INSTITUTE OF MATHEMATICS
of the
Polish Academy of Sciences

ul. Śniadeckich 8, P.O.B. 21, 00-956 Warszawa 10, Poland http://www.impan.pl

IM PAN Preprint 741 (2012)

Hassan Babiker and Stanisław Janeczko

Combinatorial cycles of tetrahedral chains

Published as manuscript

Received 05 November 2012

COMBINATORIAL CYCLES OF TETRAHEDRAL CHAINS

HASSAN BABIKER & STANISŁAW JANECZKO

Abstract. Tetrahedral chains build of face-sharing regular tetrahedrons in Euclidean three space are investigated. Explicit formula for the positions of all vertices and the complete description of the geometric structure with optimal folding of tetrahedral chains are obtained. Parametrization of chains by sequences of ordered reflections is constructed and periodicity in their combinatorial structure is found. It is based on the structure of sequences of admissible triplets of integers and their cycling properties. The corresponding numerical invariants and the indexing role of the tetrahedral group were discovered.

1. Introduction.

The simplest naturally ordered tetrahedral packing is built of an ordered sequence of regular tetrahedra glued together face to face like linear packing of tetrahedral helix (Figure 1) introduced implicitly by H.S.M. Coxeter in [1]. Such tetrahedral structures, studied already by several authors [7, 8, 10] are called *tetrahedral chains*.

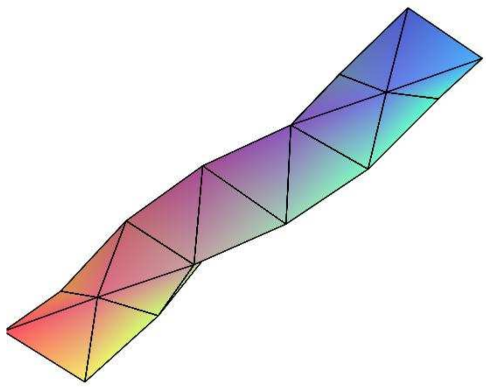

Figure 1. Tetrahelix

Date: November 5, 2012.

1991 *Mathematics Subject Classification.* 53D05, 51N10, 53D22, 70H05, 15A04.

Key words and phrases. sphere packing, tetrahedrons.

The second author was partially supported by the grant of Polish Ministry of Science MNiSzW, No. 2006-2009 and grant No. 201 397237.

Any tetrahedral chain is build of three types of simplest configurations of four consecutive tetrahedrons called *tetrahedral units*. Two types are left and right tetrahedral short spirals, U, D, and the third type, F, is a flat configuration of four tetrahedrons (Figure 2).

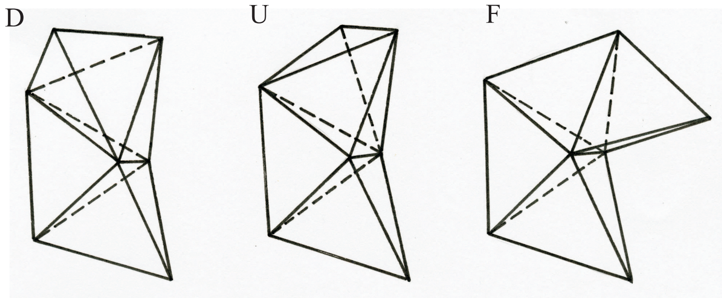

FIGURE 2. Tetrahedral units

The three strands of the left or right oriented tetrahedral helix form a spiral with irrational slope. This is a reason of effective density of tetrahedral chains and as it was proved by J.H.Mason in [7] nonexistence of closed tetrahedral chains in Euclidean space (Figure 3).

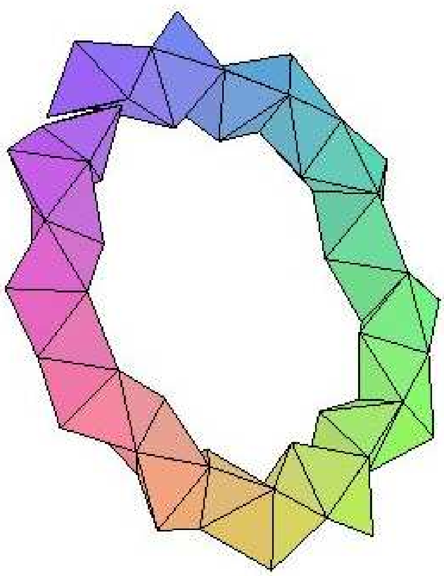

FIGURE 3. Semi-ring structure

Let us assume that the gluing process of tetrahedrons is ordered along the chain and each step is realized by reflection in a proper face of a consecutive tetrahedron. To each tetrahedron we prescribe four reflections $R_i, i = 1, \ldots, 4$ in the configurational three dimensional space V. Reflections R_i in V are represented by four corresponding operators twist-morphisms $\bar{R}_i, i = 1, \dots, 4$ acting in the space of regular tetrahedrons $\mathcal{T}$ through a reflectional transformation of their vortexes. Any tetrahedral chain of length $n+1$ is uniquely represented by an initial tetrahedron T and an ordered sequence of n-twist-morphisms

$$\bar{R}_{i_1}, \dots, \bar{R}_{i_n}, \quad i_{k+1} \neq i_k, k = 1, \dots, n-1.$$

Its structure in U, D, F elementary units is written in the form of word $UUDFUDD....$ The fact that the tetrahedral chain is so rigid in 3-space and regular tetrahedrons can not tile the space gives rise to several questions. The main question we consider in this paper is the recognition of the combinatorial and algebraic structure of tetrahedral chains. We want to investigate their geometric properties and determine what information is contained in the chain invariant of orthogonal transformations and re-numerations.

