x.*

ii) *Let $M_i^* = M_{i,\infty}^* \cup M_{i,b}^*$ where*

$$\begin{aligned} M_{i,\infty}^* &= \{m \in M_i^* : \text{if } m_k \in M_i^0, m_k \to m \text{ as } k \to \infty \text{ then} \\ &\quad |g(\Phi_i^{-1}(m_k))| \to \infty \text{ as } k \to \infty\}, \\ M_{i,b}^* &= M_i^* \setminus M_{i,\infty}^*, \end{aligned}$$

and let $m^i = q_i(x)\mu + m_s^i$ be the Lebesgue's–Nikodym decomposition of the measure m^i with respect to μ. Then $\mathrm{supp}\nu_x^i \subseteq M_{i,\infty}^*$ for m_s almost every $x \in \Omega$.

Proof. To abbreviate we will omit an index "i" in our notation.

"i):" Let $A = A_i$ and $\tilde{\nu}_x$ be the measure from Lemma 3.1. Then according to Lemma 3.4 we have $\mathrm{supp}\tilde{\nu}_x \subseteq M \setminus K$. Now the assertion follows from formulae (21) and (22) used to define measures ν_x and $\overline{\nu}_x$ and from Lemma 4.1.

"ii):" Let $F(\lambda) = g(\lambda)\mathrm{dist}(\Phi(\lambda), M_\infty^*)$ if $\lambda \in A$ and $F(\lambda) = 0$ if $\lambda \not\in A$. According to the Theorem 1.3, after extracting the subsequence, we obtain $F(u^\nu) \overset{*}{\rightharpoonup}$

$$\int_{\mathrm{int}A} F(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{\partial A \cap A} F(\lambda)\overline{\nu}_x(\lambda)\overline{m}(dx) + \int_{M^*} \mathrm{dist}(\tau, M_\infty^*)\nu_x(d\tau)m(dx)$$

in the space of measures. It suffices to prove that the singular part of the above measure (with respect to μ), equal to

$$(\int_{M^*} \mathrm{dist}(\tau, M_\infty^*)\nu_x(d\tau))m_s(dx)$$

vanishes. This will follow from the fact that the sequence $F(u^\nu)$ is weakly compact in $L^1(\Omega, \mu)$. To prove this, according to the Dunford–Pettis criterion it suffices to show that

$$\mathcal{A}_k = \limsup_{\nu \to \infty} \int_{\{|F(u^\nu)| > k\}} |F(u^\nu(x))|\mu(dx) \overset{k \to \infty}{\to} 0.$$

Define the following sets:

$$\begin{aligned} D^\nu &= \{x : |F(u^\nu(x))| > k\}, \\ C^\nu &= \{x : |F(u^\nu(x))| > k, \mathrm{dist}(\Phi(u^\nu(x)), M_\infty^*) < \epsilon\}, \\ E^\nu &= \{x : |F(u^\nu(x))| > k, \mathrm{dist}(\Phi(u^\nu(x)), M_\infty^*) \geq \epsilon\}, \end{aligned}$$

and let us decompose $\int_{D^\nu} = \int_{C^\nu} + \int_{E^\nu}$. Then the first integral is not bigger than

$$\epsilon \limsup_{\nu \to \infty} \int_{u^\nu \in A} |g(u^\nu(x))|\mu(dx),$$

while the second one is zero if we take k large enough. To see this it suffices to show that $E^\nu = \emptyset$ if k is large enough. Let

$$A_\epsilon = \{\lambda \in A : \mathrm{dist}(\Phi(\lambda), M_\infty^*) \geq \epsilon\}.$$

We will show that there exists a constant C_ϵ such that

$$|g(\lambda)| \leq C_\epsilon \text{ if } \lambda \in A_\epsilon. \tag{24}$$

Essentially, note that the function $1/g : A \to \mathbf{R}_+$ is continuous on A, M^*_∞ is closed, and M^*_∞ is the maximal set K such that the function $G = \frac{1}{g} \cdot \Phi^{-1} : M^0 \to \mathbf{R}_+$ extended by 0 to the set $K \cup M^0$ is continuous. This implies that there exists a constant $L_\epsilon > 0$ such that the extension $\tilde{G}$ of G to $M^0 \cup M^*_\infty$ satisfies $|\tilde{G}(m)| \geq L_\epsilon$ for every $m \in (M^*_\infty \cup M^0) \cap \{m : \mathrm{dist}(m, M^*_\infty) \geq \epsilon\} = M^0 \cap \{m : \mathrm{dist}(m, M^*_\infty) \geq \epsilon\}$. This gives (24) and together with the fact that the function $\mathrm{dist}(\Phi(\lambda), M^*_\infty)$ is bounded, implies that $E^\nu = \emptyset$ for k sufficiently large. The theorem is proved. □

6 From weak $*$ convergence in measures to weak convergence in L^1

Now we are going to explain when can we expect that the sequence $\{f(u^\nu)\}$ is weakly convergent in $L^1(\Omega, \mu)$. This will be done with the help of Theorem 6.1 formulated below. It generalizes Theorem 2.9 in the paper by Alibert and Bouchitte [2], who have obtained similar result with the additional assumption that the function f in the formulation of Theorem 6.1 is continuous. Our proof is based on similar techniques.

