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IM PAN Preprint 731 (2011)

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Published as manuscript

Received 5 October 2011

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Abstract

The radiation field of a laser (a collimated laser beam) in a bounded domain is considered. The paper concerns reconstruction of this field from measurements made on a part of the domain boundary. The relevant model problem of the physical system is described by the Cauchy problem for the Helmholtz equation on a rectangle in the case when noisy data are given on one side of the rectangle only. In the general case when the beam is not axially symmetric, a convergent series representation of the solution is derived. This representation is the starting point for formulation of different regularization methods. An example of a spectral type regularization method is formulated and analyzed. An error bound for the method is presented.

1 Introduction

In optoelectronics, determination of the radiation field surrounding a source of radiation (e.g. a laser or a light emitting diode) is a problem of frequent occurrence. As a rule, experimental determination of the whole radiation field is not possible. Practically, we are able to measure the electromagnetic field only on some subset of physical space (e.g. on some surfaces). So, the problem arises how to reconstruct the radiation field from such experimental data (see for instance [3, 16]).

Let us consider collimated light beams generated by some sources. In this case the sources generate the electromagnetic field in all the space $\mathbb{R}^3$ outside of the sources, but field values become very small, practically vanish far from the beam axis.

We consider a simplified mathematical model (for a stationary case) in which each component of the field in an open bounded domain D outside of the sources is a solution of the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in } D, \tag{1.1}$$

with a given real wave number k. The problem consists in reconstruction of the solution u of (1.1) in a subdomain $\Omega \subset D$ from measurement data, i.e. from inexact values of u and its normal derivatives on a part od $\partial\Omega$. Let us denote by Γ the part of boundary where measurements of u are available. So, the approximate values g_δ of $u_{|\Gamma}$ can be obtained directly from measurements of the field on the surface Γ. The approximate values of normal derivatives of u on Γ can be found by solving an auxiliary well-posed problem (see [14], Section 3).

The above-mentioned problem is an example of the ill-posed Cauchy problems for elliptic equations. In the recent literature many aspect of regularizing these problems with noisy data have been studied. For an overview, see, e.g. [1, 5, 6, 8]

For simplicity we restrict our consideration to the case of rectangular domain Ω in $\mathbb{R}^2$ where the solution has to be reconstructed. The obtained results can also be extended to the case of a cuboid. With respect to real experiments for a collimated laser beam it is reasonable to assume that measurement data are given only on the one side of rectangle (cuboid) most distant from the sources. This is the main difference between this paper and previous ones, cf. [4, 10, 11, 20] where additional homogeneous or periodic boundary conditions are assumed on the sides parallel to the beam axis. However, the homogeneous boundary conditions have no clear physical meaning, and periodic boundary conditions can be applied only in the case of symmetric beams. The model considered in this paper is more general.

In [7, 14, 15, 18, 19] the Cauchy problem for the Helmholtz equation (1.1) was considered on the infinite strip $\mathbb{R}^2 \times (0, d)$ (or $\mathbb{R} \times (0, d)$) with data given on a one strip side. The approach applied there consisted in application of Fourier transform with respect to the two variables in $\mathbb{R}^2$ (or the one variable in $\mathbb{R}$) which yields to the equivalent formulation of the problem in the form of an operator equation in the frequency space. It was shown in [15] that some spectral type methods give the optimal or order optimal error bounds on certain source sets. This approach cannot be directly applied for the case of rectangle or cuboid because the related Fourier series are not termwise differentiable (as it is in the case of homogeneous boundary conditions on the sides parallel to the beam axis). However, using the idea described in [12, 9], we replace the nonhomogeneous boundary value problem by the auxiliary one

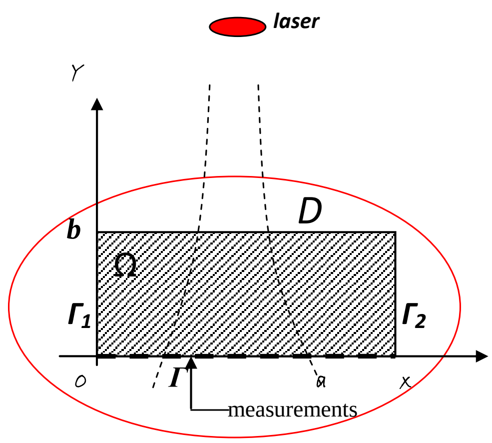

Figure 2.1:

such that the eigenfunction expansion method can be applied for it. This yields to the infinite system of differential equations which is satisfied by the Fourier coefficients of the solution expansion.

In Section 2 we derive a series representation of the solution which is the starting point to formulation of different regularization methods. An example of a spectral type regularization method is formulated in Section 3. Error bounds for regularized solutions are obtained. These estimations depend on the regularization parameter, a measurement error and a priori bounds for certain norms of the solution trace on the rectangle sides were no measurements exist.

2 Cauchy problem on a rectangle

Let us consider the two dimensional model problem presented schematically on Fig.(1). Assume that the unknown field component $u \in H^2(D)$ satisfies the Helmholtz equation on an open domain $D \subset \mathbb{R}^2$. Measurements are available on $\Gamma = (0, a) \times \{0\} \subset D$. Let g and h be the exact values of the solution u and its derivative $\frac{\partial u}{\partial y}$ on the set Γ. Therefore, u is also a solution of the Cauchy problem on the rectangle $\Omega = (0, a) \times (0, b) \subset D$

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega \\ u(x,0) = g(x);\ u_y(x,0) = h(x) & x \in (0,a). \end{cases} \tag{2.1}$$

The main features of the problem (2.1):

- The problem is ill posed in $L^2(\Omega)$: the solution does not depend continuously on the boundary data and it may well be that no solution exists even for arbitrary smooth functions $\widetilde{g} \sim g, \widetilde{h} \sim h$.

