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IM PAN Preprint 714 (2010)

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Published as manuscript

Received 10 March 2010

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March 10, 2010

Abstract

This paper is based on a paper of B. Hofmann, P. Matheé and H. von Weizsäcker. Basic notions in regularization theory for linear operators are investigated We show how one may get the fundamental regularization results with various Hilbert space methods.

1 Preface

This preprint presents the results published in [1], however the proofs differ from the original ones. The aim was to present the theory as simple as possible with hope it will be easy to understand.

This section summarize basic knowledge from the theory of operators acting in Hilbert space. The facts presented here are all what we need to present regularization theory of linear operators acting in Hilbert spaces.

The operators we consider are acting in a Hilbert space H. We assume that the domains of these operators are dense in H. The domain, range and kernel of an operator A are denoted by $D(A), \operatorname{ran} A, \ker A$, respectively. Let us recall some well known definitions.

Operator A is *closed* if for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in D(A)$, $\|x_n - x_0\| \to 0$ and $\|Ax_n - y\| \to 0$ for some $y \in H$ implies $x_0 \in D(A)$ and $Ax_0 = y$.

Operator A is *closable* if there exists a closed extension of A, i.e. there exists a closed operator B such that $D(A) \subset D(B)$ and $Ax = Bx$ for $x \in D(A)$.

An operator B is *adjoint* to A if $\langle Ax, y\rangle = \langle x, By\rangle$ for any $x \in D(A)$. The domain of B is the set of all those $y \in H$ that there exists $z \in H$ such that $\langle Ax, y\rangle = \langle x, z\rangle$. Then $z = By$ and we write $A^* = B$.

1.1 Polar decomposition

Because $A^*A \geq 0$ there exists the operator $B \geq 0$ such that $A^*A = B^2$.

Let us define $B^{\dagger} = 0_{|\ker B} \oplus \operatorname{cl}((B_{|D(B)\cap(\ker B)^{\perp}})^{-1})$, cl denotes here the closure of an operator.

We have the identity

$$\langle Ax, Ay\rangle = \langle Bx, By\rangle \quad \text{for } x, y \in D(A).$$

Note that

$$\overline{\operatorname{ran} B} = (\ker B)^\perp = (\ker A^*A)^\perp = (\ker A)^\perp = \overline{\operatorname{ran} A^*}.$$

Hence for $u, v \in \operatorname{ran} B$ we have the identity

$$\langle AB^\dagger u, AB^\dagger v\rangle = \langle u, v\rangle.$$

which shows that $AB^\dagger$ is an isometry on $\operatorname{ran} B$. The closure U of $AB^\dagger$ is partial isometry – it isometrically transforms $\overline{\operatorname{ran} B} = \overline{\operatorname{ran} A^*}$ on $\operatorname{ran} U = \overline{\operatorname{ran} AB^\dagger} = \overline{\operatorname{ran} A}$ and vanishes on its kernel – $\ker U = \ker B^\dagger = \ker B = \ker A$.

We have

$$UBx = AB^\dagger Bx = Ax \quad \text{for } x \in D(A).$$

$A = UB$ is called *right polar decomposition* of A.

U^* is also a partial isometry – it isometrically transforms $\overline{\operatorname{ran} A^*}$ on $\overline{\operatorname{ran} A}$ and $\ker U^* = (\operatorname{ran} U)^\perp = (\operatorname{ran} A)^\perp = \ker A^*$.

Moreover the equality

$$\langle U^*Uu, v\rangle = \langle Uu, Uv\rangle = \langle u, v\rangle \quad \text{valid for } u, v \in \overline{\operatorname{ran} B} = \overline{\operatorname{ran} A^*}$$

shows that

$$U^*Uu = \begin{cases} u & \text{for } u \in \overline{\operatorname{ran} A^*}, \\ 0 & \text{for } u \in \ker A. \end{cases}$$

Therefore $A = UBU^*U = CU$, where $C = UBU^*$ is also a selfadjoint non-negative operator. This is the *left polar decomposition* of A. Note that U is the same in both polar decompositions. The above in particullary implies that $AA^* = UB^2U^* = U(A^*A)U^*$.

1.2 Examples

Let $H = L^2(0,1)$ and A be the differentiation operator $Ax(t) = x'(t)$. $D(A) = \{x \in H$ such that $x' \in H\}$. A is closed. What is A^*? The equality

$$\langle Ax, y\rangle = \int x'\bar{y} = -\int x\bar{y}' + (x\bar{y})|_0^1$$

shows that if $\operatorname{supp} y \subset (0,1)$ then $A^*y = -y'$, and that for $y \in D(A^*)$ we should have additionally $y(0) = y(1)$.

