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Shannon wavelet approximations
of
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Abstract

Recent works emphasized the interest of numerical solution of PDE's with wavelets. In their works [3, 4], A. Cohen, W. Dahmen and R. DeVore focussed on the non linear approximation aspect of the wavelet approximation of PDE's to prove the relevance of such methods. In order to extend these results, we focuss on the convergence of the iterative algorithm, and we consider different possibilities offered by the wavelet theory: the tensorial wavelets and the derivation/integration of wavelet bases. We also investigate the use of wavelet packets. We apply these extended results to prove in the case of the Shannon wavelets, the convergence of the algorithm introduced in [8]. This algorithm carries out the Leray projector with divergence-free wavelets.

1   Introduction

Since the end of the 80's, the mathematical *Theory of Wavelets* has invented new tools for numerical simulations. Thanks to the Fast Wavelet Transform, wavelets provide efficient algorithms including optimal preconditionners for elliptic operators [12]. Recently, Cohen-Dahmen-De Vore's articles [3, 4] demonstrated the optimal complexity of wavelet algorithms for the solution of elliptic problems.

These works enhanced the interest for these methods applied to the solution of partial differential equations. Wavelet approach also resulted in non-linear approximations [5], and in denoising methods [10]. Wavelet methods offer the possibility to regulate both the accuracy in space and the accuracy in frequency.

In the following, we apply the Shannon decomposition to differential operators in order to investigate the convergence of wavelet algorithms to solve partial differential equations. In the first part, we recall some operator theory basic elements; we indicate how the Shannon wavelets and wavelet packets can be used to part the support of the Fourier transform of a function.

In the following part, we resume the works [3, 4] on the wavelet approximation of differential operators and state the correlated theorem of convergence for constant coefficient operators. In the frame of Shannon wavelets, we show that this convergence depends on the chosen wavelet decomposition (MRA or tensorial). We extend this study to Shannon

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wavelet packets. After that, we introduce a result on the derivation of biorthogonal wavelets due to P. G. Lemarié-Rieusset [14] that serves to construct new wavelet approximations of differential operators.

In the last two parts, we present explicit examples that are implicated in the numerical solution of the Navier-Stokes equations: the solution of the implicit Laplacian $(Id - \alpha\Delta)^{-1}$ and the Leray projector $\mathbb{P}$.

2 Symbol of an operator

The convergence of the wavelet methods involves the partition of the spectra of the operator. This partition is provided by the wavelet decomposition. Hence we need the notion of symbol as introduced by Lars Hörmander in his book *The Analysis of Partial Differencial Operators* [11].

We denote by ∂_j the derivation along the variable x_j, and by D_j the derivation $-i\partial_j$. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we write $D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}$. Let u denote a Shwartz function of d real variables (i.e. u is C^∞ and fast decreasing: $\forall N \in \mathbb{N}$, $\exists A > 0 \ / \ \forall x \in \mathbb{R}^d$, $|u(x)| < A/(1 + |x|^2)^{N/2}$). We denote by $< \cdot, \cdot >$ the scalar product either on vectors either in dual spaces.

Then $\widehat{u}$ stands for the Fourier transform of u, i.e.

$$\widehat{u}(\xi) = \int_{\xi \in \mathbb{R}^d} e^{-i<x,\xi>} u(x) dx$$

We also denote by $\mathcal{F}$ the isomorphism of $L^2(\mathbb{R}^d)$ given by $\mathcal{F} : u \mapsto (2\pi)^{-d/2} \widehat{u}$.

The inverse Fourier transform is done by:

$$u(x) = (2\pi)^{-d/2} \int_{\xi \in \mathbb{R}^d} e^{i<x,\xi>} \mathcal{F}u(\xi) d\xi \tag{2.1}$$

When we derivate the relation (2.1), it yields:

$$D^\alpha u(x) = (2\pi)^{-d/2} \int_{\xi \in \mathbb{R}^d} e^{i<x,\xi>} \xi^\alpha \mathcal{F}u(\xi) d\xi$$

Thus derivating u by D^α consists in multiplying the Fourier transform of u by $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_d^{\alpha_d}$. The function $\xi \mapsto \xi^\alpha$ is called the symbol of D^α. More generally speaking, if $a(\xi)$ is a C^∞ function slowly increasing (i.e. such that $\exists N \in \mathbb{N}$, $A > 0 \ / \ \forall \xi \in \mathbb{R}^d$, $|a(\xi)| < A(1 + |\xi|^2)^{N/2}$), $a(D)$ defines an operator of symbol $a(\xi)$ acting on the class of the Schwartz functions $\mathcal{S}$ by

$$a(D)u(x) = (2\pi)^{-d/2} \int_{\xi \in \mathbb{R}^d} e^{i<x,\xi>} a(\xi) \mathcal{F}u(\xi) d\xi \tag{2.2}$$

Let's now consider a differential operator P of order m with variable coefficients a_α in $\mathcal{S}$, $P = \sum_{|\alpha|=0}^m a_\alpha(x) D^\alpha$. Then, instead of using the formula:

$$Pu = (2\pi)^{-d/2} \int_{\xi \in \mathbb{R}^d} e^{i<x,\xi>} \mathcal{F}(Pu)(\xi) d\xi$$

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where

$$\mathcal{F}(Pu)(\xi) = (2\pi)^{d/2} \sum_{|\alpha|=0}^{m} \mathcal{F}a_\alpha * \xi^\alpha \mathcal{F}u$$

that is no more a multiplication but an integral operator on $\mathcal{F}u$, we use the formula (for $x \in \mathbb{R}^d$):

$$Pu(x) = (2\pi)^{-d/2} \int_{\xi \in \mathbb{R}^d} e^{i<x,\xi>} \left(\sum_{|\alpha|=0}^{m} a_\alpha(x)\xi^\alpha \right) \mathcal{F}(u)(\xi)d\xi$$

that we write:

$$Pu(x) = (2\pi)^{-d/2} \int_{\xi \in \mathbb{R}^d} e^{i<x,\xi>} p(x,\xi)\mathcal{F}(u)(\xi)d\xi \tag{2.3}$$

introducing the "symbol" $p(x, \xi)$ of P

$$p(x,\xi) = \sum_{|\alpha|=0}^{m} a_\alpha(x)\xi^\alpha$$

The formula (2.3) gives us the possibility to define the operators $p(x, D)$ of symbol $p(x, \xi)$ that are not polynomials in ξ. These operators are called pseudo-differential. The functions p must verify regularity and increase properties of polynomial type (see [11]).

We'll remark that the function $\mathcal{F}(Pu)$ is no more the function $p(x,\xi)\mathcal{F}u(\xi)$ which appears in (2.3) since the latter depends on x.

The following definition of an elliptic operator is given in [11]:

Definition 2.1 (elliptic operators) *From the symbol $p(x,\xi) = \sum_{|\alpha|=0}^{m} a_\alpha(x)\xi^\alpha$ we extract the principal symbol $p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$. A differential operator P is said to be elliptic iff*

$$\forall x \in \mathbb{R}^d, \quad \forall \xi \in \mathbb{R}^d \setminus \{(0,\dots,0)\}, \quad p_m(x,\xi) \neq 0$$

In the following, we'll need differential operators applied to vector functions $\mathbb{R}^d \to \mathbb{R}^m$. We denote by $\mathbf{u}$ with bold caracter the (multi-variable) vector function u of real variables when it has several components. For $\mathbf{u}$ having several components, the symbol $p(x,\xi)$ is a matrix: $\forall x, \xi \in \mathbb{R}^d$, $M(x,\xi) \in \mathbb{C}^{n \times m}$. Let $A = (A_{ij})_{1 \leq i \leq n, \ 1 \leq j \leq m}$ be a differential operator $A : \left(H^s(\mathbb{R}^d) \right)^m \to \left(H^r(\mathbb{R}^d) \right)^n$, with:

$$A_{ij} = \sum_\alpha a_{ij,\alpha}(x)D^\alpha$$

Its symbol is $M = (m_{ij})_{1 \leq i \leq n, \ 1 \leq j \leq m}$ with

$$m_{ij}(x,\xi) = \sum_\alpha a_{ij,\alpha}(x)\xi^\alpha$$

We apply the operator A componentwise as follows:

$$(A\mathbf{u})_i = \sum_{j=1}^{m} m_{ij}(x,D)u_j$$

Therefore, the multidimensional symbol can be handled in much the same way as in 1-D.