In Section 2 we demonstrate that any tetrahedral chain of length $n+1$ is characterized by one from the four twist-morphisms, say $\bar{R}_l$ and collection of pairs $\{(g_i, \sigma_{lm_i})\}_{i=1,\dots,k}$ where $g_i \in S_4$ is a cycle of order n_i, $2 \leq n_i \leq 4$, $\sum_{i=1}^{k} n_i = n$, and $\sigma_{lm_i} \in S_4$ is a transposition in the permutation group of four elements. The encoding of U, D, F elements in the triplets of consecutive tetrahedrons along a chain is investigated in Section 3. By this way any tetrahedral chain is decomposed into sequence of admissible triplets of integers (k, j, i), $\quad 1 \leq k, j, i \leq 4, \quad k \neq j \neq i$ with the adjacency procedure reconstructing the chain in the form of word in U, D, F characters. Numerical description of a chain Q_n of length n by the admissible triplets of integers is reduced to an exact numerical invariant in the form of sign $\{-1, 0, +1\}$ sequence $\Lambda(Q_n)$ and presented in Section 5. In Section 6 the folding properties of tetrahedral chain as a simplicial complex are investigated. All chains centered around one vortex with various branching orders are classified. The periodic chains are investigated in Section 7 where it was proved that any periodic tetrahedral chain can be reconstructed by transformation M (composed by twist-morphisms) and its powers M^m. Finally in Section 8 we show that minimal indexing of numerical decomposition reduces to the tetrahedral group structure of 12 elements.

2. Tetrahedral chains.

A regular tetrahedron T in Euclidean 3-space is identified with the set of its four vertices $\{p_1, p_2, p_3, p_4\}$, $p_i \in \mathbb{R}^3$. We decompose T into four faces; $T = S_1 \cup \ldots \cup S_4$, where we enumerate faces in such a way that vortex p_i lies outside the plane defined by face S_i. By (S_i, p_i) we denote an elementary pair of T. We also write $T = \{(S_1, p_1), \ldots (S_4, p_4)\}$. By $\mathcal{T}$ we denote the space of regular tetrahedrons, and in what follows we will consider only regular tetrahedra.

An ordered sequence of regular tetrahedrons $\{T^{(i)}\}_{i=0}^{n}$ is called a *tetrahedral chain* if any pair of two consecutive tetrahedrons have a common face.

By s_i we denote the center of S_i, $s_i = \frac{1}{3}(\sum_{j=1}^{4} p_j - p_i)$. Four reflections R_i are defined uniquely by S_i and their orthogonal $s_i - p_i, \quad 1 \le i \le 4$.

$$R_i(p) = p - 2\frac{(p - s_i|s_i - p_i)}{(s_i - p_i|s_i - p_i)}(s_i - p_i). \tag{2.1}$$

For a given tetrahedron T the *tetrahedral triangulation process* is an attaching a copy of T, to one of its faces. The natural step of triangulation process is defined by reflection of vertex p_i in the face S_i. The new attached tetrahedron is defined by three vertices laying on the face S_i and the reflected vertex $p_i^{(1)}$. We denote it by $T_i^{(1)}$ as it is determined uniquely by an extra vertex $p_i^{(1)}$. The new attached tetrahedron $T_i^{(1)}$ is defined by four vertices $\{R_i(p_1), R_i(p_2), R_i(p_3), R_i(p_4)\}$,

$$R_i(p_j) = p_j, \quad j \neq i$$

as $p_j - s_i$ is orthogonal to $s_i - p_i$ and for $j = i$

$$R_i(p_i) = p_i + 2(s_i - p_i). \tag{2.2}$$

The tetrahedron $T_i^{(1)}$ given by reflection of p_i in the face S_i is defined by new four vertices, $T_i^{(1)} = \{p_j^{(1)_i}\}_{j=1}^{4}$ where $T_i^{(1)} = \bar{R}_i T$ and $\bar{R}_i$ is defined by reflection R_i (2.1) and it is called a *twist-morphism* of $\mathcal{T}$, $\bar{R}_i^2 = id$.

$$p_j^{(1)_i} := R_i(p_j) = p_j + 2\delta_{ij}(\frac{1}{3}\sum_{k\neq i} p_k - p_j), \quad j = 1, \ldots, 4 \tag{2.3}$$

The tetrahedral chains of the consecutive length $1, 2, \ldots, n$ are encoded (and described in affine coordinates) by compositions of $\bar{R}_i, \quad i = 1, \ldots, 4$;

$$\begin{array}{rcl} T^{(0)} & = & T \\ T_{i_1}^{(1)} & = & \bar{R}_{i_1} T \\ T_{i_1 i_2}^{(2)} & = & \bar{R}_{i_2}\bar{R}_{i_1} T, \quad i_1 \neq i_2 \\ & & \ldots \quad \ldots \quad \ldots \\ T_{i_1 i_2 \ldots i_n}^{(n)} & = & \bar{R}_{i_n} \ldots \bar{R}_{i_2}\bar{R}_{i_1} T, \quad i_{k+1} \neq i_k, k = 1, \ldots n-1. \end{array}$$

We define the center of a tetrahedron T; $C : \mathcal{T} \to V$, $C(T) = \frac{1}{4}\sum_{i=1}^{4} p_i$. The vectors joining centers of two consecutive tetrahedra in a chain are defined by $x_r = C(T^{(r)} - T^{(r-1)}) = c^{(r)} - c^{(r-1)}$ (Figure 4).

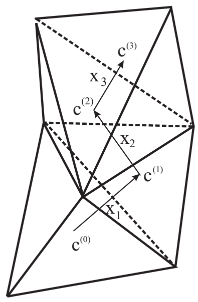

FIGURE 4. Skeleton orientation

The skeleton of the tetrahedral chain is a broken line build of consecutive segments, intervals $|x_r|$ joining the consecutive centers, $x_r = c^{(r)} - c^{(r-1)}$. In the coordinate matrix form $\bar{R}_i$ we have.

$$\bar{R}_1 = \begin{pmatrix} -1 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \bar{R}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & -1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\bar{R}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & -1 & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \bar{R}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & -1 \end{pmatrix},$$

Let S_4 be the symmetric group. We consider its matrix representation in $\mathbb{R}^4$. Then we prove the following properties of twist-morphisms.

