IM PAN Preprint 737 (2012)

Piotr M. Hajac and Bartosz Zieliński

The cocycle condition
for
multi-pullbacks of algebras

THE COCYCLE CONDITION
FOR MULTI-PULLBACKS OF ALGEBRAS

PIOTR M. HAJAC AND BARTOSZ ZIELIŃSKI

ABSTRACT. Take finitely many topological spaces and for each pair of these spaces choose a pair of corresponding closed subspaces that are identified by a homeomorpism. We note that this gluing procedure does not guarantee that the building pieces, or the gluings of some pieces, are embedded in the space obtained by putting together all given ingredients. Dually, we show that a certain sufficient condition, called the cocycle condition, is also necessary to guarantee sheaf-like properties of surjective multi-pullbacks of algebras with distributive lattices of ideals.

When constructing a topological space as the gluing of pieces, it is desirable that the parts are embedded into the described space. The gluing of three intervals $I_1 \cong I_2 \cong I_3 \cong [-1, 1]$ into the space T_* described by Fig. 1 fails this property as the endpoints of I_2 and I_3 are glued into a single point:
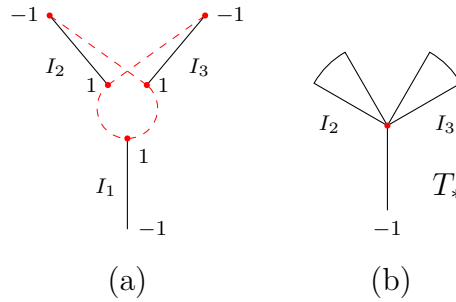

(a) (b)

FIGURE 1

There is, however, a more subtle way in which a gluing may fail to embed its parts into the whole space. To see this, consider another gluing of I_1, I_2 and I_3 depicted on Fig. 2(a) into the space $T_\circ$ pictured on Fig. 2(b). All the I_j's are embedded into $T_\circ$ but the partial
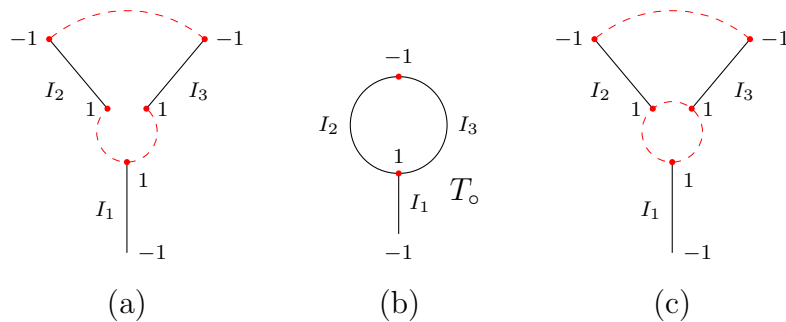

(a) (b) (c)

FIGURE 2

gluing of I_2 and I_3 is not. Of course one can define an alternative gluing procedure of I_j's into $T_\circ$ (see Fig. 2(c)) for which all partial gluings are embedded into $T_\circ$.

Let us now consider the problem of gluing from the point of view of algebras. Let J be a finite set, and let

$$\{\pi_j^i : B_i \longrightarrow B_{ij} = B_{ji}\}_{i,j \in J, i \neq j} \tag{1}$$

be a family of algebra homomorphisms.

Definition 1 ([1, 5]). *The* multi-pullback algebra B^π *of a family* (1) *of algebra homomorphisms is defined as*

$$B^\pi := \left\{ (b_i)_i \in \prod_{i \in J} B_i \ \middle| \ \pi_j^i(b_i) = \pi_i^j(b_j), \ \forall i, j \in J, \ i \neq j \right\}.$$

Definition 2. *A family* (1) *of algebra homomorphisms is called* distributive *if and only if all of them are surjective and their kernels generate a distributive lattice of ideals.*

The multi-pullback algebra of a distributive family of homomorphisms is the main mathematical concept of this note, and plays a key role in [3, 2, 4]. In particular, it includes the multi-pullbacks of all finite families of C^*-epimorphisms between unital C^*-algebras. In case of commutative unital C^*-algebras, such a multi-pullback C^*-algebra can be identified with the algebra of all continuous functions on the compact Hausdorff space obtained by the gluing procedure described in the abstract applied to compact Hausdorff spaces.