Thus

$$A^*y = -y', D(A^*) = \{x \in H; x' \in H, x(0) = x(1) = 0\}.$$

We can easily see, that $AA^*y = -y''$ and $A^*Ay = -y''$, However domains of these operators differ.

$$D(AA^*) = \{x \in H; x'' \in H \text{ and } x(0) = x(1) = 0\},$$

$$D(A^*A) = \{x \in H; x'' \in H \text{ and } x'(0) = x'(1) = 0\}.$$

The operator AA^* has eigenfunctions $\sin k\pi x$, with eigenvalues $(k\pi)^2$, $k = 1, 2, \ldots$, The operator A^*A has eigenfunctions $\cos k\pi x$, with eigenvalues $k^2\pi^2$, $k = 0, 1, 2, \ldots$, all these eigenfunctions have norm $\frac{1}{\sqrt{2}}$ except $\cos 0\pi x$, which has norm 1. Thus setting $s_k = s_k(x) = \sqrt{2}\sin k\pi x$, $c_k = c_k(x) = \sqrt{2}\cos k\pi x$, $k = 1, 2, \ldots$ and $c_0 = 1$ we have expansions:

$$AA^* = \sum_{k=1}^{\infty} k^2\pi^2 \langle \cdot, s_k \rangle s_k, \quad A^*A = \sum_{k=1}^{\infty} k^2\pi^2 \langle \cdot, c_k \rangle c_k$$

Defining

$$B = \sum_{k=1}^{\infty} k\pi \langle \cdot, s_k \rangle s_k, \quad C = \sum_{k=1}^{\infty} k\pi \langle \cdot, c_k \rangle c_k,$$

$$U = \sum_{k=1}^{\infty} \langle \cdot, s_k \rangle c_k, \quad V = -\sum_{k=1}^{\infty} \langle \cdot, c_k \rangle s_k,$$

we have $AA^* = B^2$, $A^*A = C^2$ with nonnegative selfadjoint operators B, C.

We have also $Ac_k = -k\pi s_k = BVc_k = VCc_k$, and $A^*s_k = -k\pi c_k = -CUs_k = -UBs_k$

Now it is easy to see that we have polar decompositions:

$$A = BV = VC, \quad A^* = -CU = -UB.$$

U is an isometry, but its range is not all H, V is partial isometry with kernel spanned by c_0. Here $U^* = -V$.

The operator A has a lot of eigenfunctions, namely any function $e^{\lambda x}$ with complex number λ is its eigenfunction. From this set of eigenfunctions we may get a subset, which forms an orthonormal basis of H, for example $e_k = e_k(x) = e^{2\pi k x i}$, $k = 0, \pm 1, \pm 2, \ldots$.

Hence one may write

$$A = 2\pi i \sum_{-\infty}^{\infty} k \langle \cdot, e_k \rangle e_k,$$

and such equality implies that A is a normal operator, with polar decompositions $A = U_0 B_0 = B_0 U_0$, where

$$B_0 = 2\pi \sum_{-\infty}^{\infty} |k| \langle \cdot, e_k \rangle e_k, \quad U_0 = i \sum_{-\infty}^{\infty} \operatorname{sign} k \langle \cdot, e_k \rangle e_k.$$

Of course it contradicts our previus considerations. What is wrong? We have choosen a basis which consists from periodic functons, as a derivative of a periodic function is again periodic we have silently restricted the domain of A to periodic functions only. Thus the operator A with the above expansion this time has a domain $\{x \in H; x' \in H, x(0) = x(1)\}$. A similar efect happens, when one tries to use sine basis (s_k) for approximation operator A. This shows that one has to be carefull while approximating unbounded operators.

1.3 Stone–von Neumann operator calculus

Operator calculus enables us to define $f(A)$, where f is a complex valued functions defined on a subset of the complex plane, and A an operator. If f is a polynomial $f(A)$ expands in powers of A in the same way as the polynomial.

In the case when A is a diagonalizable operator, i.e. $A = \sum \lambda_j \langle \cdot, e_k \rangle e_k$ we set $f(A) = \sum f(\lambda_j)\langle \cdot, e_k \rangle e_k$ and this definiton is consistent with the definition for polynomials.

Defining functions of selfadjoint operators is nearly the same task as that for diagonal ones. Let μ be a nonnegative Borel mesure defined on Borel subsets of real line. Let $H = L^2(\mu)$, and A be an operator defined by $Ax(t) = tx(t)$. This operator is selfadjoint, its spectrum coincides with the support of μ, and for any Borel function f the operator $f(A)$ is defined by $f(A)x(t) = f(t)x(t)$.

Operators of this kind are blocks from which any selfadjoint operator is composed. Namely, if A is a selfadjoint operator acting in a Hilbert space H then there exists a family $\{\mu_\alpha\}_\alpha$ of nonnegative Borel measures and a unitary operator $U : H \to \bigoplus_\alpha L^2(\mu_\alpha)$ such that $A = U^*(\bigoplus_\alpha A_\alpha)U$, where $A_\alpha x(t) = tx(t)$.