Theorem 6.1 *Suppose that*

1) *conditions A and B are satisfied,*

2) *$f \in \mathcal{F}$, and $M^*_{i,\infty}, M^*_{i,b}$ are the same as in Theorem 5.1,*

3) *$\overline{m}^i$, m^i, $\{\overline{\nu}^i_x\}_{x\in\Omega}$ and $\{\nu^i_x\}_{x\in\Omega}$ are the same as in Theorem 1.3,*

4) *functions $p_i, q_i \in L^1(\Omega, \mu)$ are such that*

$$\overline{m}^i(dx) = p_i(x)\mu(dx), \quad m^i(dx) = q_i(x)\mu(dx) + m^i_s(dx),$$

where both expressions describe the Lebesgue's Nikodym's decomposition of measures $\overline{m}^i$ and m^i with respect to the measure μ.

Then the following statements are equivalent.

i) *For every $i \in \{1, \ldots, k\}$, we have*

$$\limsup_{\nu\in\mathbf{N}} \int_{\{x\in\Omega: u^\nu(x)\in A_i : \mathrm{dist}(\Phi_i(u^\nu(x)), M^*_{i,\infty}) < \epsilon\}} |f(u^\nu(x))|\mu(dx) \overset{\epsilon\to 0}{\to} 0. \tag{25}$$

ii) *The sequence $|f(u^\nu)|$ is weakly convergent in $L^1(\Omega,\mu)$ to*

$$\int_{\cup_i \mathrm{int} A_i} |f(\lambda)|\mu_x(d\lambda) + \sum_{i=1}^k p_i(x)\int_{\partial A_i \cap A_i} |f(\lambda)|\overline{\nu}_x^i(d\lambda) + \sum_{i=1}^k q_i(x)\int_{M^*_{i,b}} |f_i(\lambda)|\nu_x^i(d\lambda), \tag{26}$$

iii) *For every $i \in \{1,\ldots,k\}$ we have*

$$(\int_{M^*_{i,\infty}} |f_i(\lambda)|\nu_x^i(d\lambda))m^i = 0$$

in the space of measures.

We also have the following two remarks.

Remark 6.1 The condition (25) is satisfied if it is satisfied with g substituted by f.

Remark 6.2 If one of the conditions i) or iii) in the formulation of Theorem 6.1 is satisfied, then the formulae (26) holds with $|f|$ substituted by f and $|f_i|$ substituted by f_i. This follows from the fact that $f = f^+ - f^-$ where $f^+ = max\{f, 0\}$ and $f^- = max\{-f, 0\}$.

Now let us prove the above theorem.

Proof of Theorem 6.1. Obviously, we can assume that f is nonnegative and f does not vanish on one of the A_j's in Condition A only, say on A_i. To abbreviate let us omit the index i in our notation.

"*i) $\Longrightarrow$ ii)*" This part of the proof is achieved in two steps. In Step 1 we check that two conditions in Dunford–Pettis criterion

$$\limsup_{\nu\to\infty} \int_{\{u^\nu \in A\}} f(u^\nu(x))\mu(dx) < \infty, \quad \text{and} \tag{27}$$

$$\limsup_{\nu\to\infty} \int_{\{u^\nu \in A, f(u^\nu(x))>k\}} f(u^\nu(x))\mu(dx) \overset{k\to\infty}{\to} 0. \tag{28}$$

are satisfied. In Step 2 we recognize the limit of $\{f(u^\nu)\}$.

Proof of Step 1: The first condition follows from inequality $f(\lambda) \le g(\lambda)\|f/g\|_{L^\infty(A)}$ and the assumption ii) in Condition B. Let us check the second condition. Take an arbitrary $\epsilon > 0$, and let $A_\nu = \{u^\nu \in A, f(u^\nu) > k\}$. Then $A_\nu = B_{\nu,\epsilon} \cup C_{\nu,\epsilon}$ where $B_{\nu,\epsilon}$ and $C_{\nu,\epsilon}$ are given by:

$$B_{\nu,\epsilon} = \{x \in A_\nu : \mathrm{dist}(\Phi(u^\nu(x)), M^*_\infty) < \epsilon\}, \quad C_{\nu,\epsilon} = \{x \in A_\nu : \mathrm{dist}(\Phi(u^\nu(x)), M^*_\infty) \ge \epsilon\}.$$

Thus, it suffices to prove that both expressions: $\limsup_{\nu\to\infty}\int_{B_{\nu,\epsilon}} f(u^\nu(x))\mu(dx)$ and $\limsup_{\nu\to\infty}\int_{C_{\nu,\epsilon}} f(u^\nu(x))\mu(dx)$ can be arbitrary small if k is large enough. By assumption i) the first expression is arbitrary small if we choose ϵ small enough. On the other hand, as $f \le g\|f/g\|_{L^\infty(A)}$, and according to (24) in the proof of Theorem 5.1, we see that the second expression vanishes if we let k to be large enough.