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Remark 2.1 *As the operator is applied to real functional spaces, its symbol verifies the same relation as the Fourier transform of real functions, that is:*

$$\forall i,j \qquad m_{ij}(x,-\xi) = m_{ij}(x,\xi)^*$$

where z^* denotes the complex conjugate of z.

3 Shannon wavelet decomposition

A good reference for the definition of the Shannon wavelets is Mallat's academic book [15]. We first briefly recall the construction of these wavelets.

The **biorthogonal wavelets** are based on **scale filters** m **and** n that provide the low-pass filter and the high-pass filter. As the scale function $\varphi(\frac{\cdot}{2})$ and the wavelet $\psi(\frac{\cdot}{2})$ belong to $V_0 = span\{\varphi(\cdot - k),\ k \in \mathbb{Z}\}$, there exist two sequences (a_k) and (b_k) such that:

$$\varphi(\frac{x}{2}) = \sum_{k \in \mathbb{Z}} a_k \varphi(x-k)\ , \qquad \psi(\frac{x}{2}) = \sum_{k \in \mathbb{Z}} b_k \varphi(x-k)$$

We thus get after a Fourier Transform:

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi)\ , \qquad \hat{\psi}(2\xi) = n(\xi)\hat{\varphi}(\xi)$$

with $m(\xi) = \frac{1}{2}\sum_{k \in \mathbb{Z}} a_k e^{-ik.\xi}$, $n(\xi) = \frac{1}{2}\sum_{k \in \mathbb{Z}} b_k e^{-ik.\xi}$
The scale function is inferred from the filter by :

$$\hat{\varphi}(\xi) = \hat{\varphi}(0) \prod_{j=1}^{\infty} m(\frac{\xi}{2^j})$$

A wavelet basis $\{\psi_{jk}\}_{j,k \in \mathbb{Z}}$ with $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$ forms a Riesz basis of $L^2(\mathbb{R})$. Similarly, if we denote by H^t the set of distribution functions f such that $(1+|\xi|^2)^{t/2}\hat{f} \in L^2$, $\{2^{tj}\psi_{jk}\}_{j,k \in \mathbb{Z}}$ provides a Riesz basis of the Hilbert space $H^t(\mathbb{R})$ for $-t_1 \leq t \leq t_2$. Where, if $(\tilde{\phi}, \tilde{\psi})$ is the dual wavelet basis associated to (ϕ, ψ), $t_1 \in \mathbb{R}$ is the maximal number such that $\tilde{\psi} \in H^{t_1}(\mathbb{R})$ and $t_2 \in \mathbb{R}$ the maximal number such that $\psi \in H^{t_2}(\mathbb{R})$.
We denote by ℓ_t^2 the space of sequences $(u_{jk})_{(j,k) \in \mathbb{Z}^2}$ with the norm $\|(u_{jk})_{(j,k) \in (\mathbb{Z}^2)}\|_{\ell_t^2}^2 = \sum_{(j,k) \in (Z^2)} 2^{2tj}|u_{jk}|^2$. One caracteristic that plays an important role in operator approximation is the semi-orthogonality coefficients $B_t, b_t > 0$ such that $\forall (u_{jk})_{j,k \in \mathbb{Z}} \in \ell_t^2$,

$$b_t \sum_{j \in \mathbb{Z}} \|\sum_{k \in \mathbb{Z}} u_{jk}\psi_{jk}\|_{H^t} \leq \|\sum_{j,k \in \mathbb{Z}} u_{jk}\psi_{jk}\|_{H^t} \leq B_t \sum_{j \in \mathbb{Z}} \|\sum_{k \in \mathbb{Z}} u_{jk}\psi_{jk}\|_{H^t}$$

Shannon wavelets have this particularity to have perfect low-pass and high-pass filters :

$$m(\xi) = \begin{cases} 1 & \text{if} \quad \xi \in [-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi], \quad k \in \mathbb{Z} \\ 0 & \text{if} \quad \xi \in [\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi], \quad k \in \mathbb{Z} \end{cases}$$

$$n(\xi) = \begin{cases} -e^{-i\xi} & \text{if} \quad \xi \in [\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi], \quad k \in \mathbb{Z} \\ 0 & \text{if} \quad \xi \in [-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi], \quad k \in \mathbb{Z} \end{cases}$$

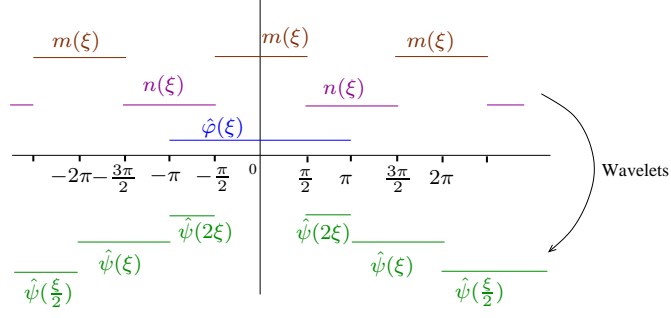Then the corresponding scaling function writes:

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Figure 1: Representation of the compact support of the Shannon filters and of the Fourier transform of the wavelets. The vertical axis has no particular meaning, but the bars represent the compact support of the functions.

$$\hat{\varphi}(\xi) = \chi_{[-\pi,\pi]}(\xi) \qquad , \qquad \varphi(x) = \frac{\sin \pi x}{\pi x}$$

and the wavelet:

$$\hat{\psi}(\xi) = e^{-i\xi/2} \, \chi_{[-2\pi,-\pi] \cup [\pi,2\pi]}(\xi) \qquad , \qquad \psi(x) = \frac{\sin 2\pi(x-1/2)}{2\pi(x-1/2)} - \frac{\sin \pi(x-1/2)}{\pi(x-1/2)}$$

where χ stands for the characteristic function i.e. $\chi_E(x) = 1$ if $x \in E, 0$ if $x \notin E$.

In the multidimensional case, the **tensorial Shannon decomposition** can be written as follows:

Let $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^m$. The Shannon decomposition of $\mathbf{u}$ is given by:

$$\mathbf{u} = \sum_{\mathbf{j} \in \mathbb{Z}^d} \mathbf{u_j} \tag{3.1}$$

with

$$\text{supp } \widehat{\mathbf{u_j}} \subset \prod_{i=1}^{d} [-2^{j_i+1}\pi, -2^{j_i}\pi] \cup [2^{j_i}\pi, 2^{j_i+1}\pi]$$

For each scale parameter $\mathbf{j} \in \mathbb{Z}^d$, and for each component $\ell = 1 \ldots m$, we have:

$$u_{\ell,\mathbf{j}}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} 2^{\mathbf{j}/2} u_{\ell \mathbf{jk}} \psi_1^\ell(2^{j_1}x_1 - k_1) \ldots \psi_i^\ell(2^{j_i}x_i - k_i) \ldots \psi_d^\ell(2^{j_d}x_d - k_d)$$

where $|\mathbf{j}| = \sum_i j_i$ and ψ_i^ℓ are wavelets of Shannon type, i.e. $\text{supp } \widehat{\psi_i^\ell} \subset [-2\pi, -\pi] \cup [\pi, 2\pi]$.

3.1 Shannon wavelet packets

With the above filters $m(\xi)$ and $n(\xi)$, we can define the Shannon wavelet packets. The wavelet packets are defined by applying the filters m and n to the wavelets. Hence we obtain two new wavelets $\psi_{(11)}$ and $\psi_{(10)}$ that are twice better localised in the Fourier space (i.e. the compact supports of their Fourier transforms are twice smaller):

$$\widehat{\psi_{(10)}}(2\xi) = m(\xi)\hat{\psi}(\xi) \tag{3.2}$$

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$$\widehat{\psi_{(11)}}(2\xi) = n(\xi)\hat{\psi}(\xi) \tag{3.3}$$

The two of them are necessary to expend $L^2(\mathbb{R})$, i.e. $L^2(\mathbb{R}) = span\{\psi_{(10)}(2^j x - 2k), \psi_{(11)}(2^j x - 2k)\}_{j,k\in\mathbb{Z}}$. More precisely, the wavelet space at level j, W_j, admits $\{\psi_{(10))}(2^j x - 2k), \psi_{(11)}(2^j x - 2k)\}_{k\in\mathbb{Z}}$ as a Riesz basis.
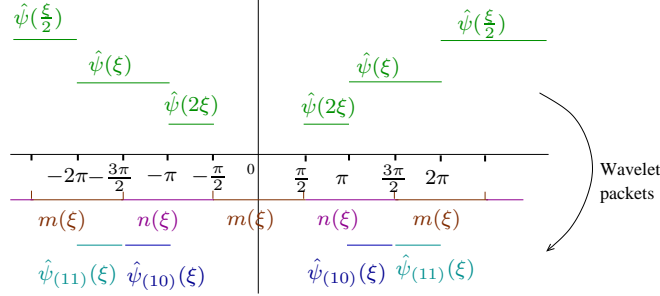

Figure 2: The construction of Shannon wavelet packets represented in the Fourier space

The operations (3.2) and (3.3) on the wavelets can be iterated as many times as one wants, and the Fourier support can be shrunk as desired. In practice this operation can also be applied to usual wavelets but doesn't come out with good results. Getting a better frequency localisation for usual wavelet packets is still a challenging problem.