Spectral measure is a useful tool for studying selfadjoint operators. For each Borel subset $\Omega \subset \mathbb{R}$ $E(\Omega)$ is an orthogonal projection acting in H. Spectral measure has properties similar to measure – $E(\Omega_1)E(\Omega_2) = E(\Omega_1 \cap \Omega_2)$, $E(\emptyset) = 0$, $E(R) = I$. Moreover $E(\Omega)A = AE(\Omega)$, and the operator A and any Borel measureable function f of this opeator may be expressed as

$$A = \int t dE, \quad f(A) = \int f(t) dE.$$

If ess sup $|f| = \sup_{t \geq 0}\{t : E(\{x : |f(x)| \geq t\}) \neq 0\}$ is bounded, then this quantity equals $\|f(A)\|$, if it is unbounded the operator $f(A)$ is unbounded.

With the representation of A as the direct sum of operatores we can write

$$E(\Omega)x = U^* \left(\bigoplus_\alpha \chi(\Omega) x_\alpha \right) U,$$

where $\chi(\Omega)$ is the charecteristic function of the set Ω.

Note also that $\sigma(A) = \text{supp } E = \overline{\bigcup_\alpha \text{supp } \mu_\alpha}$

2 Regularization

While investigating the proofs of the results in [1], we can see that the most essentiall parts are those which refer to properties of a selfadjoint operator. We shall give the proofs in the case when $H = L^2(\mu)$, and A is defined by $Ax(t) = tx(t)$. We shall refer to this case as the *model case.* The proofs in this case are the essence of the proofs in the general case. These last are more technical and will be also presented.

2.1 Basic definitions

Definition 1. *Family* $\{g_\alpha\}_{0<\alpha<\bar{\alpha}}$ *of bounded Borel functions* $g_\alpha : R^+ \to R^+$ *is* regularization *if they are piecewise continuous in* α *and*

(a) $r_\alpha(t) = 1 - tg_\alpha(t) \to 0$ *as* $\alpha \to 0$,

(b) $|r_\alpha(t)| = |1 - tg_\alpha(t)| < \gamma_1$, *for all* $\alpha \in (0, \bar{\alpha}]$, $t > 0$,

(c) $\sqrt{t}|g_\alpha(t)| < \frac{\gamma_*}{\sqrt{\alpha}}$ *for all* $t > 0$.

The *solution* of equation $Ax = y$ will be denoted by $x^\dagger$, *approximate solution* x_α^δ of equation $Ax = y$ is defined as

$$x_\alpha^\delta = A^* g_\alpha(AA^*)y^\delta, \text{ where } \|y^\delta - y\| \leq \delta.$$

The error may be splitted in two parts

$$x^\dagger - x_\alpha^\delta = (x^\dagger - x_\alpha) + (x_\alpha - x_\alpha^\delta), \text{ where } x_\alpha = A^* g_\alpha(AA^*)y. \tag{1}$$

$x^\dagger - x_\alpha$ is called the bias. The error in algorithms for ill posed problems has two sources, the bias – the error that algorithm produces on exact data, the second source is the inexactness of given data which is measured by δ.

2.2 Impact of data error

Corollary 1. *(c.f. [1, Corrolary 2]) With* $r_\alpha(t) = 1 - tg_\alpha(t)$

$$\|r_\alpha(A^*A)\| \leq \gamma_1, \quad \|A^* g_\alpha(AA^*)\| \leq \frac{\gamma_*}{\sqrt{\alpha}}.$$

Proof. In our model case the thesis reads as

$$\sup_{t>0} |1 - t^2 g_\alpha(t^2)| \leq \gamma_1, \quad \sup_{t>0} |tg_\alpha(t^2)| \leq \frac{\gamma_*}{\sqrt{\alpha}},$$

so there is nothing to prove, as this is equivalent with definitions.

In the general case let $A = BU$ be polar decomposition of A and E be spectral measure for $B \geq 0$, then $A^*A = U^*B^2U$, $AA^* = B^2$ and

$$r_\alpha(A^*A) = U^* \int r_\alpha(t^2)dEU, \quad A^* g_\alpha(AA^*) = U^* \int tg(t^2)dE.$$

Thesis follows form the fact that $\|r_\alpha(A^*A)\| = \text{ess sup}_{\,t\in\sigma(A^*A)}|r_\alpha(t)| \leq \gamma_1$ and similarily for $\|A^* g_\alpha(AA^*)\|$. □

Lemma 1. *(c.f. [1, Lemma 2]) With $x_\alpha = A^* g_\alpha(AA^*)Ax^\dagger = A^* g_\alpha(AA^*)y$ we have*

$$\|x_\alpha - x_\alpha^\delta\| \le \gamma_* \frac{\delta}{\sqrt{\alpha}}.$$

Proof. In the model case $x_\alpha(t) = tg_\alpha(t^2)y(t)$. Therefore

$$\begin{aligned} |x_\alpha(t) - x_\alpha^\delta(t)| = |tg_\alpha(t^2)(y(t) - y^\delta(t))| &\le \sup_{t>0} |tg_\alpha(t^2)||y(t) - y^\delta(t))| \\ &\le \frac{\gamma_*}{\sqrt{\alpha}} |y(t) - y^\delta(t)| \end{aligned}$$

and this implies the thesis.