Proof of Step 2: According to Theorem 1.3, after passing to a subsequence $\{f(u^\nu)\}$ converges weakly $*$ in the space of measures to

$$\int_{\mathrm{int}A} f(\lambda)\mu_x(d\lambda)\mu(dx)+\int_{\partial A\cap A} f(\lambda)\overline{\nu}_x(d\lambda)\overline{m}(dx)+\int_{M^*}(f/g)\circ\Phi^{-1}(\lambda)\nu_x(d\lambda)m(dx). \quad (29)$$

Let us recall that according to (21) the measure ν_x was constructed by

$$(F,\nu_x)=\frac{1}{\tilde{\nu}_x(M^*)}\int_{M^*}F(\lambda)\tilde{\nu}_x(d\lambda),$$

for m–almost all x, and $\tilde{\nu}_x$ is the same as that in Lemma 3.1 (where $A=A_i$). By Lemma 3.3 the measure $(f/g)\circ\Phi^{-1}\tilde{\nu}_x\angle M^*$ is supported on M_b^*. Thus the last term in (29) equals

$$\int_{M_b^*}(f/g)\circ\Phi^{-1}(\lambda)\nu_x(d\lambda)q(x)\mu(dx)+\int_{M_b^*}(f/g)\circ\Phi^{-1}(\lambda)\nu_x(d\lambda)m_s(dx).$$

On the other hand, we have just proved in Step 1 that the sequence $\{f(u^\nu)\}$ is weakly compact in $L^1(\Omega,\mu)$, so the last term in the above expression vanishes. That completes the proof of Step 2.

"*iii)* $\Longleftrightarrow$ *i)*"

Let $h_\delta : A\to[0,1]$ be defined by $h_\delta(\lambda)=0$ if $\mathrm{dist}(\Phi(\lambda),M_\infty^*)>\delta$, $h_\delta(\lambda)=1$ if $\mathrm{dist}(\Phi(\lambda),M_\infty^*)<\delta/2$, and $h_\delta(\lambda)=-2\delta^{-1}\mathrm{dist}(\Phi(\lambda),M_\infty^*)+2$ if $\mathrm{dist}(\Phi(\lambda),M_\infty^*)\in[\delta/2,\delta]$. Define

$$\begin{aligned} f_\delta(\lambda) &= \begin{cases} f(\lambda)h_\delta(\lambda) & \text{if } \lambda\in A \\ 0 & \text{if } \lambda\not\in A \end{cases}, \text{ and} \\ A_\delta &= \limsup_{\nu\in\mathbf{N}}\int_{\{x:u^\nu(x)\in A:\mathrm{dist}(\Phi(u^\nu(x)),M_\infty^*)<\delta\}} f(u^\nu(x))\mu(dx). \end{aligned}$$

According to Theorem 1.3 $f_\delta(u^\nu)$ converges weakly $*$ in measures to

$$B_\delta=\int_{\mathrm{int}A} f_\delta(\lambda)\mu_x(d\lambda)\mu+\int_{\mathrm{int}A\cap A} f_\delta(\lambda)\overline{\nu}_x(d\lambda)p(x)\mu+\int_{M^*}F_\delta(\lambda)\nu_x(d\lambda)m,$$

where $F_\delta(\lambda)=f_\delta/g\circ\Phi^{-1}$. Note that $|f_\delta|\le|f_{2\delta}|$, and $\int_\Omega B_\delta\mu(dx)\le A_\delta\le\int_\Omega B_{2\delta}\mu(dx)$. After letting $\delta\to0$ and using the Lebesque's Dominated Convergence Theorem, we get

$$\lim_{\delta\to0}A_\delta=\int_\Omega\int_{M_\infty^*} f/g\circ\Phi^{-1}(\lambda)\nu_x(d\lambda)m(dx).$$

which implies that i) is equivalent to iii).

"$ii) \Longrightarrow iii)''$ This part follows directly from Theorem 5.1. $\square$

Remark 6.3 Suppose that f in the assumptions of Theorem 6.1 does not vanish on one of compactificaton bricks $A = A_i$ only, and let us omit the index i in our notation. Assume additionally that

$$\lim\sup_{\nu\to\infty} \mu(\{x : u^\nu(x) \in A, dist(\Phi(u^\nu(x)), M^*_\infty) < \epsilon\}) \stackrel{\epsilon\to 0}{\to} 0. \tag{30}$$

Then we have two observations.

1) If the sequence $\{f(u^\nu)\}$ is weakly convergent in $L^1(\Omega, \mu)$, then its weak limit must be equal to

$$\overline{f}(x) = \int_{intA} f(\lambda)\mu_x(d\lambda) + p(x)\int_{\partial A\cap A} f(\lambda)\overline{\nu}_x(d\lambda) + q(x)\int_{M^*_b}(f/g)\circ\Phi^{-1}(\lambda)\nu_x(d\lambda).$$

2) If instead we only assume that the sequence $\{f(u^\nu)\}$ is only bounded in $L^1(\Omega, \mu)$ then we can use the Chacon's Biting Lemma (see e.g. [8]) and find a sequence $\{E_r\}$ of subsets of Ω such that $E_{r+1} \subseteq E_r$, $\mu(E_r) \to 0$ as $r \to \infty$, and a subsequence of $\{f(u^\nu)\}$ is weakly convergent in $L^1(\Omega \setminus E_r, \mu)$ for an arbitrary $r \in \mathbf{N}$ to some function $\tilde{f}$. This function $\tilde{f}$ is called the biting limit of $\{f(u^\nu)\}$. By the first observation with Ω substituted by $\Omega \setminus E_r$ we verify that $\tilde{f} = \overline{f}$. In the case when f is continuous on the whole $\mathbf{R}^m$ ($A = \mathbf{R}^m$) such property was proved by Alibert and Bouchitte in [2, Theorem 2.9].