The Shannon wavelet packet decomposition gives us more oportunities to approximate any operators than the classical Shannon decomposition (3.1). We can report to part 8 for their actual use.

4   Solution of elliptic PDE's with wavelets – the Richardson iteration

In their paper [3], A. Cohen, W. Dahmen and R. DeVore consider a simple method to solve elliptic operator equations with wavelets. Let first consider $m = n = 1$. In order to find the solution of the differential equation:

$$Au = v \tag{4.1}$$

where A is a linear differential operator and u the unknown function, they use the expansions of u and v in wavelet bases. We denote by $\underline{u} = (u_{jk})_{jk}$ the vector of wavelet coefficients: $u_{jk} = < u, \tilde{\psi}_{jk} >$ with $\{\psi_{jk}\}_{j,k}$ and $\{\tilde{\psi}_{jk}\}_{j,k}$ two dual wavelet bases. Then the expansion of u writes $u = \sum_{j,k} u_{jk}\psi_{jk}$.

Let $\underline{A}$ be the variational discretisation of A expressed in the wavelet basis $\{\psi_{jk}\}_{(j,k)}$ (it is called the Petrov-Galerkin stiffness matrix) $\underline{A} = (< A\psi_{jk}, \tilde{\psi}_{j'k'} >)_{j,j',k,k'}$ and $\underline{D}$ a wavelet preconditionner associated to the wavelet expansions (usually, it is the diagonal of $\underline{A}$, that has the form $\mathrm{Diag}(2^{tj})$). We assume that A is continuous from $H^{t/2}$ to $H^{-t/2}$ and $a = < A\cdot,\cdot >$ is coercive.

In order to diversify the considered wavelet transforms in dimension $d > 1$, let J denote the scale indice set (J countable, e.g. $\mathbb{Z} \times \{0,1\}^{d*}$ for the MRA where $*$ means deprivated

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of the element $(0, \ldots, 0)$, and $J = \mathbb{Z}^d$ for tensorial wavelets). For a function $\alpha : J \to \mathbb{R}$ we define the ℓ^2_α norm on the wavelet coefficients $\underline{u} = (u_{\mathbf{jk}})_{\mathbf{j} \in J, \mathbf{k} \in \mathbb{Z}^d}$ by

$$\|(u_{\mathbf{jk}})_{\mathbf{j} \in J, \mathbf{k} \in \mathbb{Z}^d}\|_{\ell^2_\alpha} = \sum_{\mathbf{j} \in J, \mathbf{k} \in \mathbb{Z}^d} 2^{\alpha(\mathbf{j})} |u_{\mathbf{jk}}|^2$$

Usually, the ℓ^2_t corresponds to either the case $\alpha(\mathbf{j}) = t j_1$ in the MRA case $\mathbf{j} = (j_1, \varepsilon) \in \mathbb{Z} \times \{0, 1\}^{d*}$, either $\alpha(\mathbf{j}) = t \max(j_1, \ldots, j_d)$ for the tensorial wavelets.

Then $\underline{D}^{-1}$ is continuous from $\ell^2_{-t/2}$ to $\ell^2_{t/2}$ (see [12]). We write the sequence $(\underline{u}_n)$, thanks to a Richardson iteration associated with a multiscale preconditionnning, starting from $\underline{u}_0 = 0$:

$$\underline{u}_n = \underline{u}_{n-1} + \underline{D}^{-1}(\underline{v} - \underline{A}\,\underline{u}_{n-1}) \tag{4.2}$$

Then this method is said to converge if

$$\exists \rho < 1, \quad \|\underline{v} - \underline{A}\,\underline{u}_n\|_{\ell^2_{-t/2}} \le \rho \|\underline{v} - \underline{A}u_{n-1}\|_{\ell^2_{-t/2}}$$

As we have:

$$\underline{v} - \underline{A}\,\underline{u}_n = (\underline{Id} - \underline{A}\,\underline{D}^{-1})(\underline{v} - \underline{A}\,\underline{u}_{n-1})$$

the algorithm converges if $\rho = \|\underline{Id} - \underline{A}\,\underline{D}^{-1}\| < 1$, in the operator norm. And

$$\underline{u}_n \to_{n \to \infty} \underline{u} \quad \text{with} \quad \underline{A}\,\underline{u} = \underline{v}$$

That is, as $u = \sum_{jk} u_{jk} \psi_{jk} \in H^{t/2}$,

$$Au = v$$

From now on we think of A as being an operator with constant coefficients. Let m, n be unspecified natural numbers. Hence we switch to vector spaces. If we denote by $M(\xi)$ the symbol associated to A, we can express A after a Fourier transform of the equation (4.1) as

$$M(\xi)\widehat{\mathbf{u}} = \widehat{\mathbf{v}} \qquad \text{and the pseudo-inverse solution} \qquad \widehat{\mathbf{u}} = M(\xi)^\dagger \widehat{\mathbf{v}}$$

with $\mathbf{u} \in (H^{t/2}(\mathbb{R}^d))^m$, $\mathbf{v} \in (H^{-t/2}(\mathbb{R}^d))^n$, $M(\xi) \in \mathbb{C}^{n \times m}$ and $M(\xi)^\dagger$ the pseudo-inverse of $M(\xi)$.
Remark that if $m = n$ and $M(\xi)M(\xi)^\dagger = Id$, $M(\xi)^\dagger = M(\xi)^{-1}$.

The idea for solving $A\mathbf{u} = \mathbf{v}$ is the following: we decompose $\mathbf{v}$ in a wavelet basis that splits the support of $\widehat{\mathbf{v}}$

$$\mathbf{v} = \sum_{\mathbf{j} \in J} \mathbf{v_j}$$

If we denote by $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ the canonical basis of $\mathbb{R}^n$ then $\mathbf{v_j}$ is the projection of $\mathbf{v}$ in the wavelet level $W_{\mathbf{j}} = span(\{\mathbf{\Psi}_{1,\mathbf{jk}}\}_{\mathbf{k} \in \mathbb{Z}^d}, \ldots, \{\mathbf{\Psi}_{n,\mathbf{jk}}\}_{\mathbf{k} \in \mathbb{Z}^d})$ with each component $v_\ell \mathbf{e}_\ell$ of $\mathbf{v}$ decomposed in the wavelet basis $\{\mathbf{\Psi}_{\ell,\mathbf{jk}}\}_{\mathbf{j} \in J, \mathbf{k} \in \mathbb{Z}^d}$ (further we'll need this generalisation

of [3] which uses an MRA).

For example in the tensorial case, we have $J = \mathbb{Z}^d$ and:

$$v_{\ell\mathbf{j}} = \sum_{\mathbf{k} \in \mathbb{Z}^d} v_{\ell\mathbf{j}\mathbf{k}} \psi_{j_1 k_1}(x_1) \ldots \psi_{j_d k_d}(x_d)$$

This modification of the MRA case will prove usefull in part 10.