In the general case let $BU = A$ be polar decomposition of A, and E be spectral measure for B. Then $x_\alpha = U^* B g_\alpha(B^2)y$, $x_\alpha^\delta = U^* B g_\alpha(B^2)y^\delta$ and

$$x_\alpha - x_\alpha^\delta = U^* B g_\alpha(B^2)(y - y^\delta)$$

and because

$$Bg_\alpha(B^2) = \int tg_\alpha(t^2)dE$$

where E is the spectral measure of B, we have

$$\|Bg_\alpha(B^2)\| \le \sup_{t>0} t|g_\alpha(t^2)| \le \frac{\gamma_*}{\sqrt{\alpha}}$$

and finally

$$\|x_\alpha - x_\alpha^\delta\| = \|U^* B g_\alpha(B^2)(y - y^\delta\| \le \|U^*\|\|\|Bg_\alpha(B^2)\|\|\|y - y^\delta\| \le \frac{\gamma_*}{\sqrt{\alpha}} \|y - y^\delta\|.$$

□

2.3 Bias convergence

Using the notation of Lemma 1 we have

$$\begin{aligned} x^\dagger - x_\alpha =& x^\dagger - A^* g_\alpha(AA^*)Ax^\dagger = U^*Ux^\dagger - U^*Bg_\alpha(B^2)BUx^\dagger \\ =& U^*(I - B^2 g_\alpha(B^2))Ux^\dagger = U^* r_\alpha(B^2)Ux^\dagger \end{aligned}$$

and

$$\|x^\dagger - x_\alpha\|^2 = \int r_\alpha^2(t^2)\|dEUx^\dagger\|^2.$$

($\|dEUx^\dagger\|^2 = d\mu$ where the measure μ is defined by $\mu(\Omega) = \|E(\Omega)Ux^\dagger\|^2$.) The convergence $\|x^\dagger - x_\alpha\| \to 0$ follows form definition of regularization (parts (a) and (b)) and the Lebesgue's dominated convergence theorem. However without any additional knowledge about the solution and properties of regularization no estimation are possible.

2.4 Source condition

Source condition for the solution $x^\dagger$ of the equation $Ax = y$ is $x^\dagger = \psi(A^*A)w$.

We shall assume that ψ is an *index function* i.e.
$\psi : (0, \infty) \to (0, \infty)$,
ψ is increasing (non-decreasing) and continuous
$\lim_{t\to 0} \psi(t) = 0$.

Lemma 2. *(c.f. [1, Lemma 7]) If the solution $x^\dagger$ satisfies source condition $x^\dagger = \psi(A^*A)w$ then*

$$\|x^\dagger - x_\alpha\| \le \|w\| \sup_{s\in\sigma(A^*A)} |r_\alpha(s)|\psi(s). \tag{2}$$

Proof. In the model case

$$x^\dagger(t) - x_\alpha(t) = \psi(t^2)w(t) - t^2 g_\alpha(t^2)\psi(t^2)w(t) = \psi(t^2)(1 - t^2 g_\alpha(t^2))w(t).$$

Hence

$$\|x^\dagger - x_\alpha\| \le \sup_{t\in \text{supp } \mu} |\psi(t^2)r_\alpha(t^2)| \cdot \|w\|.$$

Now the thesis follows form the fact that $\sigma(A^2) = \{t^2 : t \in \sigma(A)\}$ and in the model case $\sigma(A) = \text{supp } \mu$.

In the general case with the notation used in proof of Lemma 6 we have $A^*A = U^*BBU$ and therefore

$$x^\dagger = \psi(A^*A)w = U^*\psi(B^2)Uw,$$

$$\begin{aligned} x_\alpha =& U^*Bg_\alpha(B^2)y = U^*Bg_\alpha(B^2)Ax^\dagger \\ =& U^*Bg_\alpha(B^2)BUU^*\psi(B^2)Uw = U^*B^2g_\alpha(B^2)\psi(B^2)Uw \end{aligned}$$

Thus

$$x^\dagger - x_\alpha = U^*(I - B^2g_\alpha(B^2))\psi(B^2)Uw. \tag{3}$$

Note that

$$(I - B^2g_\alpha(B^2))\psi(B^2) = \int (1 - t^2 g_\alpha(t^2))\psi(t^2)dE$$

$$\|(I - B^2g_\alpha(B^2))\psi(B^2)\| \le \sup_{t\in\sigma(B)} |1 - t^2 g_\alpha(t^2)|\psi(t^2) = \sup_{t\in\sigma(A^*A)} |r(t)|\psi(t), \tag{4}$$

because $\sigma(A^*A)$ may differ from $\sigma(AA^*) = \sigma(B^2)$ only by 0, and by the spectral mapping theorem $\sigma(B^2) = \{t^2; t \in \sigma(B)\}$.

Because $\|U\| = \|U^*\| = 1$ (2) and (3) imply the thesis. □

3 Qualification

To effectively use the estimate in (2) we need to know some properties of the function $|r_\alpha(s)|\psi(s)$. Useful index functions φ's has diserved for a special name.