Remark 6.4 If in the previous remark we assume instead of (30) that

$$\lim\sup_{\nu\to\infty} \mu(\{x : u^\nu(x) \in A, dist(\Phi(u^\nu(x)), M^*) < \epsilon\}) \stackrel{\epsilon\to 0}{\to} 0,$$

then analogous observations as in Remark 6.3 hold with

$$\overline{f}(x) = \int_{intA} f(\lambda)\mu_x(d\lambda) + p(x)\int_{\partial A\cap A} f(\lambda)\overline{\nu}_x(d\lambda).$$

7 Special cases

In this section we are going to illustrate our Representation Theorem on concrete examples. To abbreviate our notation through this section we will assume that

1) $\Omega \subseteq \mathbf{R}^n$ is the compact subset, $\mu \in \mathcal{M}(\Omega)$ is the Borel measure,

2) $u^\nu : \Omega \to \mathbf{R}^m$ is the bounded sequence in $L^1(\Omega, \mu)$,

3) $F \subseteq \mathbf{R}^m$ is the closed subset such that $u^\nu \to F$ in the measure μ.

Let us consider at first the case when $k = 1$ and $A_1 = \mathbf{R}^m$ in Theorem 1.3. The following result was obtained in a slightly more general version by Alibert and Bouchitte (see Theorem 2.5 in [2]).

Theorem 7.1 *Let us define the following objects*

$$\begin{aligned} \mathcal{F}(\Omega \times \mathbf{R}^m) :=& \quad \{f \in \quad C(\mathbf{R}^m) : f^\infty(\lambda) := \lim_{t\to\infty} \frac{f(t\lambda)}{t} \in C(S^{m-1})\}, \\ F^\infty :=& \quad \{\lambda \in \quad \mathbf{R}^m : \exists_{\lambda_k \in \mathbf{R}^m, t_k \in \mathbf{R}} : \lambda_k \to \lambda, t_k \to \infty, t_k\lambda_k \in F\}. \end{aligned}$$

Then there exists a subsequence $\{u^\nu\}_{\nu\in\mathbf{N}}$ still denoted by the same expression, a positive measure $m \in \mathcal{M}(\Omega)$, families of probability measures $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$, $\{\nu_x^\infty\}_{x\in\Omega} \in \mathcal{P}(\Omega, S^{m-1}, m)$ such that

$$f(u^\nu) \stackrel{*}{\rightharpoonup} \int_{\mathbf{R}^m} f(\lambda)\mu_x(d\lambda)\mu + \int_{S^{m-1}} f^\infty(\lambda)\nu_x^\infty(d\lambda)m$$

in the space of measures. Moreover, the following properties hold:

i) $\int_{\mathbf{R}^m} |\lambda|\mu_x(d\lambda) \in L^1(\Omega, \mu)$ *and* $\mathrm{supp}\nu_x \in F$,

ii) $\mathrm{supp} m \subseteq \mathrm{supp}\mu$ *and* $\mathrm{supp}\nu_x^\infty \subseteq F^\infty \cap S^{m-1}$ *for m almost all $x \in \Omega$.*

Proof. This follows from Theorem 1.3. Let $A = \mathbf{R}^m$, $g(\lambda) = 1 + |\lambda|$, and $\Phi(\lambda) = \frac{\lambda}{1+|\lambda|} : \mathbf{R}^m \to B(1) \subseteq \mathbf{R}^m$. Then $M^* = S^{m-1}$, $M^*_\infty = M^*$, $M^*_b = \emptyset$, and for $\lambda \in S^{m-1}$ we have $f/g \circ \Phi^{-1}(\lambda) = f^\infty(\lambda) = \lim_{t\to\infty} \frac{f(t\lambda)}{t}$. □

Now we will study the weak limit of $\{f(u^\nu)\}_{\nu\in\mathbf{N}}$ in the case when f has a finite number of discontinuity points. We have the following theorem.

Theorem 7.2 *Assume that $A_1, \ldots, A_k \in \mathbf{R}^m$, $\theta_i(\lambda) := \frac{\lambda - A_i}{|\lambda - A_i|}$, $f \in C(\mathbf{R}^m \setminus \cup_{j=1}^k A_j)$ and f has radial limits at ∞ and at A_i for every $i \in \{1, \ldots, k\}$, which means that the expressions*

$$f^\infty(\lambda) := \lim_{t\to\infty} \frac{f(t\lambda)}{t}, \text{ and } f_i(\lambda) := \lim_{t\to 0} f(t\theta_i(\lambda) + A_i)$$

are well defined. Set

$$\begin{aligned} \mathcal{F} := \{f : \mathbf{R}^m \to \mathbf{R} : f \in C(\mathbf{R}^m \setminus \{A_1, \ldots, A_k\}), f^\infty(\lambda) \in C(S^{m-1}), \\ f_i(\lambda) \in C(S^{m-1})\} \text{ for } i \in \{1, \ldots, k\}\} \\ F^\infty := \{\lambda \in \mathbf{R}^m : \exists_{\lambda_k \in \mathbf{R}^m, t_k \in \mathbf{R}} : \lambda_k \to \lambda, t_k \to \infty, t_k\lambda_k \in F\} \\ F^i := \{\lambda \in S^{m-1} : \exists_{\lambda_k \in F\setminus A_i} : \lambda_k \to A_i, \theta_i(\lambda_k) \to \lambda \text{ as } k \to \infty\}, i = 1, \ldots, k. \end{aligned}$$