Let us assume that for each $\mathbf{j} \in J$, $\widehat{\mathbf{v}}_{\mathbf{j}}$ and $\widehat{\mathbf{u}}_{\mathbf{j}}$ are compactly supported (wavelet decompositions give us the opportunity to do this with the desired accuracy). For each $\mathbf{j} \in J$, we build a matrix $M_{\omega_{\mathbf{j}}}$ ($M_{\omega_{\mathbf{j}}} \in \mathbb{R}^{n \times m}$ depending on the compact support of $\widehat{\mathbf{v}}_{\mathbf{j}}$) such that:

$$M_{\omega_{\mathbf{j}}} \approx M(\xi) \qquad \text{for } \xi \in \text{supp}(\widehat{\mathbf{v}}_{\mathbf{j}}) \tag{4.3}$$

Then we approximate the relation $M(\xi)\widehat{\mathbf{u}} = \widehat{\mathbf{v}}$ by:

$$\forall \mathbf{j} \in J, \ \forall \mathbf{k} \in \mathbb{Z}^d, \ M_{\omega_{\mathbf{j}}} \begin{bmatrix} u_{1\mathbf{j}\mathbf{k}} \\ \vdots \\ u_{n\mathbf{j}\mathbf{k}} \end{bmatrix} = \begin{bmatrix} v_{1\mathbf{j}\mathbf{k}} \\ \vdots \\ v_{m\mathbf{j}\mathbf{k}} \end{bmatrix} \tag{4.4}$$

In the view of Richardson iteration, we take as a preconditionnner $D = \sum_{\mathbf{j} \in J} M_{\omega_{\mathbf{j}}} P_{\mathbf{j}}$, where $P_{\mathbf{j}}$ is the projector on the wavelet level $W_{\mathbf{j}}$. Then the corresponding discrete preconditionner $\underline{D}$ which applies to wavelet coefficients is $\underline{D} = \sum_{\mathbf{j} \in J} M_{\omega_{\mathbf{j}}} \underline{P}_{\mathbf{j}}$ where $\underline{P}_{\mathbf{j}}$ is a diagonal matrix with ones on the lines $(\mathbf{k}, \mathbf{j})_{\mathbf{k} \in \mathbb{Z}^d}$ and zeros everywhere else. In the case of tensorial wavelet basis ($J = \mathbb{Z}^d$), the space $W_{\mathbf{j}}$ is the L^2 closure of the space generated by the family

$$\{\mathbf{\Psi}_{\ell,\mathbf{j}\mathbf{k}}\}_{1 \leq \ell \leq m, \, \mathbf{k} \in \mathbb{Z}^d} = \{(\psi_{j_1 k_1} \times \cdots \times \psi_{j_d k_d}, \underbrace{0, \ldots, 0}_{m-1})\}_{\mathbf{k} \in \mathbb{Z}^d} \cup \cdots \cup \{(\underbrace{0, \ldots, 0}_{m-1}, \psi_{j_1 k_1} \times \cdots \times \psi_{j_d k_d})\}_{\mathbf{k} \in \mathbb{Z}^d}$$

In the following, we use the notation $M_\omega = \sum_{\mathbf{j}} M_{\omega_{\mathbf{j}}} P_{\mathbf{j}}$.

If we write the sequence (4.2) with $\underline{\mathbf{v}}_n = \underline{\mathbf{v}} - \underline{A}\,\underline{\mathbf{u}}_n$, it comes:

$$\underline{\mathbf{u}}_0 = 0, \quad \underline{\mathbf{v}}_0 = \underline{\mathbf{v}}, \qquad \underline{\mathbf{u}}_{n+1} = \underline{\mathbf{u}}_n + M_\omega^\dagger \underline{\mathbf{v}}_n \quad \text{and} \quad \underline{\mathbf{v}}_{n+1} = \underline{\mathbf{v}}_n - \underline{A}(\underline{\mathbf{u}}_{n+1} - \underline{\mathbf{u}}_n) \tag{4.5}$$

Theorem 4.1 *Let $M(\xi)$ be the symbol matrix associated to $A : H^{t/2} \to H^{-t/2}$ continuous. If the wavelet basis $\{\mathbf{\Psi}_{\ell,\mathbf{j}\mathbf{k}}\}_{1 \leq \ell \leq m, \mathbf{j} \in J, \mathbf{k} \in \mathbb{Z}^d}$ provides a Riesz basis of $H^{\pm t/2}$ (i.e. the associated decompositions $\mathbf{v} \mapsto \underline{\mathbf{v}}$, $H^{\pm t/2} \to \ell^2_{\pm t/2}$, and reconstructions $\underline{\mathbf{v}} \mapsto \mathbf{v}$, $\ell^2_{\pm t/2} \to H^{\pm t/2}$ are continuous).*

Moreover, we suppose we have constructed for all $\mathbf{j} \in J$ matrices $M_{\omega_{\mathbf{j}}} \in \mathbb{R}^{n \times m}$ such that $M_\omega^\dagger = \sum_{\mathbf{j}} M_{\omega_{\mathbf{j}}}^\dagger P_{\mathbf{j}} : H^{-t/2} \to H^{t/2}$ is continuous. We also assume that the wavelet decomposition $\mathbf{v} \mapsto (\mathbf{v}_{\mathbf{j}})_{\mathbf{j} \in J}$ satisfies:

$$\exists \tilde{B} > 0 \quad \text{such that} \ \forall \mathbf{v} \in H^{-t/2}, \qquad \|(Id - A\,M_{\omega_{\mathbf{j}}}^\dagger)\mathbf{v}\|^2_{H^{-t/2}} \leq \tilde{B} \sum_{\mathbf{j} \in J} \|(Id - A\,M_{\omega_{\mathbf{j}}}^\dagger)\mathbf{v}_{\mathbf{j}}\|^2_{H^{-t/2}}$$

If there exist a real number $\rho \geq 0$ such that:

$$\forall \mathbf{j} \in J, \ \||(Id - A\,M_{\omega_{\mathbf{j}}}^\dagger)_{|W_{\mathbf{j}}}\|| \leq \rho$$

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i.e.

$$\forall \mathbf{j} \in J, \quad \forall \mathbf{v_j} \in W_{\mathbf{j}}, \quad \|(Id - A\, M_{\omega_{\mathbf{j}}}^{\dagger})\mathbf{v_j}\|_{H^{-t/2}} \leq \rho \|\mathbf{v_j}\|_{H^{-t/2}}$$

then for ρ small enough, the sequence $(\underline{\mathbf{u}}_n)_{n \in \mathbb{N}}$ defined by (4.5) converges in $\ell^2_{t/2}$ to the wavelet expansion $\underline{\mathbf{u}}$ of $\mathbf{u}$ such that:

$$\underline{A}\,\underline{\mathbf{u}} = \underline{\mathbf{v}} \quad \text{and we have} \quad A\mathbf{u} = \mathbf{v}$$

proof:
The graph of continuous operators can be summarized as follows:

$$
\begin{array}{ccc}
& \text{wavelet transform} & \\
\mathbf{u} \in H^{t/2} & \longleftrightarrow & \underline{\mathbf{u}} \in \ell^2_{t/2} \\
& & \\
A \downarrow \quad \uparrow M_\omega^{\dagger} & & \underline{M}_\omega^{\dagger} \uparrow \quad \downarrow \underline{A} \\
& & \\
\mathbf{v} = A\,\mathbf{u} \in H^{-t/2} & \longleftrightarrow & \underline{\mathbf{v}} = \underline{A}\,\underline{\mathbf{u}} \in \ell^2_{-t/2} \\
& \text{wavelet transform} &
\end{array}
$$

The operator $M_\omega^{\dagger}$ is not the inverse of A but its approximation.

As the wavelet decompositions are continuous,

$$\exists b, B > 0 \quad \text{such that} \quad \forall \mathbf{v} \in H^{-t/2}, \quad b \sum_{\mathbf{j} \in J} \|\mathbf{v_j}\|^2_{H^{-t/2}} \leq \|\mathbf{v}\|^2_{H^{-t/2}} \leq B \sum_{\mathbf{j} \in J} \|\mathbf{v_j}\|^2_{H^{-t/2}}$$

When $b = B = 1$, the wavelet basis is said to be semi-orthogonal.
Then we have:

$$\|\mathbf{v}_{n+1}\|^2_{H^{-t/2}} \leq \tilde{B} \sum_{\mathbf{j} \in J} \|(Id - A\, M_\omega^{\dagger})\mathbf{v}_{n\mathbf{j}}\|^2_{H^{-t/2}} \leq \tilde{B} \sum_{\mathbf{j} \in J} \rho^2 \|\mathbf{v}_{n\mathbf{j}}\|^2_{H^{-t/2}} \leq \rho^2 \frac{\tilde{B}}{b} \|\mathbf{v}_n\|^2_{H^{-t/2}}$$

If $\rho^2 \tilde{B}/b < 1$, as $M_\omega^{\dagger}$ is continuous, the serie $\sum_n M_\omega^{\dagger} \mathbf{v}_n$ converges in the Banach space $H^{t/2}$ to a solution $\mathbf{u}$ of the equation $A\mathbf{u} = \mathbf{v}$.

The ideal wavelets that provide a minimal compact support for the Fourier transform are the Shannon wavelets. In this case, as the compact supports of the Fourier transforms of wavelets from different levels are disjoint, we have $\tilde{B} = b = 1$ for all A. Shannon wavelets have an infinite support and are not used in practice. But, in first approximation, all wavelets behave as Shannon wavelets with more or less accuracy.