- If g, h determine a solution of (2.1) then they determine exactly one solution (see [8], Chapter 3). This uniqueness result is shown in [2], Theorem 4.1 for the case of an arbitrary Lipschitz domain in $\mathbb{R}^d$ under the assumption, that $\exists z \in \Gamma$ and $\exists r > 0$ such that $\Gamma \supset B(z,r) \cap \partial\Omega$ where $B(z,r)$ denotes the ball with the center z and the radius r.

We are going to solve the following problem:

Problem P1

Given noisy data $g^\delta(x)$ and $h^\delta(x)$ on Γ satisfying

$$\|g - g^\delta\|_{L^2(0,a)} \le \delta, \quad \|h - h^\delta\|_{L^2(0,a)} \le \delta \tag{2.2}$$

for a given data error bound δ. For any fixed $y \in (0, b]$ find a function $u^\delta(\cdot, y) \in L^2(0,a)$ which is an approximation of the exact solution $u(\cdot, y)$ for (2.1).

We make the following assumptions on the problem under consideration

A1 The exact solution u is small on $\Gamma_1 = \{0\} \times (0,b)$ and $\Gamma_2 = \{a\} \times (0,b)$, i.e. $\exists\varepsilon$

$$\|u(0,\cdot)\|_{H^2(0,b)} \le \varepsilon, \quad \|u(a,\cdot)\|_{H^2(0,b)} \le \varepsilon. \tag{2.3}$$

It means that the distances of Γ_1, and Γ_2 from the beam axis are sufficiently large.

A2 : A constant $M < \infty$ is known such that

$$\|u'_x(\cdot, b)\|_{L^2(0,a)} \le M \tag{2.4}$$

This is a priori condition on the solution u.

We do not assume that the electromagnetic field is symmetric with respect to the beam axis.

2.1 Auxiliary problem

Let us consider the following auxiliary problem:

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega \\ u(x,0) = \widetilde{g}(x);\ u_y(x,0) = \widetilde{h}(x) & x \in (0,a)\ , \\ u(0,y) = 0,\ u(a,y) = 0 & y \in (0,b). \end{cases} \tag{2.5}$$

Let $\widetilde{g}_n$ and $\widetilde{h}_n$ be Fourier coefficients of the odd $2a$- periodic functions equal to $\widetilde{g}$ and $\widetilde{h}$ on the interval $(0,a)$, respectively, i.e.

$$\widetilde{g}_n = \frac{2}{a}\int_0^a \widetilde{g}(x)\sin\frac{n\pi x}{a}dx,\ n = 1,2,\dots, \tag{2.6}$$

$$\widetilde{h}_n = \frac{2}{a}\int_0^a \widetilde{h}(x)\sin\frac{n\pi x}{a}dx\ n = 1,2,\dots. \tag{2.7}$$

Lemma 2.1 *If the functions $\widetilde{g}$ and $\widetilde{h}$ are such that the solution of (2.5) exists in $H^2(\Omega)$ then*

$$u(x,y) = \sum_{n=1}^{\infty} U_n(y)\sin\frac{n\pi x}{a} \tag{2.8}$$

where

$$U_n(y) = \begin{cases} \widetilde{g}_n \cosh y\zeta_n + \frac{1}{\zeta_n}\widetilde{h}_n \sinh y\zeta_n, & \text{if } n \neq \frac{ak}{\pi} \\ \widetilde{g}_n + \widetilde{h}_n y, & \text{if } n = \frac{ak}{\pi} \end{cases} \tag{2.9}$$

with

$$\zeta_n := \sqrt{\frac{n^2\pi^2}{a^2} - k^2}.$$

Proof: The proof is standard but it is quoted here with respect to derive explicite coefficient formulas. Applying the method of separation of variables we seek a solution of (2.1) in the form $u(x,y) = X(x)Y(y) \neq 0$. Hence

$$-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + k^2 = \lambda \qquad \text{(a constant)}.$$

The function u satisfies the zero boundary conditions on Γ_1 and Γ_2 if and only if $X(0) = X(a) = 0$. Therefore, λ and $X(x)$ have to be the eigenvalues and eigenfunctions of the problem $X'' + \lambda X = 0$, $X(0) = X(a) = 0$, i.e.

$$\lambda_n = \frac{n^2\pi^2}{a^2}, \qquad X_n(x) = \sin\frac{n\pi x}{a},\ n = 1,2\dots. \tag{2.10}$$

For λ as given in (2.10) the general solution of the equation $Y''-(\lambda-k^2)Y=0$ is

$$Y_n = \begin{cases} \alpha_n \cosh y\sqrt{\lambda_n - k^2} + \beta_n \sinh y\sqrt{\lambda_n - k^2}, & \text{if } \lambda_n \neq k^2 \\ \alpha_n + \beta_n y & \text{if } \lambda_n = k^2. \end{cases} \qquad (2.11)$$

Thus the solution of (2.5) can be written as $\sum_{n=1}^{\infty} X_n(x)Y_n(y)$, provided α_n and β_n in (2.11) are such that the Cauchy data on Γ are satisfied, i.e.