An index function φ is a *qualification* of the regularisation g_α if there are constants $\gamma = \gamma_\varphi < \infty$, $\bar{\alpha}_\varphi$ such that

$$\sup_{s\in\sigma(A^*A)} |r_\alpha(s)|\varphi(s) \le \gamma\varphi(\alpha), \quad 0 < \alpha \le \bar{\alpha}_\varphi. \tag{5}$$

Usually we do not know $\sigma(A^*A)$ which appears in (5), all we know is that $\sigma(A^*A) \subset [0,\infty)$, moreover qualification function is not defined at 0, therefore (5) should be read as

$$\sup_{s>0} |r_\alpha(s)|\varphi(s) \le \gamma\varphi(\alpha), \quad 0 < \alpha \le \bar{\alpha}. \tag{6}$$

Proposition 1. *(c.f. [1, Proposition 2]) Let g_α be a regularization with some known qualification φ. If ψ is an index function such that*
there exists $s_0 > 0$ such that the function $s \to \psi(s)/\varphi(s)$, $0 < s \le s_0$ is non-increasing,

$$\psi(s) \le C\varphi(s) \quad \textit{for } s > s_0 \tag{GC}$$

then ψ is a qualification of g_α.

Proof. If $s \le \alpha$ then $\psi(s) \le \psi(\alpha)$ and by the regularization definition (b)

$$|r_\alpha(s)|\psi(s) \le \gamma_1\psi(\alpha) \quad \text{for } s \le \alpha. \tag{7}$$

We have

$$|r_\alpha(s)|\psi(s) = |r_\alpha(s)|\varphi(s)\frac{\psi(s)}{\varphi(s)} \le \gamma_\varphi\varphi(\alpha)\frac{\psi(s)}{\varphi(s)} \tag{8}$$

If $\alpha \le s \le s_0$ then

$$\frac{\psi(s)}{\varphi(s)} \le \frac{\psi(\alpha)}{\varphi(\alpha)}.$$

This and (8) show that

$$|r_\alpha(s)|\psi(s) \le \gamma_\varphi\varphi(\alpha)\frac{\psi(\alpha)}{\varphi(\alpha)} = \gamma_\varphi\psi(\alpha) \quad \text{if } \alpha \le s \le s_0 \tag{9}$$

We write (8) in the form

$$|r_\alpha(s)|\psi(s) \le \gamma_\varphi\psi(\alpha)\frac{\varphi(\alpha)}{\psi(\alpha)}\frac{\psi(s)}{\varphi(s)}.$$

If $\alpha \le s_0$ then

$$\frac{\psi(s_0)}{\varphi(s_0)} \le \frac{\psi(\alpha)}{\varphi(\alpha)} \quad \text{or equivalently } \frac{\varphi(\alpha)}{\psi(\alpha)} \le \frac{\varphi(s_0)}{\psi(s_0)}$$

Therefore

$$|r_\alpha(s)|\psi(s) \le \gamma_\varphi \psi(\alpha) \frac{\varphi(s_0)}{\psi(s_0)} \frac{\psi(s)}{\varphi(s)} \qquad \text{if } \alpha \le s_0$$

If $s \ge s_0$ then $\psi(s) \le C\varphi(s)$ and

$$|r_\alpha(s)|\psi(s) \le C\gamma_\varphi \frac{\varphi(s_0)}{\psi(s_0)} \psi(\alpha) \qquad \text{if } \alpha \le s_0 \text{ and } s \ge s_0. \tag{10}$$

The inequalities (7), (8) and (10) show that ψ is a qualification for g_α with constants $\bar{\alpha}_\psi = \min\{\bar{\alpha}, s_0\}$ and $\gamma_\psi = \min\{C\gamma_\varphi \frac{\varphi(s_0)}{\psi(s_0)}, \gamma_\varphi, \gamma_1\}$. □

It is easy to show, that if condition (GC) holds for some s_0 then it holds for any $s_0 > 0$, the constant C may change only. However s_0 appears also in the assumption on the monotonicity of $\frac{\psi(s)}{\varphi(s)}$. Thus we cannot ignore the constant $\bar{\alpha}_\psi$ (c.f. [1, Remark 7]) .

Proposition 2. *(c.f. [1, Proposition 3]) Let g_α be a regularization with some known qualification φ. If ψ is an index function such that*
there exists $s_0 > 0$ such that the function $s \to \psi(s)/\varphi(s)$, $0 < s \le s_0$ is non-decreasing, and (GC) holds. then

$$|r_\alpha(s)|\psi(s) \le C\varphi(\alpha) \qquad \textit{for } \alpha \in (0, \bar{\alpha}), s > 0.$$

Proof. We have

$$\frac{\psi(s)}{\varphi(s)} \le \frac{\psi(s_0)}{\varphi(s_0)} \qquad \text{for } s \le s_0,$$

hence from (8) it follows that

$$|r_\alpha(s)|\psi(s) \le \gamma_\varphi \varphi(\alpha) \frac{\psi(s)}{\varphi(s)} \le \gamma_\varphi \varphi(\alpha) \frac{\psi(s_0)}{\varphi(s_0)} \qquad \text{for } s \le s_0. \tag{11}$$