Lemma 2.1. *For every $g \in S_4$*

$$\bar{R}_i g = g \bar{R}_{g^{-1}(i)}.$$

Proof. We have relations

$$\bar{R}_i = \sigma_{ij} \bar{R}_j \sigma_{ij}, \quad \bar{R}_i \sigma_{ij} = \sigma_{ij} \bar{R}_j = \sigma_{ij} \bar{R}_{\sigma_{ij}(i)},$$

and

$$\bar{R}_i \sigma_{kl} = \sigma_{kl} \bar{R}_i = \sigma_{kl} \bar{R}_{\sigma_{kl}(i)}, \text{ where } k \neq i \neq l, \quad \sigma_{kl}(i) = i,$$

where σ_{kl} is a transposition between k and l, $k, l \leq 4$. Let $g = \sqcap_{i=1}^{n} \sigma_i$ be decomposition of g onto transpositions. Then

$$\bar{R}_i g = \bar{R}_i \sigma_1 \ldots \sigma_n = \sigma_1 \ldots \sigma_r \bar{R}_{\sigma_r \circ \ldots \circ \sigma_n(i)} = g \bar{R}_{g^{-1}(i)},$$

as $g^{-1} = (\sigma_1 \ldots \sigma_n)^{-1} = \sigma_n^{-1} \ldots \sigma_1^{-1}$ and $\sigma_i = \sigma_i^{-1}$. □

Corollary 2.2. *By Lemma (2.1) we have relation*

$$g\bar{R}_i = \bar{R}_{g(i)}g$$

By Lemma (2.1) we can prove the following decomposition.

Theorem 2.3. *For any sequence of twist-morphisms $\bar{R}_{i_n}\dots\bar{R}_{i_2}\bar{R}_{i_1}T$ there is a decomposition*

$$(g_k\bar{R}_{l_k})^{n_k}(g_{k-1}\bar{R}_{l_{k-1}})^{n_{k-1}}\dots(g_1\bar{R}_{l_1})^{n_1}T, \quad n_1+\dots+n_k=n,$$

where $1\le l_i\le 4$ and $g_i\in S_4, i=1,\dots,k$, g_i is a cycle of order n_i, $2\le n_i\le 4$.

Proof. A chain of length $n+1$ can be characterized by a map f of $\mathbb{N}_n = \{1,\dots,n\}$ to $\mathbb{N}_4=\{1,2,3,4\}$, $f:\mathbb{N}_n\to\mathbb{N}_4$ which may be decomposed into maximal monotonous units of length greater than or equal to 2. Any monotonous unit of length k, $2\le k\le 4$, can be indexed by an action of a cyclic element of S_4 of order k. We can write

$$\bar{R}_{i_k}\dots\bar{R}_{i_1} = \bar{R}_{g^{k-1}(i_1)}\dots\bar{R}_{i_1}.$$

Using Lemma 2.1 we get

$$\bar{R}_{i_k}\dots\bar{R}_{i_1} = g^k\bar{R}_{g^{k-1}(i_1)}\dots\bar{R}_{i_1} = (g\bar{R}_{i_1})^k$$

which finishes the proof. □

The twist-morphisms $\bar{R}_1,\dots,\bar{R}_4$ are related by composition with permutation matrices. As an example $\bar{R}_2,\bar{R}_3,\bar{R}_4$ can be expressed uniquely by $\bar{R}_1$,

$$\bar{R}_2=\sigma_{12}\bar{R}_1\sigma_{12}, \quad \bar{R}_3=\sigma_{13}\bar{R}_1\sigma_{13}, \quad \bar{R}_4=\sigma_{14}\bar{R}_1\sigma_{14}$$

Thus the tetrahedral chain of length $n+1$ is characterized by one of the twist-morphisms, say $\bar{R}_r$, with relations $\bar{R}_j=\sigma_{rj}\bar{R}_r\sigma_{rj}$, $j\ne r$, and a sequence of pairs of cycles $\{(g_i,\sigma_{rl_i})\in S_4\times S_4 : i=1,\dots k\}$, with the decomposition $\sum_{i=1}^k n_i = n$, where n_i is an order of g_i.

3. Coding in triplets of consecutive steps.

The three consecutive tetrahedra or two skeleton segments define the orientation plane spanned by two vectors x_{r+2} and x_{r+1}. Thus we can write the coordinate representation of the three possible reflections $\bar{R}_{i_{r+3}}, i_{r+3}\ne i_{r+2},(i_{r+2}=1,\dots,4)$. The corresponding tetrahedrons follow after the orientation unit, $T^{(r+2)}_{i_{r+1}i_{r+2}} = \bar{R}_{i_{r+2}}\bar{R}_{i_{r+1}}T^{(r)}$ build of three neighboring tetrahedra, $T^{(r)}, T^{(r+1)}_{i_{r+1}} = \bar{R}_{i_{r+1}}T^{(r)}, T^{(r+2)}_{i_{r+1}i_{r+2}} = \bar{R}_{i_{r+2}}\bar{R}_{i_{r+1}}T^{(r)}$.

The three possible consecutive reflections $\bar{R}_{i_{r+1}},\bar{R}_{i_{r+2}},\bar{R}_{i_{r+3}}$ of an initial tetrahedron $T^{(r)}$ complete the initial tetrahedron to the oriented quadruple of four tetrahedrons and determine the three skeleton segments x_{r+1},x_{r+2},x_{r+3}.

Definition 3.1. *The three units F, U, D of tetrahedral chain, build of four consecutive tetrahedrons are defined by the three possible orientations of three consecutive skeleton segments,*

$$F: \quad T^{(r+3)}; \quad det(x_{r+1}, x_{r+2}, x_{r+3}) = 0 \tag{3.1}$$

$$U: \quad T^{(r+3)}; \quad det(x_{r+1}, x_{r+2}, x_{r+3}) > 0 \tag{3.2}$$

$$D: \quad T^{(r+3)}; \quad det(x_{r+1}, x_{r+2}, x_{r+3}) < 0 \tag{3.3}$$

The initial value for continuation of the recurrence process at each step of the process is defined by $T^{(r)}$ with $r \geq 2$. A tetrahedral chain with fixed orientation in the space (i.e. with coordinates prescribed to each vortex of the chain) is written in the form of word, e.g.

$$F \dots FU \dots UD \dots D \dots$$

The initial configuration of three consecutive tetrahedra $T^{(0)}, T^{(1)}, T^{(2)}$ establish an exact position of the chain in $\mathbb{R}^3$. But the distribution of elements F,D,U along the word defines its geometric structure uniquely. In symbolic description of the chain (invariant according to SO(3) symmetries) we postpone the initial value data.