$$\widetilde{g}(x) = u(x,0) = \sum_{n=1}^{\infty} \alpha_n \sin\frac{n\pi x}{a},$$

$$\widetilde{h}(x) = u_y(x,0) = \sum_{n=1}^{\infty} \beta_n s_n \sin\frac{n\pi x}{a},$$

where $s_n = \sqrt{\lambda_n - k^2}$ for $\frac{n\pi}{a} \neq k$ and $s_n = 1$ in the case $\frac{n\pi}{a} = k$. Taking into account (2.6) and (2.7) we get (2.8) and (2.9). ∎

2.2 Series representation

The exact solution u of (2.1) belongs to $H^2(D)$ and $\partial\Omega \subset D$. The values of u on the boundaries $\Gamma_1 = 0 \times (0,b)$ and $\Gamma_2 = a \times (0,b)$ let be denoted by f_1 and f_2, respectively. Thus f_i, $i = 1,2$ are functions from $H^2(0,b)$, and according to the assumption A1

$$\|f_i\|_{H^2(0,b)} \leq \varepsilon,\ i = 1,2. \qquad (2.12)$$

Problem (2.1)is equivalent to

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \Omega \\ u(x,0) = g(x);\ u_y(x,0) = h(x) & x \in (0,a), \\ u(0,y) = f_1(y); u(a,y) = f_2(y) & y \in (0,b). \end{cases} \qquad (2.13)$$

with unknown data f_1 and f_2.

For simplicity of notation we will assume subsequently that $k \neq \frac{n\pi}{a}$.

In order to use the eigenfunction expansion method described in Subsection 2.1 we apply a simple technique for reducing the nonhomogeneous boundary conditions to the homogeneous case (cf. [12], [9], sec.6.6). We look for an auxiliary simple smooth function $P(x,y)$ satisfying only the given nonhomogeneous boundary conditions at $x = 0$ and $x = a$. We choose such a function as a polynomial with respect to x, i.e. $P(x,y) = p_0(y) + p_1(y)x$ which satisfies the conditions:

$$P(0,y) = f_1(y), \qquad P(a,y) = f_2(y).$$

Thus

$$P(x,y) = f_1(y) + \frac{f_2(y) - f_1(y)}{a}x. \tag{2.14}$$

Clearly, if u is a solution to (2.1), then the function

$$v(x,y) := u(x,y) - P(x,y) \tag{2.15}$$

is a solution of the following initial boundary value problem

$$\begin{cases} \Delta v + k^2 v = \psi, & \text{in } \Omega \\ v(x,0) = g(x) - P(x,0);\ v_y(x,0) = h(x) - P_y(x,0) & x \in (0,a)\ , \\ v(0,y) = v(a,y) = 0 & y \in (0,b), \end{cases} \tag{2.16}$$

where

$$\psi(x,y) = \psi_0(y) + \psi_1(y)x, \tag{2.17}$$

$$\psi_0(y) = -f_1''(y) - k^2 f_1(y),$$

$$\psi_1(y) = \frac{f_1''(y) - f_2''(y)}{a} + k^2 \frac{f_1(y) - f_2(y)}{a}.$$

Let

$$f_{i,0} := \lim_{y \to 0} f_i(y),\ f_{i,1} := \lim_{y \to 0} f_i'(y),\ i = 1, 2.$$

Proposition 2.2 *If $u \in H^2(D)$ and $u_{|\Omega}$ is the solution to (2.13) then for any fixed $y \in (0,b)$ u has the following convergent representation twice differentiated term by term*

$$u(x,y) = P(x,y) + \sum_{n=1}^{\infty} V_n(y) \sin \frac{n\pi x}{a}, \tag{2.18}$$

where P is given by (2.14) and $\forall n$ V_n satisfies the Cauchy problem:

$$\begin{cases} V_n'' + (-\lambda_n + k^2)V_n = \varphi_n & 0 < y < b\ , \\ V_n(0) = \widetilde{g}_n, \quad V_n'(0) = \widetilde{h}_n, \end{cases} \tag{2.19}$$

where

$$\lambda_n = \frac{n^2\pi^2}{a^2}$$

$$\varphi_n = \frac{2a}{n\pi}\left((-1)^{n+1}(\psi_0+\psi_1)+\psi_0\right),$$
$$\widetilde{g}_n = g_n - \frac{2}{n\pi}\left(f_{1,0}+f_{2,0}(-1)^{n+1}\right),$$
$$\widetilde{h}_n = h_n - \frac{2}{n\pi}\left(f_{1,1}+f_{2,1}(-1)^{n+1}\right),$$

and g_n, h_n are Fourier coefficients of the odd $2a$ periodic functions equal to g and h on $(0,a)$.

PROOF: The proof is standard but it is quoted here in order to derive the formulas for φ_n, $\widetilde{g}_n$, $\widetilde{h}_n$. Separation of variables for nonhomogeneous problem (2.16) yields to the method of eigenfunction expansion with respect to the eigenfunctions X_n of the problem $X''+\lambda X=0$, $X(0)=X(a)=0$, i.e. (cf. (2.10)) $X_n(x)=\sin\frac{n\pi x}{a}$, $n=1,2\ldots$. Since the solution of (2.16) is a twice differentiable function satisfying the homogeneous boundary conditions at $x=0$ and $x=a$, it follows that for any fixed y the solution v has the eigenfunction representation:

$$v(x,y)=\sum_{n=1}^{\infty}V_n(y)\sin\frac{n\pi x}{a}. \tag{2.20}$$

and moreover, the above series can be twice differentiated term by term with respect to x. Indeed, integrating by parts the formula for V_n we get for $n=1,\ldots$