Then there exist:

a subsequence $\{u^\nu\}_{\nu\in\mathbf{N}}$ *still denoted by the same expression;*

a positive measure $m \in \mathcal{M}(\Omega)$*;*

μ*–measurable functions* $p_i, q_i : \Omega \to [0,1]$ *where* $i = 1,\dots,k$*;*

families of probability measures $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$*,* $\{\nu_x^\infty\}_{x\in\Omega} \in \mathcal{P}(\Omega, S^{m-1}, m)$*, and* $\{\nu_x^i\}_{x\in\Omega} \in \mathcal{P}(\Omega, S^{m-1}, \mu)$ *where* $i \in \{1,\dots,k\}$

such that

i) $f(u^\nu) \overset{*}{\rightharpoonup} \int_{\mathbf{R}^m\setminus\{A_1,\dots,A_k\}} f(\lambda)\mu_x(d\lambda)\mu + \int_{S^{m-1}} f^\infty(\lambda)\nu_x^\infty(d\lambda)m+$

$$+\sum_{i=1}^{k}\int_{S^{m-1}} f^i(\lambda)\nu_x^i(d\lambda)p_i(x)\mu + \sum_{i=1}^{k} f(A_i)q_i(x)\mu$$

in the space of measures,

ii) $\{\mu_x\}_{x\in\Omega}$ *is the Young measure generated by the sequence* $\{u^\nu\}_{\nu\in\mathbf{N}}$*,* $\int_{\mathbf{R}^m} |\lambda|\mu_x(d\lambda) \in L^1(\Omega,\mu)$ *and* $\mathrm{supp}\mu_x \subseteq F$ *for* μ *almost all* $x \in \Omega$*,*

iii) $\mathrm{supp} m \subseteq \mathrm{supp}\mu$ *and* $\mathrm{supp}\nu_x^\infty \subseteq F^\infty \cap S^{m-1}$ *for* m *almost all* $x \in \Omega$*,* $\mathrm{supp}\nu_x^i \subseteq F^i$ *for* $i = 1,\dots,k$ *and* μ *almost all* $x \in \Omega$*,*

iv) $p_i(x)+q_i(x) = \mu_x(\{A_i\})$ *for* μ*–almost all* x*, in particular* $\sum_{i=1}^k p_i(x)+\sum_{i=1}^k q_i(x) = \sum_{i=1}^k \mu_x(A_i) \le 1$ *for* μ*–almost all* x*.*

Proof. We sketch the proof only, leaving easy details to the reader. Choose $r, R \in \mathbf{R}$ such that the sets $U_1,\dots,U_{k+1}$ are disjoint where $U_i = B(A_i, r)$ for $i = 1,\dots,k$, and $U_{k+1} = \mathbf{R}^m \setminus \overline{B(R)}$. Cover $\mathbf{R}^m$ by $U_0,\dots,U_{k+1}$ where $U_0 = \{\lambda \in \mathbf{R}^m : |\lambda| < 2R, |\lambda - A_i| > r/2\}$, and let $\{\phi_r\}_{r=0,\dots,k+1}$ be the continuous partition of unity subordinate to this covering. After decomposing $f = \sum_i \phi_i f = \sum f_i$, it suffices to prove that the result is true if either 1): $f \in C(\mathbf{R}^m)$ and $A_1,\dots,A_k \notin \mathrm{supp}\, f$ or 2): $f \in C(\mathbf{R}^m \setminus \{A_i\})$ and $\mathrm{supp} f \subseteq B(A_i, r)$.

In the first case it suffices to apply Theorem 7.1.

In the second one we decompose $\mathbf{R}^m$ by bricks: $A_1 = P_i := B(A_i, r) \setminus \{A_i\}$, $A_2 = \{A_i\}$, $A_3 = \mathbf{R}^m \setminus B(A_i, r)$ and apply Theorem 1.3 to two families of sequences $\{f(u^\nu)\chi_{\{u^\nu\in B(A_i,r)\setminus\{A_i\}\}}\}_{\nu\in\mathbf{N}}$ and $\{f(u^\nu)\chi_{\{u^\nu(x)=A_i\}}\}_{\nu\in\mathbf{N}}$ separately. In the case of the first sequence we compactify the ring P_i by adding the sphere S^{m-1} at $\{A_i\}$ and shrinking the sphere $S^{m-1}(A_i, r)$ to a single point. Namely, let $\phi_r(s) = -s + r : \mathbf{R} \to \mathbf{R}$, $r_i(\lambda) = |\lambda - A_i|$, $\Phi_i(\lambda) = \underline{\phi_r(r_i(\lambda))}\theta_i(\lambda) : P_i \to B(0, r) \subseteq \mathbf{R}^m$. Then we associate P_i to the triple $(\Phi_i, P(0, r), \overline{B(0, r)})$. As for the second sequence, the set $A = \{A_i\}$

is compact, so we associate it to the triple $(id, \{A_i\}, \{A_i\})$. Now it is easy to deduce assertions i), ii) and iii).