Example 1. Consider the C^*-algebra $C(T_*)$ of all continuous functions on T_* as the multi-pullback C^*-algebra corresponding to the gluing depicted on Fig. 1(a). Here we take $B_i = C(I_i)$, $i = 1, 2, 3$, $B_{12} = B_{13} = \mathbb{C}$, $B_{23} = \mathbb{C} \oplus \mathbb{C}$, and define C^*-epimorphisms by the formulae

$$\pi_2^1 = \pi_1^2 = \pi_3^1 = \pi_1^3 : f \mapsto f(1), \quad \pi_3^2 : f \mapsto (f(-1), f(1)), \quad \pi_2^3 : f \mapsto (f(1), f(-1)).$$

The fact that I_2 is not embedded in T_* corresponds to the non-surjectivity of the canonical projection $B^\pi \to B_2$.

Example 2. Consider the C^*-algebra $C(T_\circ)$ of all continuous functions on $T_\circ$ as the multi-pullback C^*-algebra corresponding to the gluing depicted on Fig. 2(a). Here we take $B_i = C(I_i)$, $B_{ij} := \mathbb{C}$, $1 \leq i, j \leq 3$, $i \neq j$, and define C^*-epimorphisms by the formulae

$$\pi_2^1 = \pi_1^2 = \pi_3^1 = \pi_1^3 : f \mapsto f(1), \quad \pi_3^2 = \pi_2^3 : f \mapsto f(-1).$$

While the canonical projections $B^\pi \to B_i$ are all surjective, the canonical projection $B^\pi \to \{(b_2, b_3) \in B_2 \times B_3 \mid \pi_3^2(b_2) = \pi_2^3(b_3)\}$ is not. Indeed, a pair $b_2 := (t \mapsto t)$, $b_3 := (t \mapsto -1)$ satisfies $\pi_3^2(b_2) = \pi_2^3(b_3)$, but there is no function $b_1 \in B_1$ such that $(b_1, b_2, b_3) \in B^\pi$. This corresponds to the fact that the gluing of I_2 and I_3 is not embedded in $T_\circ$.

Example 3. We can present the C^*-algebra $C(T_\circ)$ of all continuous functions on $T_\circ$ pictured in Fig. 2(b) by using different multi-pullbacks: one corresponding to the gluing depicted in Fig. 2(a) (see Example 2) and one corresponding to the gluing depicted in Fig. 2(c). For the latter case, we take the B_i's, B_{12}, B_{13}, π_2^1, π_1^2, π_3^1, π_1^3 as in Example 2, but we put $B_{23} := \mathbb{C} \oplus \mathbb{C}$ and $\pi_3^2 = \pi_2^3 : f \mapsto (f(-1), f(1))$. Now not only the canonical projections $B^\pi \to B_i$ are all surjective, but also, for all distinct i, j, k and all $b_i \in B_i$, $b_j \in B_j$ such that $\pi_j^i(b_i) = \pi_i^j(b_j)$, there exists $b_k \in B_k$ such that $\pi_k^i(b_i) = \pi_i^k(b_k)$ and $\pi_k^j(b_j) = \pi_j^k(b_k)$.