$$V_n(y)=\frac{2}{a}\int_0^a v(x,y)\sin\frac{n\pi x}{a}dx=-\frac{a^2}{n^2\pi^2}\frac{2}{a}\int_0^a\frac{\partial^2 v}{\partial x^2}(x,y)\sin\frac{n\pi x}{a}dx,$$

since, the corresponding terms vanish for $x=0$ and $x=a$. Thus $-\frac{n^2\pi^2}{a^2}V_n(y)$ $n=1,\ldots$ are the Fourier coefficients of convergent in $L^2(0,a)$ eigenfunction representation of $\frac{\partial^2 v}{\partial x^2}$. Substituting (2.20) into (2.16) and differentiating the series term by term imply that

$$\sum_{n=1}^{\infty}\left[V_n''-\frac{n^2\pi^2}{a^2}V_n+k^2V_n\right]\sin\frac{n\pi x}{a}=\psi_0+\psi_1 x. \tag{2.21}$$

For any fixed y the right hand side of the equation above can be expanded into the convergent series

$$\widetilde{\psi}(x,y)=\sum_{n=1}^{\infty}\varphi_n(y)\sin\frac{n\pi x}{a}\quad\text{with }\varphi_n(y)=\frac{2}{a}\int_0^a\psi(x,y)\sin\frac{n\pi x}{a}dx,$$

where $\widetilde{\psi}$ is the $2a$ periodic odd function such that $\widetilde{\psi}(x,y) = \psi(x,y)$ for $x \in (0,a)$. Similarly, the functions $P(x,0)$ and $P_y^{'}(x,0)$ have the series representations for $x \in (0,a)$ of the form

$$P(x,0) = \sum_{n=1}^{\infty} p_{n,0} \sin \frac{n\pi x}{a} \text{ with } p_{n,0} = \frac{2}{a} \int_0^a P(x,0) \sin \frac{n\pi x}{a},$$

$$P_y^{'}(x,0) = \sum_{n=1}^{\infty} p_{n,1} \sin \frac{n\pi x}{a} \text{ with } p_{n,1} = \frac{2}{a} \int_0^a P_y^{'}(x,0) \sin \frac{n\pi x}{a}$$

Thus the Fourier coefficients $V_n(y)$, $n = 0, 1, \ldots$ should satisfy the following Cauchy problems

$$\begin{cases} V_n^{''}(y) + (-\lambda_n + k^2)V_n = \varphi_n(y) & 0 < y < b \ , \\ V_n(0) = g_n - p_{n,0}, \quad V_n^{'}(0) = h_n - p_{n,1} \end{cases} \tag{2.22}$$

We easily find that

$$\varphi_n(y) = \psi_0(y) \frac{2}{n\pi}((-1)^{n+1} + 1) + \psi_1(y) \frac{2a}{n\pi}(-1)^{n+1}.$$

Moreover, we get

$$p_{n,s} = f_{1,s} \frac{2}{n\pi} + f_{2,s} \frac{2}{n\pi}(-1)^{n+1}, \ s = 0, 1,$$

which ends the proof. ∎

Let us derive explicite formulas for V_n, $n = 1, 2, \cdots$.

Formally, for any $n < \infty$ the problem (2.16) splits into the pair of equations:

$$\begin{cases} \Delta \widetilde{w} + k^2 \widetilde{w} = 0, \\ \widetilde{w}(x,0) = \widetilde{g}(x); \ \widetilde{w}_y(x,0) = \widetilde{h}(x), \\ \widetilde{w}(0,y) = \widetilde{w}(a,y) = 0, \end{cases} \qquad \begin{cases} \Delta \widetilde{s} + k^2 \widetilde{s} = \psi, \\ \widetilde{s}(x,0) = \widetilde{s}_y(x,0) = 0, \\ \widetilde{s}(0,y) = \widetilde{s}(a,y) = 0 \end{cases}$$

However, these problems are ill posed and they may have no solutions.

Therefore, it is more convenient to split (2.16) into the well posed non-homogeneous Dirichlet problem

$$\begin{cases} \Delta s + k^2 s = \psi, \\ s(x,0) = s_y(x,b) = 0, \\ s(0,y) = s(a,y) = 0, \end{cases} \tag{2.23}$$

and the ill-posed Cauchy problem

$$\begin{cases} \Delta w + k^2 w = 0, \\ w(x,0) = \widetilde{g}(x),\ \ w_y(x,0) = \widetilde{h}_1(x), \\ w(0,y) = w(a,y) = 0, \end{cases} \tag{2.24}$$

with $\widetilde{g} = g - P(\cdot, 0)$, $\widetilde{h}_1 = h - P_y(\cdot, 0) - s_y(\cdot, 0)$. The solution w to (2.24) exists, since $w = v - s$ and v exist by the assumptions.