Proposition 3.2. *At each element of a tetrahedral chain, say $T^{(r+3)}$, its position, F, U, D is defined uniquely by the triplet of three consecutive tetrahedrons $T^{(r)}, T^{(r+1)}, T^{(r+2)}$ and one of the three twist-morphisms acting on $T^{(r+2)}$.*

Proof. At each step of the triangulation process of the tetrahedral chain we have three preceding operations by twist-morphisms,

$$T_k^{(r+1)} = \bar{R}_k T^{(r)} \tag{3.4}$$

$$T_{kj}^{(r+2)} = \bar{R}_j \bar{R}_k T^{(r)} \tag{3.5}$$

$$T_{kji}^{(r+3)} = \bar{R}_i \bar{R}_j \bar{R}_k T^{(r)}. \tag{3.6}$$

Let $T^{(r)} = \{p_l\}_{l=1}^4$ Now we distinguish uniquely the three configurations depending on the last twist-morphism $\bar{R}_i$. Assuming $c^{(r)} = \frac{1}{4}\sum_{i=1}^4 p_i = 0$ we get,

$$(F): \quad i = k, x_{r+1} = -\frac{2}{3}p_k, x_{r+2} = -\frac{2}{3}(p_j + \frac{2}{3}p_k), x_{r+3} = -\frac{4}{9}(p_j + \frac{5}{6}p_k)$$

$$(U): \quad i \neq k, x_{r+1} = -\frac{2}{3}p_k, x_{r+2} = -\frac{2}{3}(p_j + \frac{2}{3}p_k), x_{r+3} = -\frac{2}{3}(p_i + \frac{2}{3}p_j + \frac{14}{9}p_k), \quad det(p_k, p_j, p_i) < 0$$

$$(D): \quad i \neq k, x_{r+1} = -\frac{2}{3}p_k, x_{r+2} = -\frac{2}{3}(p_j + \frac{2}{3}p_k), x_{r+3} = -\frac{2}{3}(p_i + \frac{2}{3}p_j + \frac{14}{9}p_k), \quad det(p_k, p_j, p_i) > 0.$$

These configurations uniquely define the three possible elements of a tetrahedral chain. □

Definition 3.3. *A triplet of integers (k, j, i), $1 \le i, j, k \le 4$ such that $k \ne j \ne i$ is called an admissible triplet. The set of all admissible triplets we denote by A.*

On the basis of Proposition 3.2 ,to each admissible triplet of integers we associate an encoding (for F, U, D) triplet of twist-morphisms,

$$(k, j, i) \mapsto (\bar{R}_k, \bar{R}_j, \bar{R}_i)$$

4. Admissible triplets along the chain.

As it is seen from the proof of Proposition 3.2 to complete orientation in the directed chain of tetrahedra and define especially U and D chain elements we need to orient at least one tetrahedron of the chain. It means that if we fix the sign, say $det[p_1, p_2, p_3] > 0$, then the orientation is defined for all other admissible triplets. In what follows we will assume this orientation for an initial tetrahedron. By direct calculation based on the proof of Proposition 3.2 we get the following result,

Proposition 4.1. *Assume the normalization conditions, $\sum_{i=1}^{4} p_i = 0$, and $det(p_1, p_2, p_3) > 0$, are fulfilled. Then in Table 1 we get the classes of admissible triplets representing the first step $\bar{U}$, $\bar{D}$, and $\bar{F}$ chain oriented elements.*

$\bar{U}$	$\bar{D}$	$\bar{F}$
$det(x_1, x_2, x_3) = 32\sqrt{3}/243$	$det(x_1, x_2, x_3) = -32\sqrt{3}/243$	$det(x_1, x_2, x_3) = 0$
(k, j, i)	(k, j, i)	(k, j, i)
$(3, 2, 1)$	$(4, 2, 1)$	$(1, 2, 1)$
$(4, 3, 1)$	$(2, 3, 1)$	$(1, 3, 1)$
$(2, 4, 1)$	$(3, 4, 1)$	$(1, 4, 1)$
$(4, 1, 2)$	$(3, 1, 2)$	$(2, 1, 2)$
$(1, 3, 2)$	$(4, 3, 2)$	$(2, 3, 2)$
$(3, 4, 2)$	$(1, 4, 2)$	$(2, 4, 2)$
$(2, 1, 3)$	$(4, 1, 3)$	$(3, 1, 3)$
$(4, 2, 3)$	$(1, 2, 3)$	$(3, 2, 3)$
$(1, 4, 3)$	$(2, 4, 3)$	$(3, 4, 3)$
$(3, 1, 4)$	$(2, 1, 4)$	$(4, 1, 4)$
$(1, 2, 4)$	$(3, 2, 4)$	$(4, 2, 4)$
$(2, 3, 4)$	$(1, 3, 4)$	$(4, 3, 4)$

Table 1. Classification of admissible triplets.

Each tetrahedron of a tetrahedral chain is given by reflection from the previous one. At each step of determining of U, D, F positions we have to normalize the initial conditions (initial tetrahedron) as it is described in Proposition 4.1 and repeat the part of the proof of Proposition 3.2. Indeed if the initial tetrahedron (say $T^{(r)}$) was oriented by $det[p_1, p_2, p_3] > 0$ then the next one ($T^{(r+1)}$) (serving as an initial for the next step of determination of U, D, F element), is given as reflection of a previous one so after normalization of its coordinates, $\sum_{i=1}^{4} p_i = 0$ we get the opposite orientation of the initial condition, i.e. $det[p_1, p_2, p_3] < 0$. And this orientation changes in any even step of our construction. Thus choosing an admissible triplet, say for D (in the second step) we choose it from the $\bar{U}$ column of admissible triplets of Table 1. In the next step (odd) of this procedure the initial data orientation comes to the first one, i.e. $det[p_1, p_2, p_3] > 0$ and we use the Table 1 as it is.

To avoid this varying initial orientation data along the chaining process we will introduce the notion of chain representing *adjoint word*.

Definition 4.2. *To each tetrahedral chain written in the form of word W (possibly infinite), for example*

$$W = UUDDUUFFU...$$

we prescribe uniquely a new word $\bar{W} = I(W)$ which will be called an adjoint word. Adjoint word is defined by replacing each character U (or D) by $\bar{D}$ (or $\bar{U}$ respectively) if it stays on the even place in the word W. All F characters are preserved in its adjoint form, for example

$$I(W) = \bar{W} = \bar{U}\bar{D}\bar{D}\bar{U}\bar{U}\bar{D}\bar{F}\bar{F}\bar{U}...$$

The adjoint word has a numerical meaning reducing the word characters along the chain to the ones listed in Table 1 which are the first step characters represented numerically. This way each tetrahedral chain can be represented by the sequence of admissible triplets uniquely defined by the initial triplet and the triplets of Table 1.