Let us prove the assertion iv). Take $f \in C_0(\mathbf{R}^m)$ such that $\mathrm{supp} f \subseteq B(A_i, r)$ and $f \equiv 1$ in some neighborhood of A_i. According to the just proved part i), we have

$$f(u^\nu) \rightharpoonup \int_{B(A_i,r)\setminus\{A_i\}} f(\lambda)\mu_x(d\lambda) + p_i(x) + q_i(x) \text{ in } L^1(\Omega, \mu).$$

On the other hand, using the classical Young's theorem (see Theorem 2.1), we observe that

$$f(u^\nu) \rightharpoonup \int_{B(A_i,r)} f(\lambda)\mu_x(d\lambda) \text{ in } L^1(\Omega, \mu).$$

Thus $p_i(x) + q_i(x) = \int_{\{A_i\}} f(\lambda)\mu_x(d\lambda) = \mu_x(\{A_i\})$. This ends the proof of theorem. $\square$

Remark 7.1 In the case when $m = 1$ we have $\int_{S^0} f^i(\lambda)\nu_x^i(d\lambda) = f^-(A_i)\nu_x^i(\{-1\}) + f^+(A_i)\nu_x^i(\{+1\})$ where $f^-(A_i)$ and $f^+(A_i)$ are left and right hand side limits of f at A_i.

We end this section with the following theorem. Its simple but technical proof is left to the reader. Obviously, it is possible to construct many examples of similar nature and generalize the presented ones in various directions.

Theorem 7.3 *Assume that* $\sup_\nu \|u^\nu\|_{L^\infty(\Omega,\mu)} < \infty$ *and* $M \subseteq \mathbf{R}^m$ *is a smooth and closed* k*–dimensional submanifold. Let* $(Y_1)_\lambda, \ldots, (Y_{m-k})_\lambda$ *be an orthonormal basis in the normal space to* M *at* $\lambda \in M$ *such that the mapping* $M \ni \lambda \mapsto ((Y_1)_\lambda, \ldots, (Y_{m-k})_\lambda)$ *is continuous. Define the following expression for* $f : \mathbf{R}^m \to \mathbf{R}$

$$\hat{f}(\lambda, \theta) := \lim_{t\to 0^+} f(\lambda + t\sum_{i=1}^{m-k} \theta_i (Y_i)_\lambda) \quad \text{where } \theta = (\theta_1, \ldots, \theta_{m-k}) \in S^{m-k-1}.$$

and let us consider the following class of functions

$$\mathcal{F} := \{f : \mathbf{R}^m \to \mathbf{R} : f \in C(\mathbf{R}^m \setminus M) \cap C(M) \text{ and for every } (\lambda, \theta) \in M \times S^{m-k-1}$$
$$\textit{the mapping } \hat{f}(\lambda, \theta) \textit{ is well defined and } \hat{f}(\lambda, \theta) \in C(M \times S^{m-k-1}), \lim_{\lambda\to\infty} f(\lambda) = 0\}.$$

Take $f \in \mathcal{F}$*. Then the following statements hold.*

i) *There exists a subsequence* $\{u^\nu\}_{\nu\in\mathbf{N}}$ *still denoted by the same expression, the family of probability measures* $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbf{R}^m, \mu)$*,* $\{\overline{\nu}_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, M, \mu)$*,* $\{\nu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, M \times S^{m-k-1}, \mu)$ *and* μ*–measurable functions* $p, q : \Omega \to [0, 1]$ *such that*

$$f(u^\nu) \rightharpoonup \int_{\mathbf{R}^m\setminus M} f(\lambda)\mu_x(d\lambda) + \int_M f(\lambda)\overline{\nu}_x(d\lambda)p(x) +$$
$$+ \int_{M\times S^{m-k-1}} \hat{f}(\lambda, \theta)\nu_x(d\lambda, d\theta)q(x)$$

in $L^1(\Omega, \mu)$*. Moreover,* $\{\mu_x\}_{x\in\Omega}$ *is the Young's measure generated by the sequence* $\{u^\nu\}_{\nu\in\mathbf{N}}$*, and* $p(x) + q(x) = \mu_x(\{M\})$ *for* μ*–almost all* $x \in \mathbf{R}^m$*.*

ii) *If* $k = m-1$ *there exist families of measures* $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega,\mathbf{R}^m,\mu)$, $\{\nu_x^1\}_{x\in\Omega}, \{\nu_x^2\}_{x\in\Omega}, \{\overline{\nu}_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, M,\mu)$, *and* μ*–measurable functions* $q_1, q_2, p : \Omega \to [0,1]$ *such that*

$$f(u^\nu) \rightharpoonup \int_{\mathbf{R}^m\setminus M} f(\lambda)\mu_x(d\lambda) + \int_M f(\lambda)\overline{\nu}_x(d\lambda)p(x) + \\ + \int_M f^+(\lambda)\nu_x^1(d\lambda)q_1(x) + + \int_M f^-(\lambda)\nu_x^2(d\lambda)q_2(x).$$

Moreover, we have $p(x)+q_1(x)+q_2(x) = \mu_x(M) \le 1$ *for* μ *almost all* x, *and* $f^+(\lambda) = \lim_{t\to 0, t>0} f(\lambda + tY_\lambda)$, $f^-(\lambda) = \lim_{t\to 0,t>0} f(\lambda - tY_\lambda)$ *where* Y_λ *is the orthonormal vector to* M *at* λ.