Let $S_n(y)$ and $W_n(y)$ denote the Fourier series coefficients for $s(\cdot, y)$ and $w(\cdot, y)$, respectively, i.e.

$$s(x,y) = \sum_{n=1}^{\infty} S_n(y) \sin \frac{n\pi x}{a}, \qquad w(x,y) = \sum_{n=1}^{\infty} W_n(y) \sin \frac{n\pi x}{a}.$$

The same kind of reasoning as in the proof of Proposition 2.2 shows that S_n is the solution to

$$\begin{cases} S_n'' + (k^2 - \lambda_n) S_n = \varphi_n, \\ S_n(0) = S_n(b) = 0. \end{cases} \tag{2.25}$$

Thus, for $k^2 \neq \lambda_n$, $n = 1, \ldots$ and $\zeta_n := \sqrt{\lambda_n - k^2}$

$$S_n(y) = \frac{1}{\zeta_n} \int_0^y \sinh(y - \tau)\zeta_n \varphi_n(\tau) d\tau - \frac{\sinh y\zeta_n}{\zeta_n \sinh b\zeta_n} \int_0^b \sinh(b - \tau)\zeta_n \varphi_n(\tau) d\tau. \tag{2.26}$$

Moreover, according to Lemma 2.1 W_n is the solution to

$$\begin{cases} W_n'' + (k^2 - \lambda_n) W_n = 0, \\ W_n(0) = \widetilde{g}_n, \quad W_n'(0) = \widetilde{h}_{1n}, \end{cases} \tag{2.27}$$

where $\widetilde{h}_{1n}$ are Fourier coefficients of the odd $2a$-periodic function equal to $\widetilde{h}_1$ on the interval $(0, a)$, and

$$W_n(y) = \widetilde{g}_n \cosh y\zeta_n + \frac{1}{\zeta_n} \widetilde{h}_{1n} \sinh y\zeta_n. \tag{2.28}$$

Since $\widetilde{h}_{1n} = \widetilde{h}_n - S_n'(0)$ and

$$S_n'(0) = -\int_0^b \frac{\sinh(b - \tau)\zeta_n}{\sinh b\zeta_n} \varphi_n(\tau) d\tau,$$

from (2.26) and (2.28) follows the following

Lemma 2.3 *If the assumption of Proposition 2.2 are satisfied and $k \neq n\frac{\pi}{a}$ then*

$$V_n(y) = \widetilde{g}_n \cosh y\zeta_n + \frac{\widetilde{h}_n}{\zeta_n} \sinh y\zeta_n + \frac{1}{\zeta_n} \int_0^y \sinh((\tau - y)\zeta_n) \varphi_n(\tau) d\tau, \tag{2.29}$$

where $\zeta_n := \sqrt{\lambda_n - k^2}$.

3 Identification u from inexact boundary data

The representation (2.18) of u depends on g, h as well as on the unknown traces of u onto Γ_1 and Γ_2. For solving the reconstruction problem P1 we are going to propose a spectral type regularization method which does not use unknown f_i, $i = 1, 2$

Let $\alpha \in (0, 1)$ and

$$n_\alpha := \max\{n : \cosh b\sqrt{\lambda_n - k^2} \le \frac{1}{\alpha}\}. \tag{3.1}$$

A regularization solution let be defined as follows:

$$u_\alpha^\delta := \sum_{n=1}^{n_\alpha} V_n^\delta(y) \sin\frac{n\pi x}{a}, \tag{3.2}$$

where V_n^δ is the solution of the equation (2.19) with the right hand side equals 0 and the initial data $V_n^\delta(0) = g_n^\delta$ and $\frac{d}{dy}V_n^\delta(0) = h_n^\delta$. Thus

$$V_n^\delta = g_n^\delta \cosh y\zeta_n + \frac{1}{\zeta_n} h_n^\delta \sinh y\zeta_n. \tag{3.3}$$

Let us observe that for any $n_\alpha < \infty$ the function u_α^δ is well defined.

Moreover, let us introduce an auxiliary function

$$u_\alpha := P(x, y) + \sum_{n=1}^{n_\alpha} V_n(y) \sin\frac{n\pi x}{a}. \tag{3.4}$$

Since the series (2.18) is convergent, $\forall y \in (0, b]$

$$\|u(\cdot, y) - u_\alpha(\cdot, y)\|_{L^2(0,a)} \to 0 \text{ as } \alpha \to 0. \tag{3.5}$$

Proposition 3.1 *If the assumptions A1 and (2.2) are satisfied and $k \neq n\frac{\pi}{a}$, then $\forall y \in (0, b]$ and $\alpha \in (0, 1)$*

$$\|u_\alpha^\delta(\cdot, y) - u_\alpha(\cdot, y)\|_{L^2(0,a)} \le c_1\frac{\delta}{\alpha} + c_2\frac{\varepsilon}{\alpha}, \tag{3.6}$$

where $c_1 = \sqrt{3 + 3b^2}$, $c_2 = 2\sqrt{2}(1 + b) + 4\sqrt{b^3}(a + 1) + \sqrt{a}$.

PROOF: According to (3.2) and (3.4)

$$\|u_\alpha^\delta(\cdot, y) - u_\alpha(\cdot, y)\| \le \|P(\cdot, y)\| + \left(\sum_{n=1}^{n_\alpha} |V_n(y) - V_n^\delta(y)|^2\right)^{\frac{1}{2}}.$$

For $n \le n_\alpha$ $\forall y \in (0, b]$ $\cosh y\zeta_n \le \frac{1}{\alpha}$ and, since the function $z \cosh z - \sinh z$ is increasing for $z \in (0, \infty)$ and vanishes at 0, for $\lambda_n \ge k^2$

$$\frac{\sinh y\zeta_n}{y\zeta_n} \le \frac{1}{\alpha}.$$

The same inequality holds for $\lambda_n < k^2$, since $\sin z < z$ for $z > 0$. Thus, using formulas (3.3) and (2.29), we get

$$|V_n(y) - V_n^\delta(y)| \le \frac{1}{\alpha} \left[|\widetilde{g}_n - g_n^\delta| + y|\widetilde{h}_n - h_n^\delta| + y^{\frac{3}{2}} \|\varphi_n\|_{L^2(0,b)} \right].$$