On the other hand

$$\psi(s) \le C\varphi(s) \qquad \text{for } s > s_0,$$

then again form (8) we have

$$|r_\alpha(s)|\psi(s) \le \gamma_\varphi \varphi(\alpha) \frac{\psi(s)}{\varphi(s)} \le C\gamma_\varphi \varphi(\alpha) \qquad \text{for } s > s_0. \tag{12}$$

The thesis follows form (11) and (12) with C replaced by $\gamma_\varphi \max\{C, \frac{\psi(s_0)}{\varphi(s_0)}\}$. □

Lemma 2 and Propositions 1 and 2 lead to the bias estimation.

Proposition 3. *Let g_α be a regularization with qualification φ and $x^\dagger = \psi(A^*A)w$ a source condition with index function ψ, which satisfies (GC).*

a) If the function $\frac{\psi(s)}{\varphi(s)}$ is non-increasing in $(0, s_0]$ then for some $C, \bar{\alpha} > 0$

$$\|x^\dagger - x_\alpha\| \le C\psi(\alpha)\|w\|, \quad \alpha \in (0, \bar{\alpha}]$$

b) If the function $\frac{\psi(s)}{\varphi(s)}$ *is non-decreasing in* $(0, s_0]$ *then for some* $C, \bar{\alpha} > 0$

$$\|x^\dagger - x_\alpha\| \le C\varphi(\alpha)\|w\|, \quad \alpha \in (0, \bar{\alpha}]$$

Remark 1. *Note that if we set*

$$\begin{array}{lll} \varphi_0(s) = \varphi(s), & \psi_0(s) = \psi(s) & \text{for } s \in (0, s_0] \\ \varphi_0(s) = \varphi(s_0), & \psi_0(s) = \psi(s_0) & \text{for } s \in (s_0, \infty) \end{array}$$

then φ_0 *is also the qualification for* g_α *(by Proposition 2) and the functions* φ_0, ψ_0 *satisfy the same assumptions as the functions* φ, ψ *in Propositions 2-4, therefore also the same claims for these functions hold.*

The expected solutions of ill posed problems may be very smooth, this smoothness is measured with help of an index function of the source conditions. We end this section showing that set of source conditions for any index function is nonvoid.

Theorem 1. *(c.f. [1, Theorem 1]) Let* A *be a nonnegative selfadjoint operator acting in* H *with* $\ker A = \{0\}$. *Then*

(a) For every $x \in H$ *and* $\varepsilon > 0$ *there exists a bounded index function* ψ *such that the general source condition*

$$x = \psi(A)w \text{ with } w \in H \text{ and } \|w\| \le (1 + \varepsilon)\|x\|$$

is satisfied.

(b) If $x \in \operatorname{ran} \psi(A)$ *for some unbounded index function* ψ*, then* $x \in \operatorname{ran} \psi_0(A)$ *for every bounded index function* ψ_0 *which coincides with* ψ *on* $(0, t_0]$ *for some* $t_0 > 0$.

Proof. of Th. 1 part (a) – model case version. We assume $H = L^2(\mu)$, $Ax(t) = tx(t)$ and $\|x\| = 1$. We have $\|x\|^2 = \int_0^\infty |x(t)|^2 d\mu = 1$, therefore for any $\alpha \in (0, 1)$ there exists decreasing and converging to 0 sequence of numbers $\{\tau_n\}_{n=0}^\infty$ such that

$$\int_{(0,\tau_n)} |x(t)|^2 d\mu \le \varepsilon\alpha^n, \text{for } n = 0, 1, \ldots.$$

Define with $\beta > 1$ and such that $\alpha\beta^2 < 1$

$$\psi_0(t) = \begin{cases} 1 \text{ for } t \ge \tau_0 \\ \beta^{-n} \text{ for } t \in [\tau_n, \tau_{n-1}), \quad n = 1, 2, \ldots,. \end{cases} \tag{13}$$

Then

$$\int_{[\tau_n,\tau_{n-1})} |\psi_0^{-1}(t)x(t)|^2 d\mu \le \varepsilon\beta^{2n}\alpha^{n-1}$$

and

$$\begin{aligned}\|\psi_0^{-1}(A)x\|^2 =& \int_{(0,\infty)} |\psi_0^{-1}(t)x(t)|^2 d\mu \\ =& \int_{[\tau_0,\infty)} |x(t)|^2 d\mu + \sum_{n-1}^{\infty} \int_{[\tau_n,\tau_{n-1})} |\psi_0^{-1}(t)x(t)|^2 d\mu \\ \leq& 1 + \frac{\varepsilon}{\alpha} \sum_{n=1}^{\infty} (\alpha\beta^2)^n = 1 + \varepsilon \frac{\beta^2}{1-\alpha\beta^2}.\end{aligned}$$

Thus with $\alpha = \frac{1}{4}$, $\beta^2 = \frac{4}{3}$ (then $\frac{\beta^2}{1-\alpha\beta^2} = 2$) we have $\|\psi_0^{-1}(A)x\| \leq \sqrt{1+2\varepsilon} < 1+\varepsilon$ and therefore $w = \psi_0^{-1}(A)x$ satisfies the thesis (part (a)).