5. Numerical invariants of tetrahedral chains.

Any tetrahedral chain is described as a composition (ordered collection, sequence) of admissible ordered triplets representing the corresponding characters (U,D,F) of the defining word, e.g. for $UUDFD$ we have $I(UUDFD) = \bar{U}\bar{D}\bar{D}\bar{F}\bar{D}$ and respectively,

$$(3, 4, 2) \to (4, 2, 1) \to (2, 1, 4) \to (1, 4, 1) \to (4, 1, 3).$$

and

$$T^{(7)}_{3421413} = \bar{R}_3\bar{R}_1\bar{R}_4\bar{R}_1\bar{R}_2\bar{R}_4\bar{R}_3T.$$

As an ordered set each admissible triplet (i_1, i_2, i_3) is a mapping from the ordered set $\Delta_3 = \{1, 2, 3\}$ to the set $\Delta_4 = \{1, 2, 3, 4\}$ denoted by $f_{i_1i_2i_3}$. There is a natural action of the simple product of symmetric groups $S_4 \times S_3$ in the space of mappings,

$$(\xi, \sigma).f_{i_1i_2i_3} = \xi \circ f_{i_1i_2i_3} \circ \sigma, \quad \xi \in S_4, \sigma \in S_3.$$

The set of mappings into admissible triplets we will denote by $\mathcal{A}$. Obviously $\mathcal{A}$ is not preserved by the above defined group action.

Now we fix an orientation of the chain choosing a reference triplet $(i, j, k), i \neq j \neq k \neq i$. We call this triplet positive. We easily see that any other mapping $f_{i_1 i_2 i_3} \in \mathcal{A}$ can be given from the reference mapping f_{ijk} through the group action, i.e.

$$f_{i_1 i_2 i_3} = \xi \circ f_{ijk} \circ \sigma$$

Lemma 5.1.

1. *If* $i_1 \neq i_2 \neq i_3 \neq i_1, \quad f_{i_1 i_2 i_3}(s) \in \{i, j, k\}, 1 \leq s \leq 3$, *then there is a uniquely defined* $\sigma \in S_3$ *such that* $f_{i_1 i_2 i_3} = (id, \sigma).f_{ijk}$
2. *If* $i_1 \neq i_2 \neq i_3 \neq i_1, \quad \{i_1, i_2, i_3\} \neq \{i, j, k\}$, *then there are uniquely defined, transposition* $\tau \in S_4$ *and permutation* $\sigma \in S_3$ *such that* $f_{i_1 i_2 i_3} = (\tau, \sigma).f_{ijk}$.

Proof. The case 1.) is straightforward as the image set of $f_{i_1 i_2 i_3}$ is equal to $\{i, j, k\}$. In the case 2.) there are three possibilities $\{i_1, i_2, i_3\} = \{i, j, l\}, \quad \{i_1, i_2, i_3\} = \{i, l, k\}, \quad \{i_1, i_2, i_3\} = \{l, j, k\}$, where $\{l\} = \Delta_4 - \{i, j, k\}$. In each case there is a transposition $\tau_{kl}, \quad \tau_{jl}, \quad \tau_{il}$ such that the image of $f_{i_1 i_2 i_3}$ is equal $\tau_* \circ f_{ijk}$, where τ_* denotes one of the permutations $\tau_{kl}, \quad \tau_{jl}$ and τ_{il}. Then there is a uniquely defined element $\sigma \in S_3$ such that $f_{i_1 i_2 i_3} = \tau_* \circ f_{ijk} \circ \sigma$. □

On the basis of Lemma 5.1 and straightforward checking we get the new numerical characteristic of admissible triplets.

Proposition 5.2. *To each admissible triplet* $f_{i_1 i_2 i_3}$ *we associate uniquely its sign* Λ, *i.e. there is a mapping* $\Lambda : \mathcal{A} \to \{-1, 0, +1\}$, *such that*

1. $\Lambda(f_{ijk}) = +1$
2. *If* $i_1 \neq i_2 \neq i_3 \neq i_1, \quad f_{i_1 i_2 i_3}(s) \in \{i, j, k\}, 1 \leq s \leq 3$, *then* $f_{i_1 i_2 i_3} = (id, \sigma).f_{ijk}$ *and* $\Lambda(f_{i_1 i_2 i_3}) = sgn(\sigma)$.
3. *If* $i_1 \neq i_2 \neq i_3 \neq i_1, \quad \{i_1, i_2, i_3\} \neq \{i, j, k\}$, *then* $f_{i_1 i_2 i_3} = (\tau_*, \sigma).f_{ijk}$ *and* $\Lambda(f_{i_1 i_2 i_3}) = sgn(\tau_*) sgn(\sigma) = -sgn(\sigma)$.
4. *If* $i_1 = i_3$, $\Lambda(f_{i_1 i_2 i_3}) = 0$.

For a formal notion of Λ we propose its representation as a sign of the determinant build of three vortexes of the first tetrahedron of the tetrahedral chain, i.e. the positive reference triplet means $det(p_i, p_j, p_k) > 0$ ($sgn(det(p_i, p_j, p_k)) = +1$), and

$$\Lambda(f_{i_1 i_2 i_3}) = sgn(det(p_{i_1}, p_{i_2}, p_{i_3})).$$

Now we show the properties of Λ represented by determinant and displayed in Proposition 5.2. We show that

$$\Lambda(f_{i_1 i_2 i_3}) = sgn(det(p_{i_1}, p_{i_2}, p_{i_3})) = \Lambda(\xi \circ f_{ijk} \circ \sigma) = -sgn(\sigma) sgn(det(p_i, p_j, p_k)).$$