If the assumption A1 is satisfied then (2.12) holds and it follows

$$|\widetilde{g}_n - g_n^\delta| \le |g_n - g_n^\delta| + \frac{4\varepsilon}{n\pi}, \quad |\widetilde{h}_n - h_n^\delta| \le |h_n - h_n^\delta| + \frac{4\varepsilon}{n\pi},$$

and from (2.2)

$$\sum_{n=1}^{\infty} |g_n - g_n^\delta|^2 \le \delta^2, \quad \sum_{n=1}^{\infty} |h_n - h_n^\delta|^2 \le \delta^2.$$

Moreover, according to the definition of φ_n (cf. Proposition 2.2)

$$\|\varphi_n\|_{L^2(0,b)} \le \frac{2\sqrt{2}}{n\pi} \left[(2a+1)\|f_1\|_{H^2(0,b)} + \|f_2\|_{H^2(0,b)} \right] \le \frac{\varepsilon}{n} \frac{4\sqrt{2}}{\pi}(a+1).$$

Thus we have

$$|V_n(y) - V_n^\delta(y)| \le \frac{1}{\alpha} \left[|g_n - g_n^\delta| + b|h_n - h_n^\delta| + \frac{\varepsilon}{n} C(a, b) \right],$$

where $C(a, b) = \frac{4}{\pi} \left(1 + b + \sqrt{2b^3}(a+1) \right)$. So,

$$\sum_{n=1}^{n_\alpha} |V_n(y) - V_n^\delta(y)|^2 \le \frac{3}{\alpha^2} \left((1+b^2)\delta^2 + \varepsilon^2 C^2(a, b) \sum_{n=1}^{n_\alpha} \frac{1}{n^2} \right) \le$$

$$\le \frac{\delta^2}{\alpha^2} 3 \left(1 + b^2\right) + \frac{\varepsilon^2}{\alpha^2} C^2 \frac{\pi^2}{2}.$$

Moreover,

$$\|P(\cdot, y)\|_{L^2(0,a)}^2 = \frac{a}{3} \left(f_1(y)^2 + f_1(y) f_2(y) + f_2(y)^2 \right) \le a\varepsilon^2$$

which ends the proof. ∎

Now, we are going to estimate an order of convergence for (3.5). We have

$$\|u(\cdot,y)-u_\alpha(\cdot,y)\| \le \left(\sum_{n>n_\alpha} S_n^2(y)\right)^{\frac{1}{2}} + \left(\sum_{n>n_\alpha} W_n^2(y)\right)^{\frac{1}{2}}, \qquad (3.7)$$

where S_n and W_n are given by formulas (2.26), (2.28), respectively.

Due to definition (3.1), if $n > n_\alpha$ then $n > k\frac{a}{\pi}$, i.e. $\zeta_n > 0$ and

$$0 < \frac{1}{\sinh b\zeta_n} < 2\alpha. \qquad (3.8)$$

Lemma 3.2 *If S_n is a solution to (2.25), then for $n > n_\alpha$*

$$|S_n(y)| \le \frac{2\sqrt{b}}{\zeta_n}\|\varphi_n\|_{L^2(0,b)} \qquad (3.9)$$

PROOF: For any fixed $y \in (0,b]$ let us define the following two auxiliary functions on the intervals $(0,y)$ and (y,b), respectively:

$$q_{1,n}(y,\tau) := \sinh(y-\tau)\zeta_n - \frac{\sinh y\zeta_n}{\sinh b\zeta_n} sinh(b-\tau)\zeta_n \qquad \text{for } \tau \in (0,y),$$

$$q_{2,n}(y,\tau) := -\frac{\sinh y\zeta_n}{\zeta_n \sinh b\zeta_n}\sinh(b-\tau)\zeta_n \qquad \text{for } \tau \in (y,b).$$

According to (2.26), $S_n(y)$ may be split into two parts:

$$S_{1,n}(y) := \frac{1}{\zeta_n}\int_0^y q_{1,n}(y,\tau)\varphi_n(\tau)d\tau,$$

$$S_{2,n}(y) := \frac{1}{\zeta_n}\int_y^b q_{2,n}(y,\tau)\varphi_n(\tau)d\tau.$$

It is easy to see that

$$\sup_{0<\tau<y}|q_{1,n}(y,\tau)| = |q_{1,n}(y,y|) = \frac{\sinh y\zeta_n \sinh(b-y)\zeta_n}{\sinh b\zeta_n}$$

and

$$\sup_{y<\tau<b}|q_{2,n}(y,\tau)| = |q_{2,n}(y,y) = \frac{\sinh y\zeta_n \sinh(b-y)\zeta_n}{\sinh b\zeta_n}.$$

Thus

$$|S_n(y)| \le \frac{\sqrt{b}}{\zeta_n}\|\varphi_n\|_{L^2(0,b)}|q_{1,n}(y,y)|$$

The function $|q_{1,n}(y,y)|$ attains its supremum at $y=\frac{b}{2}$, which is equal to

$$\frac{\sinh^2 \frac{b}{2}\zeta_n}{\sinh b\zeta_n} < 2.$$

Hence (3.9) is proved. ∎

Lemma 3.3 *Let W_n be a solution to (2.27). If A1 is satisfied then for $n > n_\alpha$*

$$|W_n(y)|^2 \le |W_n(b)|^2 + g_n^2 + \frac{16\varepsilon^2}{n^2\pi^2} + \frac{1}{\zeta_n^2}\left(h_n^2 + 2\frac{16\varepsilon}{n^2\pi^2} + 2b\|\varphi_n\|^2\right). \quad (3.10)$$