8 Links with Convergence Theorem

Our goal now obtain the following version of Convergence Theorem, holding for the single–valued function. This assumption will be relaxed later.

Theorem 8.1 *Suppose that Conditions A and B are satisfied with* $g \equiv 1$ *and additionally the sequence* $\{u^\nu\}$ *in Condition B converges* μ*–almost everywhere to some function* u. *Let* $f \in \mathcal{F}$ *and assume that the sequence* $\{f(u^\nu)\}$ *is convergent weakly in* $L^1(\Omega,\mu)$. *Then its limit equals* μ*–almost everywhere to*

$$\begin{aligned}\overline{f}(x) &= f(u(x))\chi_{u(x)\in\cup_{i=1}^k intA_i} + \sum_{i=1}^k p_i(x)\int_{\partial A_i\cap A_i} f(\lambda)\overline{\nu}_x^i(d\lambda) + \\ &+ \sum_{i=1}^k q_i(x)\int_{M_i^*} f_i(\lambda)\nu_x^i(d\lambda) \end{aligned} \tag{31}$$

where $\{\overline{\nu}_x^i\}_{x\in\Omega} \in \mathcal{P}(\Omega,\partial A_i\cap A_i,\mu)$ *and* $\{\nu_x^i\}_{x\in\Omega} \in \mathcal{P}(\Omega, M_i^*,\mu)$ *are the same as in Theorem 1.3,* $p_i(x), q_i(x) \in L^1(\Omega,\mu)$ *and are nonnegative, moreover,*

$$\sum_{i=1}^k p_i(x) + \sum_{i=1}^k q_i(x) = \chi_{u(x)\in\mathbf{R}^m\setminus\cup_{i=1}^k intA_i}, \text{ for } \mu - \text{almost all } x\in\Omega. \tag{32}$$

In particular the limit function $\overline{f}$ *satisfies* $\overline{f} \in conv(f^\#(u(x)))$ *for* μ*–almost all* x.

Remark 8.1 The representation (31) describes precisely the limit function $\overline{f}$ and quantities $p_i, q_i, \{\overline{\nu}_x^i\}_{x\in\Omega}$ and $\{\nu_x^i\}_{x\in\Omega}$ are independent on the choice of f. This will follow from the proof. Studying supports of measures $\overline{\nu}_x^i$ and ν_x^i with the help of Theorem 5.1 one can precisely describe the smallest subset $F(x)$ in the set of accumulation points $f^\#(u(x))$ such that $\overline{f}(x) \in conv(F(x))$.

Proof of Theorem 8.1. At first we note that the Young measure $\{\mu_x\}_{x\in\Omega}$ generated by $\{u^\nu\}$ is the Dirack delta concentrated at $u(x)$ for μ–almost all x. Then we apply Theorem 1.3 with $\overline{m}^i = p_i(x)\mu$ and $m^i = q_i(x)\mu + m^i_s$ in the Lebesgue's–Nikodym's decomposition of measures $\overline{m}^i$ and m^i with respect to μ to identify the limit of $\{f(u^\nu)\}$ in the space of measures. The fact that this limit equals $\overline{f}\mu$ follows from Theorem 6.1 and Remark 6.2. The formula (32) follows from substitution $f \equiv 1$ to (31), while the last statement is the direct consequence of (32). □

At the end of this section let us remark that the assumption that f is single–valued in the formulation of Theorem 8.1 can be relaxed. Namely, we have the following remark.

Remark 8.2 If we assume that f is the selection of some multifunction F and take $w^\nu(x) = f(u^\nu(x))$ we obtain the version of Convergence Theorem holding for the restricted class of multifunctions.

9 Some other applications

Let me briefly describe some possible applications of the nonclassical Young measures.

1) *PDE's with discontinuous datas.* It is well known that the classical Young measures are widely applied to nonlinear PDE's with continuous coefficients (see e.g. [17, 44]). On the other hand some discontinuous problems are treated with the help of Convergence Theorem in Set–Valued Analysis; most often one has to do with differential inclusions (see e. g. Chapter 10 in [4] and Chapter 2 in [3]). Then one must be sure that the sequence $\{u^\nu\}$ converges almost everywhere. Now one can relax this assumption and apply our Representation Theorem to problems with discontinuous datas or to differential inclusions. This can be done in the similar way as the classical Young measure theory is applied to nonlinear problems with continuous datas. But now we apply the nonclassical Young measures. Interesting applications to differential inclusions with discontinuous datas are object of the forthcoming paper of P. Gwiazda and A. Zatorska [26].

2). *Measure–valued solutions.* In many non–convex optimization problems and in nonlinear PDE's there does not exists any classical solution to the problem, but only the generalized solution the so–called measure–valued solution which involves Young measures (see e. g. [13, 44, 38, 58, 53] and their references). Now one can consider also discontinuous problems and apply the nonclassical Young measures to obtain the measure–valued solutions to such problems. This will be done in the work of P. Gwiazda [25] who applies the nonclassical Young measures to construct the measure–valued solutions to the Savage–Hutter model of the granular flow.