Indeed,

$$\begin{aligned} & det(p_{\xi\circ f_{ijk}\circ\sigma(1)}, p_{\xi\circ f_{ijk}\circ\sigma(2)}, p_{\xi\circ f_{ijk}\circ\sigma(3)}) \\ & = sgn(\sigma) det(p_{\xi\circ f_{ijk}(1)}, p_{\xi\circ f_{ijk}(2)}, p_{\xi\circ f_{ijk}(3)}) \\ & = sgn(\sigma) det(p_{\xi(i)}, p_{\xi(j)}, p_{\xi(k)}) = sgn(\sigma) det(p_i, p_j, p_l), \end{aligned}$$

because we choose (as an example) $\xi = \tau_{kl}$. And taking $p_l = -(p_i + p_j + p_k)$ we get

$$sgn(\sigma)det(p_i, p_j, p_l) = sgn(\sigma)det(p_i, p_j, -p_i - p_j - p_k) = -sgn(\sigma)det(p_i, p_j, p_k).$$

In the sequence of admissible triplets defining a tetrahedral chain for each triplet the two last numbers repeat in the subsequent triplet in the same order. To each tetrahedral chain W_n we associate the chain of admissible triplets.

Definition 5.3. *The chain of admissible triplets*

$$\dots \xrightarrow{\tilde{L}_*} (i_1, i_2, i_3) \xrightarrow{\tilde{L}_*} (i_2, i_3, i_4) \xrightarrow{\tilde{L}_*} (i_3, i_4, i_5) \xrightarrow{\tilde{L}_*} \dots,$$

where $\tilde{L}_*$ is defined by the corresponding transposition $\tau_* \in S_4$, maximal order element (maximal cycle) γ of S_3 and the unique relation

$$f_{i_{r+2}i_{r+3}i_{r+4}} = \tau_* \circ f_{i_{r+1}i_{r+2}i_{r+3}} \circ \gamma \tag{5.1}$$

is called an admissible chain (pre-complex) and we denote it by Q_n.

The action of above defined pairs (τ_*, γ) preserve the space of admissible mappings $\mathcal{A}$. The relation of sign Λ in an admissible chain reads

$$\Lambda(f_{i_2i_3i_4}) = \Lambda(\tau_* \circ f_{i_1i_2i_3} \circ \gamma) = -sgn(\gamma)\Lambda(f_{i_1i_2i_3})$$

In the space of all finite admissible chains we introduce the following equivalency,

Definition 5.4. *Two admissible chains Q_n and Q'_n*

$$Q_n = \dots \xrightarrow{\tilde{L}_*} (i_{r+1}, i_{r+2}, i_{r+3}) \xrightarrow{\tilde{L}_*} (i_{r+2}, i_{r+3}, i_{r+4}) \xrightarrow{\tilde{L}_*} (i_{r+3}, i_{r+4}, i_{r+5}) \xrightarrow{\tilde{L}_*} \dots,$$

$$Q'_n = \dots \xrightarrow{\tilde{L}_*} (j_{r+1}, j_{r+2}, j_{r+3}) \xrightarrow{\tilde{L}_*} (j_{r+2}, j_{r+3}, j_{r+4}) \xrightarrow{\tilde{L}_*} (j_{r+3}, j_{r+4}, j_{r+5}) \xrightarrow{\tilde{L}_*} \dots,$$

are equivalent, $Q_n \sim Q'_n$, if there exists $\xi \in S_4$ such that $f_{i_{k+1}i_{k+2}i_{k+3}} = \xi \circ f_{j_{k+1}j_{k+2}j_{k+3}}$ for $k = 0, \dots, n-1$.

To each admissible chain, say Q_n, we associate its sign sequence

$$\Lambda(Q_n) = \{\Lambda(f_{i_1i_2i_3}), \dots, \Lambda(f_{i_ni_{n+1}i_{n+2}})\}$$

Proposition 5.5. *The sign sequence is a numerical invariant, i.e.*

$$\Lambda(Q_n) = \Lambda(Q'_n) \ \text{ if and only if } \ \mathrm{Q_n \sim Q'_n}.$$

Thus the sign sequences corresponding to admissible sequences describe in 1-1 correspondence the tetrahedral chains. There is a natural correspondence of sequences

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\tilde{L}_*} & \mathcal{A} & \xrightarrow{\tilde{L}_*} & \mathcal{A} & \xrightarrow{\tilde{L}_*} & \mathcal{A} & \xrightarrow{\tilde{L}_*} & \dots \\ {\scriptstyle\Lambda}\downarrow & & {\scriptstyle\Lambda}\downarrow & & {\scriptstyle\Lambda}\downarrow & & {\scriptstyle\Lambda}\downarrow & & {\scriptstyle\Lambda}\downarrow \\ \dots & \xrightarrow{\tilde{S}_*} & G & \xrightarrow{\tilde{S}_*} & G & \xrightarrow{\tilde{S}_*} & G & \xrightarrow{\tilde{S}_*} & \dots \end{array}$$

where $G = \{-1, 0, +1\}$ and $\tilde{S}_*$ is an operator $G \to G$ corresponding to $\tilde{L}_*$.

6. Geometric characteristics of short tetrahedral chains.

An important notion of tetrahedral chains is their clustering characteristic. First we introduce the notion of vortex order. If p is a common vortex for a number of tetrahedrons we call this number an order of vortex p. Vortex order is a function $P : V_{C_n} \to \mathbb{N}$ defined on the ordered sequence of vortices V_{C_n} of chain C_n (of length n) which to each vortex $p \in V_{C_n}$ along the chain C_n prescribes its order $P(p)$. In the same way we define an edge order function $G : E_{C_n} \to \mathbb{N}$ defined on the ordered sequence of edges E_{C_n} ordered along the chain. $G(l)$ is a number of tetrahedrons which share an edge l, $l \in E_{C_n}$. We can easily see the following

Lemma 6.1. *1. The total vortex order $\tilde{V} = \sum_{p \in V_{C_n}} P(p)$ for a chain does not depend on a form of chain and for the chain of length n it is equal to $4n$.*
2. Maximal vortex order is 20 and the chains of length 20 with such vortex we will call ico-clusters (icosahedral clusters). Distribution of values of vortex order function along the chain uniquely up to orientation defines the geometric shape of the chain, i.e. if two functions P_1 and P_2 for two chains of the same length are equal then their shapes are identical up to the change of orientation.