Remark 4.1 *In the case of Shannon wavelets, as*

$$\widehat{\mathbf{v}}_{|\cup_\ell \text{supp}\,(\widehat{\mathbf{\Psi}_{\ell \mathbf{jk}}})} = \sum_{\ell, \mathbf{k} \in \mathbb{Z}^d} v_{\ell \mathbf{jk}} \widehat{\mathbf{\Psi}}_{\ell \mathbf{jk}}$$

the equation 4.4 is equivalent to

$$\forall \mathbf{j} \in J, \quad M_{\omega_{\mathbf{j}}} \widehat{\mathbf{u}}(\xi) = \widehat{\mathbf{v}}(\xi) \quad \text{for} \quad \xi \in \cup_\ell \text{supp}\,(\widehat{\mathbf{\Psi}_{\ell \mathbf{jk}}})$$

In the futur, that will allow us to express this relation using the components $\mathbf{v_j}$ of the Shannon decomposition of $\mathbf{v}$.
One can also remark that as M_ω doesn't depend on ξ, we can apply the operator $M_\omega^{\dagger}$ in the physical space (expressed with wavelets) and not in the Fourier space.

5 Multiresolution analysis (MRA) versus tensorial basis

There are two main different kinds of wavelet decompositions for a function on $\mathbb{R}^d$ with $d \geq 2$. It can be decomposed either in a multidimensional multiresolution analysis or in a tensorial basis. In an MRA, the wavelet decomposition of a function u in 2D writes:

$$
u(x_1, x_2) = \sum_{j \in \mathbb{Z}} \left(\sum_{(k_1, k_2) \in \mathbb{Z}^2} u^{(1,0)}_{j,k_1,k_2} \, \psi_{0\,j,k_1}(x_1) \, \varphi_{1\,j,k_2}(x_2) + \sum_{(k_1, k_2) \in \mathbb{Z}^2} u^{(0,1)}_{j,k_1,k_2} \, \varphi_{0\,j,k_1}(x_1) \, \psi_{1\,j,k_2}(x_2) \right.
$$
$$
\left. + \sum_{(k_1, k_2) \in \mathbb{Z}^2} u^{(1,1)}_{j,k_1,k_2} \, \psi_{0\,j,k_1}(x_1) \, \psi_{1\,j,k_2}(x_2) \right)
$$

where we used the notation $\psi_{j,k}(x) = \psi(2^j x - k)$.
While, in a tensorial basis it writes:

$$
u(x_1, x_2) = \sum_{j_1 \in \mathbb{Z}} \sum_{j_2 \in \mathbb{Z}} \sum_{(k_1, k_2) \in \mathbb{Z}^2} u_{j_1, j_2, k_1, k_2} \, \psi_0(2^{j_1} x_1 - k_1) \, \psi_1(2^{j_2} x_2 - k_2)
$$

These two decompositions correspond to two different partitions of the Fourier space (i.e. frequency domain). Both of them are represented in figures 3 and 4. On each figure, in the last square, which corresponds to the wavelet transform, the low frequencies are localised in the upper left corner of the square of coefficients, and the high frequencies in the bottom right.
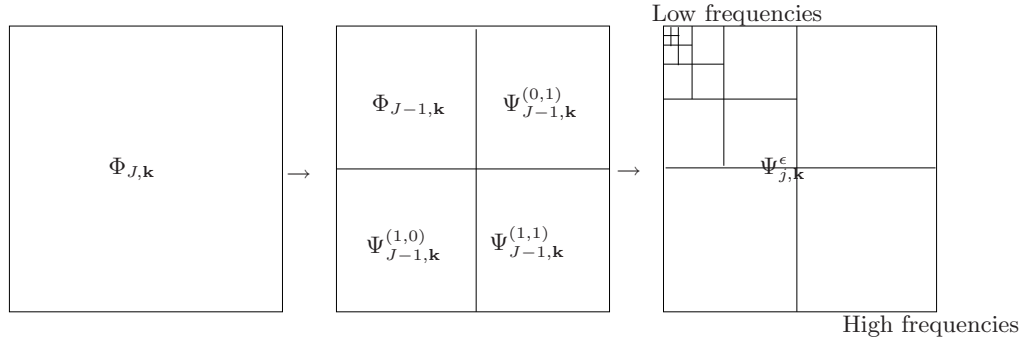

Figure 3: Splitting of the Fourier modes induced by the 2D-MRA wavelet decomposition

5.1 Convergence theorems with Shannon wavelets

To begin with, we'll only consider approximation matrices that are constant over each frequency domain indexed by $\mathbf{j} \in \mathbb{Z}^d$.

The two previous decompositions induce different conditions for the approximation of the matrix $M(\xi)$. Adding to part 4, we distinguish MRA and tensorial wavelet convergence theorems as follows:

Theorem 5.1 (MRA) *If the symbol matix $M(\xi)$ admits a pseudo-inverse $M(\xi)^\dagger$ such that $M(\xi)M(\xi)^\dagger = Id \ \forall \xi \neq (0, \dots, 0)$, and if $\exists \rho < 1$ such that $\forall j \in \mathbb{Z}$ and $\forall \varepsilon \in \{0,1\}^d \setminus$*
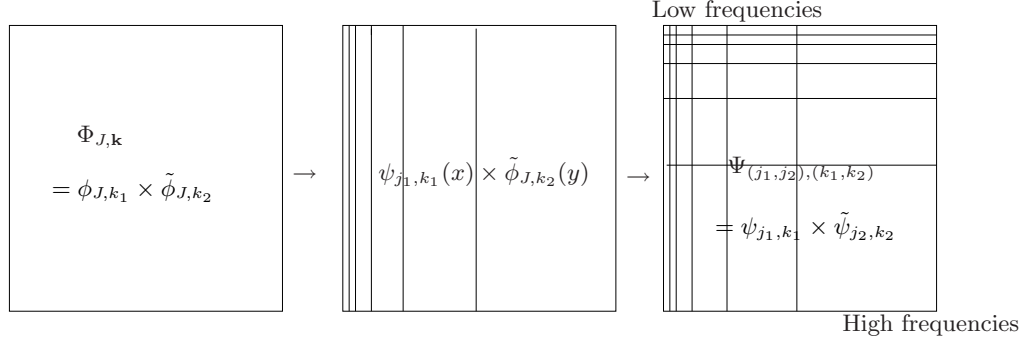
10

Figure 4: Fourier splitting induced by the tensorial wavelet decomposition

$\{(0,\dots,0)\}$ $\exists M_{\omega_{j,\varepsilon}} \in \mathbb{R}^{n\times m}$ such that $\forall \xi \in \prod_{i=1}^{d} \pm[\varepsilon_i 2^j \pi, (\varepsilon_i+1)2^j \pi]$, $\|Id - M(\xi)M_{\omega_{j,\varepsilon}}^\dagger\| \leq \rho$ then the sequence (4.2) using the MRA decomposition with Shannon wavelets, converges.

proof:
We recall that we are in the case $b = \tilde{B} = 1$ of theorem 4.1 since we deal with Shannon wavelets.
The partition of the support of $\widehat{\mathbf{v}}$ operated by the MRA decomposition is the following:

$$J = \{(j,\varepsilon) \in \mathbb{Z} \times \{0,1\}^{d,*}\}$$

$$\mathbf{v} = \sum_{(j,\varepsilon)\in J} \mathbf{v}_j^\varepsilon \quad \text{with} \quad \operatorname{supp}\widehat{\mathbf{v}}_j^\varepsilon \subset \pm[\varepsilon_i 2^j \pi, (\varepsilon_i + 1)2^j \pi]$$

This writting is due to the fact that the case $\varepsilon_i = 0$ corresponds to a scaling function ϕ_j for the variable x_i, and $\varepsilon_i = 1$ to a wavelet function ψ_j. Owing the fact that $\operatorname{supp}\widehat{\phi}_j \subset [-2^j\pi, 2^j\pi]$ and $\operatorname{supp}\widehat{\psi}_j \subset [-2^{j+1}\pi, -2^j\pi] \cup [2^j\pi, 2^{j+1}\pi]$, we obtain the sets indicated in the theorem. This case is represented in figure 3.
Then we apply theorem 4.1 to obtain the convergence.