3). *Numerical approximation of Young measures.* Many nonlinear PDE's are treated by numerical methods. This methods require to discritize the problem to the sequence of related problems, each of them having the solution $\{u^\nu\}$ ($\nu \in \mathbf{N}$) in some finite–dimensional space. Then one applies theoretical methods ensuring that the sequence is convergent in some sense. To verify that the limit function u is the solution of the main problem one needs to compute Young measures generated by $\{u^\nu\}$, or by its subsequence. Now also equations with discontinuous constrains can be treated by similar techniques. We refer for example to [11, 40, 51, 52] and their references for related results involving classical Young measures. Let me mention that recently also M. Kružík, T. Roubíček in [39] have pointed out that there a the need to introduce Young measures controlling discontinuous functions and apply them in numerical methods in PDE's.

4). *Three modelling examples.*

a) *The Savage–Hutter model of the granular flow.* In the two–dimensional Savage–Hutter model of the granular flow (see [19, 32, 33]) one considers the system of equations

$$\begin{aligned} h_t \quad &+ \quad \mathrm{div}_x(hu) = 0, \\ (hu)_t \quad &+ \quad \mathrm{div}_x(hu^2) + \frac{1}{2}h^2 = hs(x,u), \end{aligned}$$

where all $u = u(t,x)$ is unknown function ($t \in \mathbf{R}_+, x \in \mathbf{R}^2$), $s = s(x,u)$ and $h = h(t,x)$ are given functions, and s is discontinuous with respect to the second variable. The discontinuity is similar to that of signum function: $\mathrm{sign}(u) = \frac{u}{\|u\|}$ where $u = (u_1, u_2)$.

b) *Non–Newtonian fluids.* Our next example comes from the theory of non–Newtonian fluids (see e. g. [60]). It is usefull in many research fields, for example in chemistry, glaciology, biology and geology. Non–Newtonian fluids are solutions (global in time) to systems describing the motion of both: incompressible liquids and compressible isothermal gases in a bounded domain $\Omega \subseteq \mathbf{R}^d$, $d \geq 2$. In the simplified version the system reads as (see (1.55) and (1.56) on page 12 and Chapter 5.2 in [44])

$$\begin{aligned} \mathrm{div}_x v \quad &= \quad 0, \\ \rho_0(v_i)_t \quad + \quad \rho_0 \sum_{j=1}^{d} v_j \frac{\partial v_i}{\partial x_j} \quad &= \quad -\frac{\partial \pi}{\partial x_i} + \mathrm{div}_x \tau_i^E(e) + \rho_0 f_i, \quad i = 1, \ldots, d, \end{aligned}$$

where $\tau_i^E = (\tau_{i1}^E, \ldots, \tau_{id}^E)$, is the stress tensor, $v = (v_1, \ldots, v_d)$, $v_i = v_i(t,x)$, $t \in \mathbf{R}_+, x \in \mathbf{R}^d$, $\rho_0 \in \mathbf{R}$, $e = e(v) = \nabla v + (\nabla v)^T$ is the symmetric part of the velocity gradient (with respect to x variable), and π is the so–called undetermined preasure. Usually one assumes that the tensor τ^E is the continuous function. Now one can relax this assumption and generalize many results of Chapter 5.2 in [44], such as construction of measure–valued solutions (and some other results contained in this book) to systems with discontinuous constraints.

c) *Hyperelasticity theory.* Let us consider the total energy functional of the elasticity theory

$$I(u) = \int_\Omega W(\nabla u(x))dx,$$

where $u : \Omega \to \mathbf{R}^3$, is the displacement, $\Omega \subseteq \mathbf{R}^3$ and $W : \mathbf{R}^3 \to \mathbf{R}_+$ is the stored energy function. In the hyperelasticity theory stationary point u of such functional (with the specified function W) and the given boundery condition $u = u_0$ on $\partial\Omega$ is the solution of the related Euler–Lagrange equilibrium equations, where by physical reason the adissible deformations u satisfy the constraint $\det\nabla u(x) > 0$ almost everywhere. One also has to assume that $W(F) \to +\infty$ as $\det F \to 0^+$, in particular the typical W is discontinuous, as one cannot compress the material with finite energy, see e. g. Chapter 4, pages 137–138 in [9].

5). *One more remark.* It may happen that the given partial differential equation does not contain any quantity of the form $f(u)$ where f is discontinuous, but it is still solvable with the help of the nonclassical Young measures. This happens for example in the paper of Crandall [10] who considers the Cauchy problem $u_t + \sum_{i=1}^n (\Phi_i(u))_{x_i} = 0$ where Φ_i are continuous and treats it via the theory of semigroups of nonlinear transformations. This treatment requires to consider the time–independent equation $u + \sum_{i=1}^n (\Phi_i(u))_{x_i} = h$ where $h \in L^1(\mathbf{R}^n)$. For the sake of this equation he introduces the linear operator acting on $L^1(\mathbf{R}^n)$, whose domain is defined with the help of discontinuous function $sig_0 : \mathbf{R} \to \mathbf{R}$, $\mathrm{sign}(r) = \frac{r}{|r|}$ if $r \neq 0$ and $\mathrm{sign}(0) = 0$ (see Definition 1.1 in [10]). Several results contained there are based on the version of Convergence Theorem, the cousin of our Representation Theorem.

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