PROOF: The formula (2.28) may be written as follows

$$W_n(y) = \frac{1}{2}\left[\left(\widetilde{g}_n + \frac{\widetilde{h}_{1,n}}{\zeta_n}\right)e^{y\zeta_n} + \left(\widetilde{g}_n - \frac{\widetilde{h}_{1,n}}{\zeta_n}\right)e^{-y\zeta_n}\right].$$

Thus,

$$|W_n(y)|^2 \le |W_n(b)|^2 + \frac{1}{4}|\widetilde{g}_n - \frac{\widetilde{h}_{1,n}}{\zeta_n}|^2. \quad (3.11)$$

Since by A1 $|f_{1,i}| + |f_{2,i}| \le 2\varepsilon$ for $i = 0,1$, we get

$$|\widetilde{g}_n| \le |g_n| + C\frac{\varepsilon}{n}, \qquad |\widetilde{h}_n| \le |h_n| + C\frac{\varepsilon}{n},$$

where $C=\frac{4}{\pi}$. Moreover, due to (2.24) and (2.27),

$$\widetilde{h}_{1,n} = \widetilde{h}_n - S_n'(0).$$

From 2.26 it follows

$$|S_n'(0)| = \left|\int_0^b \frac{\sinh(b-\tau)\zeta_n}{\sinh b\zeta_n}\varphi_n(\tau)d\tau\right| \le \sqrt{b}\|\varphi_n\|_{L^2(0,b)}.$$

Thus

$$|\widetilde{h}_{1,n}| \le |h_n| + \frac{C\varepsilon}{n} + \sqrt{b}\|\varphi_n\|_{L^2(0,b)},$$

and finally

$$\frac{1}{4}|\widetilde{g}_n - \frac{\widetilde{h}_{1,n}}{\zeta_n}|^2 \le g_n^2 + \frac{C^2\varepsilon^2}{n^2} + \frac{1}{\zeta_n^2}\left(h_n^2 + \frac{2C^2\varepsilon^2}{n^2} + 2b\|\varphi_n\|^2\right),$$

which together with (3.11) give the desired estimation. ∎

Proposition 3.4 *Let $k \neq n\frac{\pi}{a}$. If the assumptions A1 and A2 are satisfied then* $\exists C_1, C_2\ \forall y \in (0, b]$

$$\|u(\cdot, y) - u_\alpha(\cdot, y)\|_{L^2(0,a)} \leq C_1(arcosh\frac{1}{\alpha})^{-1} + C_2\varepsilon(arcosh\frac{1}{\alpha})^{-\frac{1}{2}} \tag{3.12}$$

and the constants C_1, C_2 depend on a, b and k.

PROOF: First, let as observe that from the definition of n_a it follows that

$$n_\alpha = E\left(\frac{a}{\pi}\sqrt{k^2 + \frac{1}{b^2}arcosh^2\frac{1}{\alpha}}\right) \geq E\left(\frac{a}{b\pi}arcosh\frac{1}{\alpha}\right),$$

where $E(x) := max\{n \in \mathbb{N} : n \leq x\}$. Thus

$$\frac{1}{n_\alpha} < \frac{1}{\frac{a}{b\pi}arcosh\frac{1}{\alpha} - 1},$$

and for $n > n_\alpha$

$$\frac{1}{\zeta_n} < \frac{b}{arcosh\frac{1}{\alpha}} \text{ and } \frac{1}{n} < \frac{\pi}{a}\frac{b}{arcosh\frac{1}{\alpha}}. \tag{3.13}$$

Therefore,

$$\sum_{n>n_\alpha}\frac{1}{n^2} \leq \frac{1}{n_\alpha} \leq \frac{1}{\frac{a}{b\pi}arcosh\frac{1}{\alpha} - 1}.$$

Now, let us return to the inequality (3.7). First, observe that by the assumption A1

$$\|\varphi_n\| \leq \frac{1}{n}\frac{2a}{\pi}(2\|\psi_0\| + \|\psi_1\|) \leq \frac{\varepsilon}{n}\frac{6a}{\pi}. \tag{3.14}$$

and according to (3.13)

$$\sum_{n>n_\alpha}\frac{1}{\zeta_n^2}\|\varphi_n\|^2 \leq \frac{(6a)^2}{\pi^2}\frac{\varepsilon^2}{(arcosh\frac{1}{\alpha})^2}\sum_{n>n_\alpha}\frac{1}{n^2}.$$

Thus from Lemma 3.2

$$\sum_{n>n_\alpha}S_n^2(y) \leq 4b\sum_{n>n_\alpha}\frac{1}{\zeta_n^2}\|\varphi_n\|^2 \leq (5a\sqrt{b})^2\frac{\varepsilon^2}{(arcosh\frac{1}{\alpha})^2}. \tag{3.15}$$

For estimating the second term of (3.7) we have to use the assumption A2, i.e. $\|u'_x(\cdot, b)\| \leq M$. Since

$$u'_x(x, b) = P'_x(x, b) + \frac{\pi}{a}\sum_1^\infty n(W_n(b) + S_n(b))\cosh\frac{n\pi x}{a}$$

and $\|P'_x(\cdot,b)\| \leq \frac{\varepsilon}{\sqrt{a}}$, we have

$$\frac{\pi^2}{a^2}\sum_1^\infty n^2\left(W_n(b)+S_n(b)\right)^2 \leq (M+\varepsilon(\sqrt{a})^{-1})^2.$$