Remark 6.2. *If the vortex order function is constant and equal 4 (beside the vortices for two initial and two final tetrahedrons) then the chain is a tetrahelix with U or D orientation. In this case the level sets of edge order function G, $G^{-1}(1), G^{-1}(2), G^{-1}(3)$ form the three helices build of external edges, two helices and central broken spiral.*

Let E denote the set of edges of a chain C_n. By S we denote the set of its faces and $\hat{S}$ the set of its external faces. Then we have an Euler characteristic relation

$$\#\hat{S} - \#E + \#V = 2$$

end relations for the total orders

$$\frac{1}{n}(\tilde{S}-\tilde{E}+\tilde{V})=2.$$

Then on each space $\mathcal{C}_n$ of proper chains of length n, we have a clustering function $Cl : \mathcal{C}_n \to \mathbb{N}$. For a chain C_n we prescribe the set $\bar{V}_{C_n}$ of vortices with vortex order greater than or equal to 4,

$$Cl(C_n) = \sum_{p\in\bar{V}_{C_n}} (P(p)-4)$$

We see that the clustering function vanishes on tetrahelix and has a maximal value for chains build of ico-clusters completed by k-clusters (chains composed of one common vortex of order k) with $k = n(mod 20)$.

In the chain growing process at each final tetrahedron one may have maximally three possibilities to attache tightly the next tetrahedron if it has no common points with any other tetrahedron than the last one. Thus we see that every tetrahedron (element of the chain) along a chain has its *branching order* which is the number $b, \quad 0 \le b \le 3$ of possible concurrent continuations of the chain at the given tetrahedron. The zero branching order terminates the chain (see Figure 5).

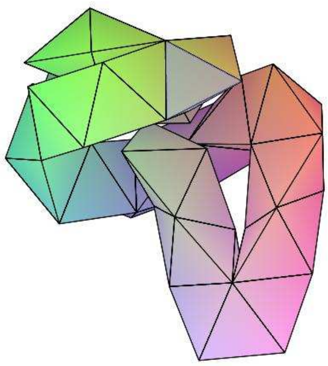

FIGURE 5. Zero branching order example

If branching order is one at that tetrahedron it is a deterministic continuation of the chain. Such chains form the basic units with substantially reduced space of possible configurations. Classification of fixed vortex clusters is given in the following

Theorem 6.3. *The tetrahedral chains sharing one fixed geometrical vortex, with different branching orders b = 1, 2 and 3 are listed in the Table 2.*

$b\backslash n$	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	*total*
1	0	0	0	0	0	1	2	6	9	19	38	49	69	79	71	34	**6**	383
2	0	0	1	4	6	10	24	46	78	113	137	153	132	85	36	6	0	831
3	2	4	6	9	16	27	38	48	55	56	50	35	22	12	2	0	0	380
total	2	4	7	13	22	38	64	100	142	188	225	237	223	176	109	40	6	1594

TABLE 2. Table of one vortex clusters.

Example 6.4. We show the first listing of short chains centered at one fixed vortex with $4 \leq n \leq 8$:

$n = 4:$ F, D
$n = 5:$ FF, FD, DU, DF,
$n = 6:$ FFD, FDF, FDU, DUF, DUD, DFF, DFU
$n = 7:$ $FFDF$, $FFDU$, $FDFF$, $FDFU$, $FDUF$, $FDUD$, $DUFF$, $DUFD$, $DUDF$, $DUDU$, $DFFU$, $DFUF$, $DFUD$
$n = 8:$ $FFDFU$, $FFDFF$, $FFDUD$, $FDFFU$, $FDFUD$, $FDFUF$, $FDUFD$ $FDUDU$, $FDUDF$, $DUFFD$, $DUFDU$, $DUFDF$, $DUDFU$, $DUDFF$ $DUDUD$, $DUDUF$, $DFFUF$, $DFFUD$, $DFUFD$, $DFUFF$, $DFUDU$, $DFUDF$.

Remark 6.5.
1. The smallest basic cluster of branching order 1 is given by the following code (Figure 6):

$$UDFUDF$$

2. The ico-clusters are listed in the following codes (Figure 7):

$$FFUFFDUDUDFFUFFDU, FFUFFDUDUDUDFFUFF$$
$$UFFDFFUDUDUFFDFFU, UFFDUDFFUFFDUDFFU$$
$$UDFFUFFDUDFFUFFDU, UDFFUFFDUDUDFFUFF$$

3. Table 2 displays one vortex chains sharing one geometrically fixed point as a vortex. This gives us the part of the clusters, namely those who has fixed orientation. This table extended by all codes with U replaced by D and vice versa completes the set of all one vortex clusters.

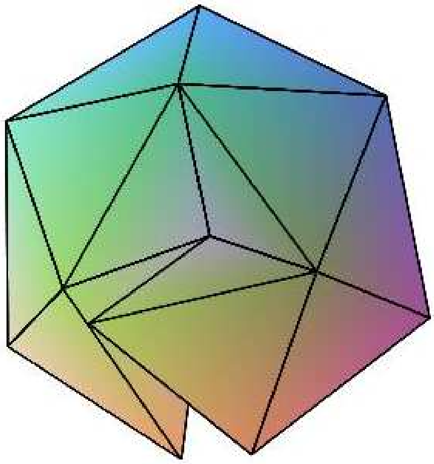

FIGURE 6. Smallest basic cluster, $n = 9, b = 1$

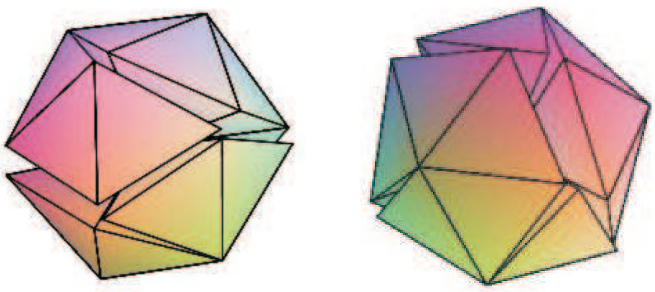

FIGURE 7. Examples of Ico-clusters $n = 20, b = 1$

7. Periodic tetrahedral chains.

We call tetrahedral chains periodic if configurations of elements F, U, D in the defining sequence (word) $D \ldots DU \ldots UF \ldots$ are periodic. Any periodic tetrahedral chain is uniquely defined by its period which forms the shortest finite tetrahedral chain written in the form of "word" of length n of U, D, F configuration, say $W_n = UDFU \ldots DU$. We will call this word the *basic period* of the periodic tetrahedral