If we require ψ to be a continuous function we may define it as a continuous piece-wise linear function, linear in the intervals $[\tau_n, \tau_{n-1}]$ and such that $\psi(\tau_n) = \psi_0(\tau_n)$. Then $\psi_0(t) \geq \psi(t)$ and $\|\psi^{-1}(A)x\|^2 = \int |\psi^{-1}(t)x(t)|^2 \leq \int |\psi_0^{-1}(t)x(t)|^2 = \|\psi_0^{-1}(A)x\|^2$ and the thesis is satisfied for ψ. □

Proof. of Th. 1 part (a) – general version Let E be spectral measure for operator A, $\varepsilon > 0$ and $\alpha = \frac{1}{4}$. We can find decreasing and converging to 0 sequence of numbers $\{\tau_n\}_{n=0}^{\infty}$ such that $\|E((0,\tau_n))x\|^2 < \varepsilon\alpha^n$. With ψ_0 defined by (13) and $\beta^2 = \frac{4}{3}$ we have

$$\|\psi_0(\tau_n)^{-1}E([\tau_n,\tau_{n-1})x\|^2 \leq \varepsilon\alpha^{n-1}\beta^{2n}$$

Because

$$\sum_{n=1}^{\infty} \psi_0^{-1}(\tau_n)^{-1}E([\tau_n,\tau_{n-1}) + E((\tau_0,\infty)) = \psi_0^{-1}(A)$$

we have

$$\begin{aligned}\|\psi_0^{-1}(A)x\|^2 =& \sum_{n=1}^{\infty} \|\psi_0^{-1}(\tau_n)^{-1}E([\tau_n,\tau_{n-1})x\|^2 + \|E((\tau_0,\infty))x\|^2 \\ <& 1 + 2\varepsilon < (1+\varepsilon)^2\end{aligned}$$

Thus $w = \psi_0^{-1}(A)x$ satisfies part (a) of the thesis. □

Proof. of part (b). Assume $H = L^2(\mu)$ and action of A on a function is its multiplication by the argument. Then

$$\begin{aligned}\|\psi_0 f\|^2 =& \left(\int_{(0,t_0)} + \int_{[t_0,\infty)}\right) |\psi_0(t)f(t)|^2 \leq \int\limits_{(0,t_0)} |\psi(t)f(t)|^2 \\ &+ \sup_t \psi_0^2(t) \int\limits_{[t_0,\infty)} |f(t)|^2 \leq \|\psi f\|^2 + \sup_t \psi_0^2(t)\|f\|^2.\end{aligned}$$

In the general case, with each nonzero $x \in H$ we may associate Borel measure on the line by $\mu(\Omega) = \|E(\Omega)x\|^2$. For any Borel measureable function ψ we then have

$$\|\psi(A)x\|^2 = \|\int \psi(t)xdE\|^2 = \int \|\psi(t)xdE\|^2 = \int |\psi(t)|^2\|xdE\|^2$$
$$= \int |\psi(t)|^2 d\mu.$$

The proof is analogous to the proof for $H = L^2(\mu)$ with $f = f(t) = 1$. □

4 Convergence rates

With assumptions of Proposition 3 and (1) we get error estimates in the form

$$\|x^\dagger - x_\alpha^\delta\| \le C\psi(\alpha)\|w\| + \gamma_* \frac{\delta}{\sqrt{\alpha}}, \quad \alpha \in (0, \bar{\alpha}]$$

$$\|x^\dagger - x_\alpha^\delta\| \le C\varphi(\alpha)\|w\| + \gamma_* \frac{\delta}{\sqrt{\alpha}}, \quad \alpha \in (0, \bar{\alpha}]$$

If $\psi(s) = s^p$ then for each fixed δ we may find $\alpha = \alpha(\delta)$ which minimizes the right hand side in these equalities. With such ψ or φ we get estimation of the form

$$\|x^\dagger - x_\alpha^\delta\| \le C\left(\alpha^p + \frac{c\delta}{\sqrt{\alpha}}\right).$$

The minimum is attained for $\alpha = \left(\frac{c\delta}{2p}\right)^{\frac{2}{2p+1}}$ and equals $c\delta^{\frac{2p}{2p+1}}$ with some new constant c. With this choice of α we get

$$\|x^\dagger - x_\alpha^\delta\| \le C\delta^{\frac{2p}{2p+1}}.$$

5 Splitting an operator

While working with an unbounded operator A we investigate what happens in a neighborhood of zero of its spectrum and then we add some auxilary conditions so that our analysis may be applied to the operator considered. Why not to split the operator A into a bounded and an unbounded part? This idea has been used in [3], however in a particular case. The general approach may be realised with help of the polar decomosition and the spectral measure.