Remark 5.1 *The fact that $M(\xi)$ admits a pseudo-inverse $M(\xi)^\dagger$ such that $M(\xi)M(\xi)^\dagger = Id$ is implied by $\exists \rho < 1$, $\exists M_{\omega_{\mathbf{j}}} \in \mathbb{R}^{n\times m}$ such that $\|Id - M(\xi)M_{\omega_{\mathbf{j}}}^\dagger\| \leq \rho$.*

Theorem 5.2 (Tensorial wavelets) *If the symbol matix $M(\xi)$ admits a pseudo-inverse $M(\xi)^\dagger$ such that $M(\xi)M(\xi)^\dagger = Id$ $\forall \xi \in \mathbb{R}^d \setminus \{\xi \in \mathbb{R}^d$ such that $\xi_1 \dots \xi_d = 0\}$, and $\exists \rho < 1$ such that $\forall \mathbf{j} \in \mathbb{Z}^d$ $\exists M_{\omega_{\mathbf{j}}} \in \mathbb{R}^{n\times m}$ such that $\forall \xi \in \prod_{i=1}^{d} \pm[2^{j_i}\pi, 2^{j_i+1}\pi]$, $\|Id - M(\xi)M_{\omega_{\mathbf{j}}}^\dagger\| \leq \rho$ then the method converges with the tensorial wavelet decomposition.*

proof:
Anew, we use theorem 4.1 with $J = \mathbb{Z}^d$ and

$$\mathbf{v} = \sum_{\mathbf{j}\in J} \mathbf{v_j} \quad \text{with} \quad \operatorname{supp}\widehat{\mathbf{v_j}} \subset \pm[2^{j_i}\pi, 2^{j_i+1}\pi]$$

This wavelet decomposition part the frequency domain as represented in figure 4.

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Remark 5.2 *If we consider only constant matrices operating on the wavelet coefficients, the resulting operations on the Fourier transform of the functions are symetric by reflection along all axes:*

$$\forall i \in \{1, \ldots, d\}, \ M_\omega(\xi_1, \ldots, -\xi_i, \ldots, \xi_d) = M_\omega(\xi_1, \ldots, \xi_i, \ldots, \xi_d) \tag{5.1}$$

On the other hand, as we deal with real functions, the approximation matrix must be real.

Remark 5.3 *The best aproximation $M_{\omega_\mathbf{j}}$ of $M(\xi)$, for the inversion is given by*

$$M_{\omega_\mathbf{j}} = \arg \min_{\mu \in \mathbb{R}^{n \times m}} \sup_{\xi \in supp(\mathbf{v_j})} \|Id - M(\xi)\mu^\dagger\|$$

Example 5.1 *As we'll see in part 9, the operator Δ^{-1} matches the two cases. Wavelet algorithms converge in the MRA context and in the tensorial one.*

Example 5.2 *If we consider the 1-D symbols $p : \mathbb{R} \to \mathbb{C}$, $\xi \mapsto p(\xi)$ that are continuous, an example that doesn't match the conditions of the theorem is given by a symbol p such that: $p(\xi) = e^{i\xi}$ for $\xi \in [\pi, 2\pi]$. Then the operator whose symbol is p can't satisfy $\exists \rho < 1$, $\exists \mu \in \mathbb{R}$ such that $\forall \xi \in [\pi, 2\pi]$, $\|1 - p(\xi)\mu^{-1}\| \le \rho$.*

Example 5.3 *On the other hand, the 1-D symbols $p : \mathbb{R} \to \mathbb{R}$, $\xi \mapsto p(\xi)$ that are continuous, verify $\forall \xi \ne 0$, $p(\xi) \ne 0$ and*

$$\sup_{j \in \mathbb{Z}} \frac{\sup_{\xi \in [2^j \pi, 2^{j+1} \pi]}(|p(\xi)|)}{\inf_{\xi \in [2^j \pi, 2^{j+1} \pi]}(|p(\xi)|)} < +\infty$$

can be approximated by a constant ω_j on each interval $\pm[2^j \pi, 2^{j+1} \pi]$ with optimal value verifying

$$\omega_j^{-1} = \frac{1}{2}(m_j^{-1} + M_j^{-1})$$

with $m_j = \inf_{\xi \in [2^j \pi, 2^{j+1} \pi]}(|p(\xi)|)$ and $M_j = \sup_{\xi \in [2^j \pi, 2^{j+1} \pi]}(|p(\xi)|)$
That is the case for functions with polynomial increase, since for ξ^α, $\frac{(2^{j+1} \pi)^\alpha}{(2^j \pi)^\alpha} = 2^\alpha$.

Example 5.4 *In 2-D, even for real operator matrices, the approximation by constant matrices can fail. For instance, if we consider a symbol matrix M such that:*

$$M(\xi_1, \xi_2) = \begin{pmatrix} \cos(\xi_1) & -\sin(\xi_1) \\ \sin(\xi_1) & \cos(\xi_1) \end{pmatrix} \quad \text{for } \xi \in [\pi, 2\pi]^2$$

any wavelet approximation by constant matrices $\mu \in \mathbb{R}^{2 \times 2}$ fails: either $\|Id + \mu\| \ge 1$ or $\|Id - \mu\| \ge 1$.

6 Derivation of wavelets

P. G. Lemarié-Rieusset [14] showed that derivating or integrating a biorthogonal wavelet basis provided a new wavelet basis. It allows us to construct two different one-dimensional multiresolution analyses of $L^2(\mathbb{R})$ related by differentiation and integration.

Theorem 6.1 (Derivation of wavelets) *[14] Let $(V_j^1)_{j \in \mathbb{Z}}$ be a one-dimensional MRA, with a differentiable scaling function φ_1, $(V_0^1 = span\{\varphi_1(x-k), k \in \mathbb{Z}\})$, and a wavelet ψ_1. There exists a second MRA $(V_j^0)_{j \in \mathbb{Z}}$ with a scaling function φ_0 $(V_0^0 = span\{\varphi_0(x-k), k \in \mathbb{Z}\})$ and a wavelet ψ_0 satisfying:*

$$\varphi_1'(x) = \varphi_0(x) - \varphi_0(x-1) \qquad \psi_1'(x) = 4 \; \psi_0(x) \tag{6.1}$$

Expressed with its Fourier transform this relation writes:

$$i\xi \widehat{\psi_1}(\xi) = 4\widehat{\psi_0}(\xi)$$

The filters (m_0, m_0^*) and (m_1, m_1^*) attached respectively to the MRA's $(V_j^0)_{j \in \mathbb{Z}}$ and $(V_j^1)_{j \in \mathbb{Z}}$ verify:

$$m_0(\xi) = \frac{2}{1 + e^{-i\xi}} \; m_1(\xi) \quad and \quad m_0^*(\xi) = \frac{1 + e^{i\xi}}{2} \; m_1^*(\xi)$$

If the wavelet ψ_1 is C^n and has p zero moments (i.e. $\widehat{\psi}_1$ is derivable $p-1$ times in a neighborhood of 0 and $\widehat{\psi}_1^{(k)}(0) = 0$ for $0 \le k \le p-1$) after such an operation, the wavelet ψ_0 has regularity C^{n-1} and $p+1$ zero moments.

Remark 6.1 *As the Shannon wavelets are C^∞ and have an infinite number of zero moments, they can be derivated or integrated in order to obtain biorthogonal wavelets satisfying the relations (6.1) of the theorem 6.1. And we can iterate the derivation or the integration of these wavelets as many times as we wish in order to obtain derivatives of arbitrary order: $\dots$, ψ_{-2}, ψ_{-1}, $\dots$, ψ_2, $\dots$, with ψ_0 the original Shannon wavelet.*

On account of the above remark, we can introduce a new operation thanks to the wavelet decomposition of a function v. Indeed, if we use the tensorial wavelet decomposition, we can derivate or integrate in every directions. For instance, if we write the wavelet decomposition of v with wavelets ψ_0 for each tensorial components except for i for which we take ψ_1 where ψ_0 and ψ_1 are related by the derivation relation (6.1) as in theorem 6.1.

$$v(x) = \sum_{\mathbf{k}, \mathbf{j} \in \mathbb{Z}^d} d_{\mathbf{j}\,\mathbf{k}} \psi_0(2^{j_1} x_1 - k_1) \dots \psi_1(2^{j_i} x_i - k_i) \dots \psi_0(2^{j_d} x_d - k_d)$$

Then if we put for u:

$$u(x) = \sum_{\mathbf{k}, \mathbf{j} \in \mathbb{Z}^d} 4 \cdot 2^{j_i} d_{\mathbf{j}\,\mathbf{k}} \psi_0(2^{j_1} x_1 - k_1) \dots \psi_0(2^{j_i} x_i - k_i) \dots \psi_0(2^{j_d} x_d - k_d)$$

We obtain:

$$u(x) = \frac{\partial v}{\partial x_i}(x) \quad \text{or, in Fourier} \quad \widehat{u}(\xi) = i\xi_i \widehat{v}(\xi)$$

7 Constructible approximations

Here we restict ourselves to the case when $m = n$. From the results of the previous section, it comes:

Theorem 7.1 (Set of constructible operators) *The set of symbol matrices that are constructible by multiplying the wavelet coefficients by some constants depending on the parameter $\mathbf{j}$ and by derivating wavelets as in theorem 6.1 is the $\mathbb{R}$-algebra of $\xi \mapsto \mathbb{C}^{n \times n}(\xi)$ generated by the elements $\{(\delta_{i,j})\}_{1 \leq i,j \leq n}$, $\{i\xi_i I\}_{1 \leq i \leq d}$ and $\{i\xi_i^{-1} I\}_{1 \leq i \leq d}$, with $(\delta_{i,j})$ denoting the matrix which is zero everywhere except at line i and column j where it is 1.*

This theorem enables us to diversify our wavelet approximations of differential operators and extends the result of section 4. But it still remains rather limited since for instance, in dimension larger than 2, we cannot reach Δ^{-1}, the inverse Laplacian, with these operations.