It turns out that the cocycle condition defined below is a perfect tool to understand the differences between the above examples. To define the cocycle condition, for any distinct i, j, k we put $B^i_{jk} := B_i/(\ker \pi^i_j + \ker \pi^i_k)$ and take $[\cdot]^i_{jk} : B_i \to B^i_{jk}$ to be the canonical surjections. Next, we introduce the family of isomorphisms

$$(2) \qquad \pi^{ij}_k : B^i_{jk} \longrightarrow B_i/\pi^i_j(\ker \pi^i_k), \qquad [b_i]^i_{jk} \longmapsto \pi^i_j(b_i) + \pi^i_j(\ker \pi^i_k).$$

Now we are ready for:

Definition 3. [1, in Proposition 9] *We say that a family* (1) *of algebra epimorphisms satisfies the* cocycle condition *if and only if, for all distinct* $i, j, k \in J$,

(1) $\pi^i_j(\ker \pi^i_k) = \pi^j_i(\ker \pi^j_k)$,

(2) *the isomorphisms* $\phi^{ij}_k := (\pi^{ij}_k)^{-1} \circ \pi^{ji}_k : B^j_{ik} \to B^i_{jk}$ *satisfy* $\phi^{ik}_j = \phi^{ij}_k \circ \phi^{jk}_i$.

It was proven in [1] that if a distributive family of π^i_j's satisfies the cocycle condition, then the canonical projections $B^\pi \to B_i$ are all surjective. In particular, the multi-pullback from Example 1 cannot satisfy the cocycle condition. The multi-pullback presentation of $C(T_\circ)$ from Example 2 demonstrates, however, that the cocycle condition is not necessary for the canonical projections $B^\pi \to B_i$ to be surjective. Indeed, they are all clearly surjective in this case, but π^i_j's do not satisfy the cocycle condition because $\pi^1_2(\ker \pi^1_3) = \{0\}$ whereas $\pi^2_1(\ker \pi^2_3) = \mathbb{C}$. On the other hand, the cocycle condition is satisfied by an alternative multi-pullback presentation of $C(T_\circ)$ given in Example 3. This suggests that the cocycle condition is related to the possibility of extending partial multi-pullbacks. Thus we arrive at the main result of this note:

Theorem 1. *The following statements about a distributive family* (1) *of algebra homomorphisms are equivalent:*

(1) *The family* (1) *satisfies the cocycle condition.*

(2) *For any $K \subsetneq J$, $k \in J \setminus K$ and $(b_l)_{l \in K} \in \prod_{l \in K} B_l$ such that $\pi^i_j(b_i) = \pi^j_i(b_j)$ for all distinct $i, j \in K$, there exists $b_k \in B_k$ such that $\pi^l_k(b_l) = \pi^k_l(b_k)$ for all $l \in K$.*

(3) *For all distinct $i, j, k \in J$ and all $b_i \in B_i$, $b_j \in B_j$ such that $\pi^i_j(b_i) = \pi^j_i(b_j)$, there exists $b_k \in B_k$ such that also $\pi^i_k(b_i) = \pi^k_i(b_k)$ and $\pi^j_k(b_j) = \pi^k_j(b_k)$.*

Proof. The proof of (1) $\Rightarrow$ (2) is essentially identical with the proof of [1, Proposition 9], and (3) is obviously a special case of (2). In order to prove (3) $\Rightarrow$ (1) and close the loop of implications, assume that for any distinct $i, j, k \in J$ and for arbitrary elements $b_i \in B_i$ and $b_j \in B_j$ such that $\pi^i_j(b_i) = \pi^j_i(b_j)$ there exists $b_k \in B_k$ such that also $\pi^i_k(b_i) = \pi^k_i(b_k)$ and $\pi^j_k(b_j) = \pi^k_j(b_k)$. Specializing this condition for $b_j = 0$ yields that for any $b_i \in \ker \pi^i_j$ there exists a $b_k \in \ker \pi^k_j$ such that $\pi^i_k(b_i) = \pi^k_i(b_k)$, that is $\pi^i_k(\ker \pi^i_j) \subseteq \pi^k_i(\ker \pi^k_j)$. Exchanging i and k we obtain the set equality. This proves Condition (1) defining the cocycle condition.