Let $A = BU$ be the polar decomposition of A, $A : D(A) \to H_2$, $D(A) \subset H_1$, where H_1, H_2 are Hilbert spaces. $B : D(B) \to H_2$, $D(B) \subset H_2$ is a positive selfadjoint operator, and $U : H_1 \to H_2$ is an isometry.

With E - the spectral measure of B we set

$$H_{2,b} = E([0, s_0])H_2, \quad H_{2,u} = E((s_0, \infty))H_2,$$
$$H_{1,b} = U^{-1}H_{2,b}, \quad H_{1,u} = U^{-1}H_{2,u}.$$

Now

$$\begin{aligned}A_b = A|_{H_{1,b}} &:H_{1,b} \to H_{2,b},\\ A_u = A|_{H_{1,u}} &:D(A_u) \to H_{2,u}, D(A_u) = D(A) \cap H_{1,u}.\end{aligned}$$

In the model case $H_{1,b} = H_{2,b} = L^2_\mu([0, s_0])$, $H_{1,u} = H_{2,u} = L^2_\mu((s_0, \infty))$

A_b is a bounded operator $\|A_b\| \leq s_0$, A_u may be unbounded, however it has a bounded inverse, because for $x \in D(A_u) \subset H_{1,u}$ we have

$$\|A_u x\| = \|BUx\| \geq s_0 \|Ux\| = s_0 \|x\|.$$

With this splitting regularization splits also

$$x^\delta_\alpha = A^* g_\alpha(AA^*) y^\delta = A^*_b g_\alpha(B^2) y^\delta_b \oplus A^*_u g_\alpha(B^2) y^\delta_u,$$

where $y^\delta_b = U^{-1}E([0, s_0])Uy^\delta$, $y^\delta_u = U^{-1}E((s_0, \infty))Uy^\delta$, and therefore $\|y^\delta_u\|^2 + \|y^\delta_b\|^2 = \|y^\delta\|^2$.

Regularization theory for bounded operators is known it suffices to check, how it may be applied for unbounded operators with bounded inverse.

If φ is a qualification for $\{g_\alpha\}$ then for any $s_0 > 0$

$$|r_\alpha(\xi)|\varphi(s_0) \leq |r_\alpha(\xi)|\varphi(\xi) \leq \sup_{s>0} |r_\alpha(s)|\varphi(s) \leq \gamma\varphi(\alpha), \quad 0 < \alpha \leq \bar{\alpha}, \xi > s_0.$$

Then we can estimate some part of bias.

In the model case we have

$$E([s_0, \infty))(x^\dagger - x_\alpha) = (I - A^2_u g_\alpha(A^2_u))E([s_0, \infty))x^\dagger$$

and

$$\|E([s_0, \infty))(x^\dagger - x_\alpha)\|^2 = \int\limits_{[s_0,\infty)} |r_\alpha(t^2)x^\dagger(t)|^2 \leq \frac{\gamma^2\varphi^2(\alpha)}{\varphi^2(s^2_0)} \|E([s_0, \infty))x^\dagger\|^2$$

In the general case we have

$$U^* E([s_0, \infty))U(x^\dagger - x_\alpha) = U^*(I - B^2 g_\alpha(B^2))E([s_0, \infty))Ux^\dagger$$

and therefore

$$\begin{aligned}\|U^* E([s_0, \infty))U(x^\dagger - x_\alpha)\|^2 &= \int_{[s_0,\infty)} r^2_\alpha(t^2)\|dEUx^\dagger\|^2\\ &\leq \frac{\gamma^2\varphi^2(\alpha)}{\varphi^2(s^2_0)} \|E([s_0, \infty))Ux^\dagger\|^2.\end{aligned}$$

Hence

$$\|U^* E([s_0, \infty))U(x^\dagger - x_\alpha)\| \leq \frac{\gamma\varphi(\alpha)}{\varphi(s^2_0)} \|x^\dagger\|.$$

Part (c) of regularization definition is mainly applicable to operators for which their positive part in polar decompsition is not strictly bounded by 0 from below. It is not the case for A_u.

$$\|U^* E([s_0, \infty))U(x_\alpha - x_\alpha^\delta)\| \leq \sup_{t \geq s_0} t|g_\alpha(t^2)\|y - y^\delta\| \leq \delta \sup_{t \geq s_0} t|g_\alpha(t^2).$$

For Tikhonov regularization $g_\alpha(t) = \frac{1}{t+\alpha}$ and

$$\sup_{t \geq s_0} t|g_\alpha(t^2) = \frac{s_0}{s_0^2 + \alpha} \leq s_0^{-1} \quad \text{for } \alpha \leq s_0^2$$

because the derivative or $\frac{t}{t^2+\alpha}$ is negative for $\alpha \leq s_0^2$. Thus the bound does not depend on α.

The results of this paper have been presented at the Numerical Analysis Seminar in IM PAN. Let me thank prof. Regińska for encouraging me to write down these results and helpful comments.

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