8 Convergence with the wavelet packets

Let A be an operator from $(H^{t/2}(\mathbb{R}^d))^n$ to $(H^{-t/2}(\mathbb{R}^d))^n$, having a continuous symbol $M(\xi)$ almost everywhere invertible on $\mathbb{R}^d$ in the sens of Riemann measure (i.e. such that for all compact sets K of $\mathbb{R}^d$, $K \cap (\det(M))^{-1}(\{0\})$ the subset of K where $M(\xi)$ is not invertible has a vanishing Riemann measure), and verifies the condition (5.1). Then it can be approximated by constant matrices $M_{\omega_{\mathbf{j}}}$ with Shannon wavelet packets providing an ad hoc partition of the frequency domain.

Theorem 8.1 *For all linear operator A satisfying the above conditions, we can numerically solve the equation $A\mathbf{u} = \mathbf{v}$ with a wavelet packet method. i.e. $\forall \varepsilon > 0$, we can find $\mathbf{u}_\varepsilon$ such that $\|\mathbf{v} - A\mathbf{u}_\epsilon\| < \varepsilon$, thanks to the wavelet algorithms described in part 4.*

Proof: First we build a finite set of rectangles $\{\omega_{\mathbf{j}}\}_{\mathbf{j} \in J}$ of the type

$$\omega_{\mathbf{j}} = \prod_{i=1}^{d} \left[2^{j_i} \ell_i, 2^{j_i}(\ell_i + 1) \right]$$

with $\mathbf{j} = (j_1, \ldots, j_d, \ell_1, \ldots, \ell_d)$, $j_i \in \mathbb{Z}$ and $k_i \in \mathbb{N}$, such that if $\Omega = \bigcup_{\mathbf{j} \in J} \omega_{\mathbf{j}}$ then $\|\widehat{v} - \widehat{v}_{|\Omega}\| < \varepsilon/2$, and for $\xi_{\mathbf{j}} = (2^{j_1} \ell_1, \ldots, 2^{j_i} \ell_i, \ldots, 2^{j_d} \ell_d) \in \omega_{\mathbf{j}}$,

$$\sup_{\mathbf{j}} \left(\sup_{\xi \in \omega_{\mathbf{j}}} \|Id - M(\xi)M^{-1}(\xi_{\mathbf{j}})\| \right) < 1$$

Remark that, as a consequence, $M(\xi)$ is invertible on Ω.
On the set Ω, we can apply the wavelet algotithm of part 4 with Shannon wavelet packets, and get $\mathbf{u}_\epsilon$ such that $\|\widehat{\mathbf{v}}_{|\Omega} - \widehat{A}_{|\Omega}\widehat{\mathbf{u}}_{\epsilon|\Omega}\| < \varepsilon/2$, then we extend $\widehat{\mathbf{u}}_\epsilon$ to $\mathbb{R}^d$ by taking $\widehat{\mathbf{u}}_{\epsilon|^c\Omega} = 0$.

Remark 8.1 *This approach is valid for Shannon wavelets. But in practice we would like to use other wavelets for which the frequency partition induced by the wavelet packets is very difficult to control.*

Remark 8.2 *Nevertheless, this method should help us to improve the existing algorithms (see parts 9 and 10).*

9   Implicit Laplacian

First, let us begin with the elliptic operator $A = (Id - \alpha\Delta)$, $\alpha \geq 0$. The above study allows us to state precise properties for the **wavelet iterative algorithm** solving $(Id - \alpha\Delta)\mathbf{u} = \mathbf{v}$ (in Navier-Stokes $\alpha = \nu\delta t$).

First, we consider a Shannon wavelet decomposition for $\mathbf{u}$ and $\mathbf{v}$:

$$(Id - \alpha\Delta) \sum_{j \in \mathbb{Z}^d} \mathbf{u_j} = \sum_{j \in \mathbb{Z}^d} \mathbf{v_j}$$

then, for each $\mathbf{j} \in \mathbb{Z}^d$, we solve

$$(Id - \alpha\Delta)\mathbf{u_j} = \mathbf{v_j}$$

with $\mathrm{supp}\,(\widehat{\mathbf{u_j}}) = \mathrm{supp}\,(\widehat{\mathbf{v_j}}) \subset \prod_{i=1}^{d} \pm[2^{j_i+1}\pi, 2^{j_i}\pi]$
We approximate $M(\xi) = (1 + \alpha|\xi|^2)\,Id$ with $\xi \in \prod_{i=1}^{d} \pm[2^{j_i+1}\pi, 2^{j_i}\pi]$ by $M_{\omega_\mathbf{j}} = (1 + \alpha\omega_\mathbf{j}^2)\,Id$ with $\omega_\mathbf{j} \in \mathbb{R}$ properly chosen.

Now, we'll see what is the most convenient value for the parameter $\omega_\mathbf{j}$. The symbol matrix $(Id - M(\xi)M_{\omega_\mathbf{j}}^{-1})$ of the operator $(Id - AM_{\omega_\mathbf{j}}^{-1})$ is a diagonal matrix λId whose element $\lambda \in \mathbb{R}$ is equal to:

$$\lambda = 1 - \frac{1 + \alpha|\xi|^2}{1 + \alpha\omega_\mathbf{j}^2}$$

This expression has a minimal maximum for $|\xi|^2 \in [a_\mathbf{j}^2, b_\mathbf{j}^2]$ at $\omega_\mathbf{j}^2 = \frac{a_\mathbf{j}^2 + b_\mathbf{j}^2}{2}$.
If we take this value for $\omega_\mathbf{j}$, then

$$\forall \xi \in \mathbb{R}^d \,/\, a_\mathbf{j}^2 \leq |\xi|^2 \leq b_\mathbf{j}^2, \quad |\lambda(Id - M(\xi)M_{\omega_\mathbf{j}}^{-1})| \leq \frac{\alpha(b_\mathbf{j}^2 - a_\mathbf{j}^2)}{2 + \alpha(b_\mathbf{j}^2 + a_\mathbf{j}^2)}$$

Then in the tensorial case, with the Shannon wavelets, as for fixed $\mathbf{j}$, $\forall i \in \{1, \ldots, d\}$, $\xi_i \in \pm[2^j\pi, 2^{j+1}\pi]$, then $b_\mathbf{j} = 2a_\mathbf{j}$ and $|\lambda| \leq \frac{3\alpha}{2a_\mathbf{j}^{-2} + 5\alpha}$. The worst case appears for $\alpha \to +\infty$, then the convergence rate tends to $\rho = \frac{3}{5} = 0.6$.
In the case of the MRA, we have always $b_{i,\mathbf{j}}^2 \leq (d+1)a_{i,\mathbf{j}}^2$, that allows the algorithm to converge too: $\rho = \frac{d\alpha}{2a_\mathbf{j}^{-2} + (d+2)\alpha} \to_{\alpha \to +\infty} \frac{d}{d+2}$.

If we now consider what should be obtained with the Shannon wavelet packets, we take $b = 3/2\,a$ in the tensorial case, then if $\alpha \to \infty$, $\rho = \frac{5}{13}$ (~ 0.38). Roughly speaking, we improve the convergence by a factor 2 for each wavelet packet refinement.

Observed convergence for the implicit Laplacian
For the operator $Id - \alpha\Delta$, the convergence of the wavelet algorithm is very fast if α is small compared to the smallest computed scale. Anyway, for the Laplacian operator Δ ($\alpha \to +\infty$) the same algorithm still converges, and the observed convergence rate is around 0.75 for spline wavelets of order 4.