To prove the second condition observe that, for all distinct $i, j, k \in J$ and any $b_i \in B_i$, $b_j \in B_j$,

$$(3) \qquad [b_i]^i_{jk} = \phi^{ij}_k([b_j]^j_{ik}) \iff \pi^{ji}_k([b_j]^j_{ik}) = \pi^{ij}_k([b_i]^i_{jk}) \iff \pi^i_j(b_i) - \pi^j_i(b_j) \in \pi^i_j(\ker \pi^i_k).$$

Now let us pick any distinct $i, j, k \in J$ and any $b_j \in B_j$. Since π^k_j is surjective, there exists $b_k \in B_k$ such that $\pi^k_j(b_k) = \pi^j_k(b_j)$, so that $[b_k]^k_{ji} = \phi^{kj}_i([b_j]^j_{ik})$ by (3). Furthermore, by

assumption, there exists $b_i \in B_i$ such that $\pi_k^i(b_i) = \pi_i^k(b_k)$ and $\pi_j^i(b_i) = \pi_i^j(b_j)$. Therefore, again by (3), we obtain

$$(4) \qquad [b_i]_{jk}^i = \phi_j^{ik}([b_k]_{ji}^k) = \phi_j^{ik}(\phi_i^{kj}([b_j]_{ik}^j)) \quad \text{and} \quad [b_i]_{jk}^i = \phi_k^{ij}([b_j]_{ik}^j).$$

Plugging in the second equality to the first one, we get $\phi_k^{ij}([b_j]_{ik}^j) = \phi_j^{ik}(\phi_i^{kj}([b_j]_{ik}^j))$ for any $[b_j]_{ik}^j \in B_{ik}^j$, as needed. $\qquad\square$

Finally, let us remark that the fact that in Example 3 we could remedy the lack of the cocycle condition in Example 2 is not a coincidence. Indeed, following [1, Proposition 4], one sees that, if B^π is the multi-pullback a family (1) such that the canonical projections $B^\pi \to B_i$ are all surjective and their kernels generate a distributive lattice of ideals, then B^π can also be presented as the multi-pullback of a family satisfying the cocycle condition even if the original family failed to do so. Herein the new family is defined via the canonical surjections:

$$(5) \qquad \{\pi_j^{'i} : B_i \cong B^\pi / \ker(B^\pi \to B_i) \longrightarrow B^\pi / (\ker(B^\pi \to B_i) + \ker(B^\pi \to B_j))\}_{i,j \in J, i \neq j}.$$

The aforementioned example is a special case of this general claim because C^*-ideals always generate a distributive lattice.

REFERENCES

[1] D. Calow and R. Matthes. *Covering and gluing of algebras and differential algebras.* J. Geom. Phys. 32 (2000), 364–396.

[2] P.M. Hajac, A. Kaygun and B. Zieliński. *Quantum projective spaces from Toeplitz cubes.* To appear in J. Noncommut. Geom.

[3] P.M. Hajac, U. Krähmer, R. Matthes and B. Zieliński. *Piecewise principal comodule algebras.* J. Noncommut. Geom. 5 (2011), 591–614.

[4] J. Rudnik. *The K-Theory of the triple-Toeplitz deformation of the complex projective plane.* Submitted to the same proceedings.

[5] G. K. Pedersen. *Pullback and pushout constructions in C*-algebra theory.* J. Funct. Anal. 267 (1999), 243–344.

INSTYTUT MATEMATYCZNY, POLSKA AKADEMIA NAUK, UL. ŚNIADECKICH 8, WARSZAWA, 00-956 POLAND, KATEDRA METOD MATEMATYCZNYCH FIZYKI, UNIWERSYTET WARSZAWSKI, UL. HOŻA 74, WARSZAWA, 00-682 POLAND

E-mail address: http://www.impan.pl/~pmh

DEPARTMENT OF THEORETICAL PHYSICS AND COMPUTER SCIENCE, UNIVERSITY OF ŁÓDŹ, POMORSKA 149/153 90-236 ŁÓDŹ, POLAND

E-mail address: bzielinski@uni.lodz.pl