Moreover, from Lemma 3.2 and from (3.14) it follows

$$n^2 S_n^2(b) \leq \left(\frac{12a\varepsilon}{\pi}\right)^2 \frac{1}{\zeta_n^2}.$$

Since for $n < k\frac{a}{\pi}$ $\frac{1}{\zeta_n} < \frac{1}{k}$ and for $n > k\frac{a}{\pi}$ $\frac{1}{\zeta_n} < \frac{a}{n\pi}$,

$$\sum_1^\infty \frac{1}{\zeta_n^2} \leq \frac{a}{k\pi} + \frac{a^2}{6}.$$

Thus

$$\sum_1^\infty n^2 S_n^2(b) \leq \varepsilon^2\frac{(12a\sqrt{b})^2}{\pi^2}(\frac{a}{k\pi}+\frac{a^2}{6}).$$

where $C = 12\sqrt{\frac{b^2\pi}{ak^2}+\frac{a^2b}{6}}$. Therefore

$$\sum_1^\infty n^2 W_n^2(b) \leq M_\varepsilon := 2\frac{a^2}{\pi^2}\left((M+\varepsilon(\sqrt{\alpha})^{-1})^2 + C^2\varepsilon^2\right).$$

It follows that

$$\sum_{n>n_\alpha} W_n^2(b) \leq \frac{1}{n_\alpha^2+1}\sum_{n>n_\alpha} n^2 W_n^2(b) \leq \frac{b^2\pi^2}{a^2 arcos^2\frac{1}{\alpha}} M_\varepsilon.$$

Let G and H denote the following upper bounds: $\|g'\|_{L^2(0,a)} \leq G$ and $\|h\|_{L^2(0,a)} \leq H$. We get

$$\sum_{n>n_\alpha} g_n^2 \leq \frac{1}{n_\alpha^2+1}\sum_{n>n_\alpha} n^2 g_n^2 \leq \frac{G^2b^2\pi}{a^2}\left(\frac{1}{arcosh\frac{1}{\alpha}}\right)^2,$$

$$\sum_{n>n_\alpha}\frac{1}{\zeta_n^2}h_n^2 \leq H^2b^2\left(\frac{1}{arcosh\frac{1}{\alpha}}\right)^2.$$

Finally,

$$\sum_{n>n_\alpha}\frac{\varepsilon^2}{n^2} \leq C\frac{\varepsilon^2}{arcosh\frac{1}{\alpha}}$$

So, we conclude that the sums of all terms appearing on the right hand sides of (3.9) and (3.10) are at least of the order

$$O\left((arcosh\frac{1}{\alpha})^{-2}\right) \qquad \text{or} \qquad O\left(\varepsilon^2(arcosh\frac{1}{\alpha})^{-1}\right),$$

which completes the proof. ∎

Finally, summarizing the results above, we come to the following error estimate:

Theorem 3.5 *Let $u \in H^2(D)$ be the exact solution to (2.1) and u_α^δ be the regularized solution defined by (3.2) for noisy data (2.2). If the assumptions A1 and A2 are satisfied then there exists constants C_1, C_2 such that $\forall y \in (0,b]$*

$$\|u(\cdot,y)-u_\alpha^\delta(\cdot,y)\|_{L^2(0,a)} \leq C_1\frac{\delta+\varepsilon}{\alpha}+C_2\frac{1}{arcosh\frac{1}{\alpha}}\left(1+\varepsilon\sqrt{arcosh\frac{1}{\alpha}}\right) \quad (3.16)$$

An open question is how to choose the regularization parameter α in order to minimize the above error bound for given δ and ε. Naturally, we have no convergence when the data error bound δ tends to 0 because of ε. However, in the model considered, ε decrease when the length of Γ increase, so we may formulate the following remark:

Remark 3.6 *Let D be an infinite strip and $\Omega := (x_-, x_+) \times (0,d) \subset D$. Let $\forall \varepsilon \exists x_-(\varepsilon), x_+(\varepsilon)$ such that*

$$\|u(x_\pm.\cdot)\|_{H^2(o,b)} \leq \varepsilon.$$

Thus if $\|u'(\cdot,b)\|_{L^2(\mathbb{R})} \leq M$ and $\varepsilon = \delta$ then

$$\|u(\cdot,y)-u_\alpha^\delta(\cdot,y)\|_{L^2(x_-,x_+)} \leq \widetilde{C}_1\frac{\delta}{\alpha}+\widetilde{C}_2\frac{1}{arcosh\frac{1}{\alpha}}\left(1+\delta\sqrt{arcosh\frac{1}{\alpha}}\right).$$

4 Conclusion

The difference between our formulation of the Cauchy problem for the Helmholtz equation on a rectangle and previous ones consists in the fact that data are only given on the one side of the rectangle. In such a case, additional homogeneous or periodic boundary conditions on the sides parallel to the beam axis have no clear physical meaning and, usually, the problem is formulated on infinite strip which allow to apply the Fourier transform. The approach presented in the paper is an alternative way to analyze the problem. Under the assumption that the collimated laser beam is such that A1 is satisfied, we propose a series expansion approach which yields to series representation of the exact solution. This representation can be used for formulation of different regularization methods. An example of such a method is proposed and its stability and error bound are shown. The problem of choice of regularization parameter for this method is not undertaken here and will be a subject of a subsequent paper.

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