10   Leray projector

Principle of the Helmholtz decomposition:
Let $\mathbf{u} \in (L^2(\mathbb{R}^n))^n$ be a vector field. We can decompose $\mathbf{u}$ as follows:

$$\mathbf{u} = \mathbf{u}_{\text{div}} + \mathbf{u}_{\text{curl}} \quad \text{where} \quad \mathbf{u}_{\text{div}} = \mathbf{curl}\ \psi\ , \quad \mathbf{u}_{\text{curl}} = \nabla p$$

The functions $\mathbf{curl}\ \psi$ and ∇p are orthogonal in $(L^2(\mathbb{R}^n))^n$ and are unique.

$$(L^2(\mathbb{R}^n))^n = \mathbf{H}_{\text{div}\,0}(\mathbb{R}^n) \oplus^{\perp} \mathbf{H}_{\text{curl}\,0}(\mathbb{R}^n)$$

In Navier-Stokes, this decomposition is very important to project the term $\mathbf{u}.\nabla\mathbf{u}$ onto $\mathbf{H}_{\text{div}\,0}(\mathbb{R}^n)$ the space of divergence free vector functions (see [1]).

The wavelet algorithm that will be studied here was originally designed in [8]. The proof of its convergence for the Shannon wavelet is new. The **wavelet iterative algorithm for** $\mathbb{P}$, the orthogonal projector (in L^2) on $\mathbf{H}_{\text{div}\,0}(\mathbb{R}^n)$, uses the result on the wavelet derivation of part 6. We approximate the Leray projector $\mathbb{P}$: $\mathbb{P}\widehat{\mathbf{u}} = M(\xi)\widehat{\mathbf{u}}$ with

$$M(\xi) = Id - \frac{1}{|\xi|^2} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \times [\xi_1 \ldots \xi_n]$$

by

$$M_\omega = \left(Id - \frac{1}{|\omega|^2} \begin{bmatrix} \frac{\omega_1^2}{\xi_1} \\ \vdots \\ \frac{\omega_n^2}{\xi_n} \end{bmatrix} \times [\xi_1 \ldots \xi_n] \right) \left(Id - \frac{1}{|\omega|^2} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \times \left[\frac{\omega_1^2}{\xi_1} \ldots \frac{\omega_n^2}{\xi_n} \right] \right)$$

where we used the notation $\omega_i = 2^{j_i}$.

Remark 10.1 *One can notice that this projector M_ω actually projects the vector field $\mathbf{u}$ on the space of divergence free vector functions (indeed $< [\xi_1 \ldots \xi_n], M_\omega \widehat{u}(\xi) > = 0$).*

We also have to extract the gradient part of $\mathbf{u}$ by the approximation L_ω of $\nabla\Delta^{-1}(\text{div}\cdot)$:

$$L_\omega = \frac{1}{|\omega|^2} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \times \left[\frac{\omega_1^2}{\xi_1} \ldots \frac{\omega_n^2}{\xi_n} \right]$$

Theorem 10.1 *The sequence (4.5) writes:*

$$\mathbf{v}_0 = \mathbf{u} \quad \text{and} \quad \mathbf{v}_{n+1} = \mathbf{v}_n - M_\omega \mathbf{v}_n - L_\omega \mathbf{v}_n \tag{10.1}$$

This sequence $(\mathbf{v}_n)$ goes to 0 in L^2 norm in the case of Shannon wavelets. The iteration of the operation (10.1) allows us to separate the divergence free part $\mathbf{u}_{\text{div}} = \sum_{n\in\mathbb{N}} M_\omega\,\mathbf{v}_n$ of the function $\mathbf{u}$ from its gradient part $\mathbf{u}_{\text{curl}} = \sum_{n\in\mathbb{N}} L_\omega\,\mathbf{v}_n$.
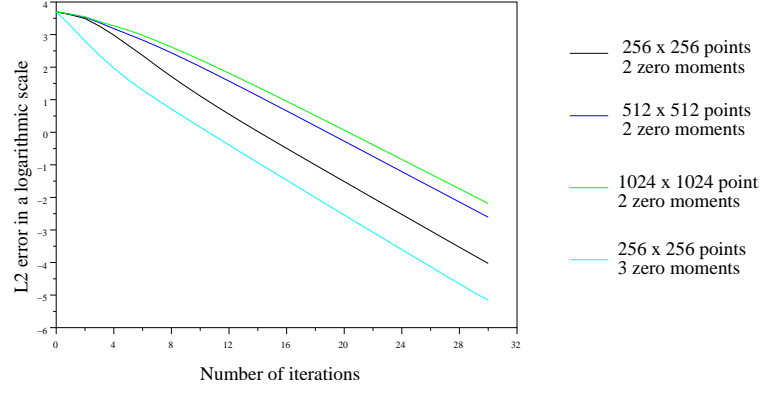
Figure 5: Observed convergence rates for various wavelets and numbers of grid points

proof:
The matrix $(Id - M_\omega - L_\omega)$ can be written, for $\xi \in \prod_i \pm [2^{j_i}\pi, 2^{j_i+1}\pi]$

$$(Id - M_\omega - L_\omega) = \frac{1}{|\omega|^2} \begin{bmatrix} \frac{\omega_1^2}{\xi_1} \\ \vdots \\ \frac{\omega_n^2}{\xi_n} \end{bmatrix} \times [\xi_1 \ldots \xi_n] \times (Id - L_\omega)$$

All its eigenvalues are equal to zero except one that equals

$$\lambda = 1 - \left(\sum_{k=1}^{n} \xi_k^2 \right) \left(\sum_{k=1}^{n} \frac{\omega_k^4}{|\omega|^4 \xi_k^2} \right)$$

i.e., with $\zeta_k = \frac{\xi_k}{\omega_k}$,

$$\lambda = 1 - \left(\sum_{k=1}^{n} \frac{\omega_k^2}{|\omega|^2} \zeta_k^2 \right) \left(\sum_{k=1}^{n} \frac{\omega_k^2}{|\omega|^2} \zeta_k^{-2} \right)$$

Then the Kantorovitch inequality yields

$$|\lambda| \leq \frac{1}{4} \left(\frac{\min_k |\zeta_k|}{\max_k |\zeta_k|} + \frac{\max_k |\zeta_k|}{\min_k |\zeta_k|} \right)^2 - 1$$

If $|\zeta_k| \in [a_k, b_k]$ for each k, we put $a = \min_k a_k$ and $b = \max_k b_k$. Then

$$|\lambda| \leq \frac{1}{4} \left(\frac{a}{b} + \frac{b}{a} \right)^2 - 1$$

For the Shannon wavelets, $b = 2a$, then convergence is assured since $|\lambda| \leq \frac{9}{16}$ (~ 0.56). In the case of the Shannon wavelet packets, $b = \frac{3}{2}a$, the convergence rate should be $\rho = |\lambda| \leq \frac{25}{144}$ (~ 0.17).

Observed convergence for the Leray projector
The convergence has been tested successfully on variate 2D and 3D fields. The observed convergence rates with spline wavelets of order 2 and 3 are around 0.5 (see figure 5). The reference [8] provides technical explanations for the impementation of this algorithm.

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11 Conclusion, perspectives

This work provides an original point of view on the wavelet algorithms and links two achievements of the wavelet theory:

- the wavelet approximation of operators like it is presented in A. Cohen's, W. Dahmen's and R. DeVore's work [2, 3, 4, 6],

- the divergence-free wavelet transform derived from P.-G. Lemarié's and K. Urban's works [14, 17] as well as the author's [7, 8, 9].

The main results of this paper are the establishment of general conditions for the convergence of wavelet algorithms with Shannon wavelets, the theoretical construction of wavelet algorithms in order to approximate operators with constant coefficients, the exact computation of the convergence rates and their optimisation for the implicit Laplacian operator and the Leray projector. It also gives a simple view of the wavelet preconditionning and a glance to what could be done thanks to wavelet packets. Indeed, wavelet packets seem to provide a powerfull solver for many kinds of PDE's but their use is still theoretical and prospective.

The progress to achieve should go toward a more general frame for the proofs of the convergence of wavelet algorithms that would allow us not to restrict ourselves to Shannon wavelets. As a result we should be able to introduce wavelets on the interval in this frame. We are still missing efficient wavelet packets concerning the frequency localisation. An interesting perspective would be to study what happens in the case of operators with non constant coefficients

These theoretical assertions have already an application since these technics are derived to simulate the Navier-Stokes equations (see [7], and a forthcoming paper in Siam Multiscale Simulation and Analysis).

Aknowledgements

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