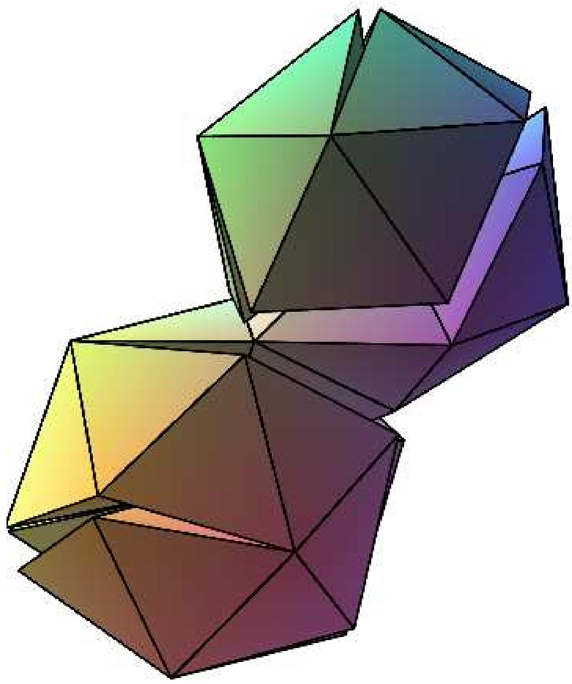

FIGURE 8. Multi ico-cluster

chain. In the twist-morphism representation for each of U, D, F characters to be defined we need two preceding twist-morphisms which define an admissible triplet of integers (indexes). Thus the period in the numerical characterization of the periodic tetrahedral chains by their sequence of admissible triplets corresponds to the shortest cycle of the triplets which continues repeating along the sequence. The cycle of triplets is directly represented by the twist-morphisms and finally by their composition, which forms the kind of *master matrix* for the infinite periodic chain. In this section we show existence of such matrix for a periodic chain and display some of its properties. For construction of coordinate representation of the periodic chains, i.e. its twist-morphism composition for the cycle of admissible triplets we will define the corresponding *adjoint defining sequence of periodic tetrahedral chain* and its *adjoint period*

Definition 7.1. *The shortest word W_n will be called period of the tetrahedral chain if it is also period for the corresponding adjoint defining sequence and its adjoint word $\bar{W}_n$ is also period for the adjoint defining sequence.*

We notice that if length of a basic period W is even, then the adjoint word $I(W)$ is also a basic period for the adjoint defining sequence of the chain. Obviously the doubled basic period is a period for the adjoint defining sequence in the case of odd length of the basic period.

Let us assume the positive orientation of an initial tetrahedron T, i.e. $det[p_1, p_2, p_3] > 0$, then using Table 1 for any two indices $i_1, i_2, \quad i_1 \neq i_2, 1 \leq i_1, i_2 \leq 4$ we can define uniquely $i_3, 1 \leq i_3 \leq 4$ such that (i_1, i_2, i_3) is an admissible triplet of the fixed type "U, D, F". Thus we have mappings $L_U, L_D, L_F : \mathbb{I} \to \Delta$, where $\Delta = \{1, 2, 3, 4\}$ and $\mathbb{I} = \{(\alpha, \beta) \in \Delta \times \Delta : \alpha \neq \beta\}$.

Any word W_n of length n can be represented as a composition of $n+2$ twist-morphisms $\bar{R}_{i_{n+2}} \ldots \bar{R}_{i_1}$. The sequence of indices $i_1, \ldots i_{n+2}$ is defined by the sequence of L_*-mappings directly corresponding to the sequence of characters in the dual word $\bar{W}_n$. For example the L_*- sequence for the period $W_n = UDFU \ldots DU$ is following:

$$\begin{aligned} L_U(i_1, i_2) &= i_3, \\ L_U(i_2, i_3) &= i_4, \\ L_F(i_3, i_4) &= i_5, \\ L_D(i_4, i_5) &= i_6, \\ \ldots &. \ldots \\ L_D(i_{n-1}, i_n) &= i_{n+1}, \\ L_D(i_n, i_{n+1}) &= i_{n+2}. \end{aligned}$$

For the twist-morphisms representation of a periodic tetrahedral chain with W_n to be a period the corresponding composition $\bar{R}_{i_{n+2}} \ldots \bar{R}_{i_1}$ is not necessary continuing along the representing sequence. It would be so if the L_*-sequence is cyclic, i.e. if $i_1 = i_{n+1}, i_2 = i_{n+2}$. This fact may be easily seen for three basic periodic chains,

$$\begin{aligned} &U \text{ periodic} : U \ldots U \ldots \\ &D \text{ periodic} : D \ldots D \ldots \\ &F \text{ periodic} : F \ldots F \ldots \end{aligned}$$

In first two cases the period is a two characters word, UU, DD. In U (and D) periodic chains the defining sequences are cyclic after two iterations of the L_*-sequence

$$\begin{aligned} L_U(i_1, i_2) &= i_3 \\ L_D(i_2, i_3) &= i_4 \\ L_U(i_3, i_4) &= i_1 \\ L_D(i_4, i_1) &= i_2, \end{aligned}$$

as displayed in Table 2. Obviously the L_*-sequence for the period of $F-$ periodic chain is cyclic, i.e. $L_F(i_1, i_2) = i_1, L_F(i_2, i_1) = i_2$.

Corollary 7.2. *The examples of cycles of admissible triplets defining the basically periodic chains are displayed in Table 3.*

U-chains	D-chains
$(3,2,1) \to (2,1,4) \to (1,4,3) \to (4,3,2)$	$(2,1,4) \to (1,4,3) \to (4,3,2) \to (3,2,1)$
$(4,3,1) \to (3,1,2) \to (1,2,4) \to (2,4,3)$	$(3,1,2) \to (1,2,4) \to (2,4,3) \to (4,3,1)$
$(2,4,1) \to (4,1,3) \to (1,3,2) \to (3,2,4)$	$(4,1,3) \to (1,3,2) \to (3,2,4) \to (2,4,1)$
$(3,4,2) \to (4,2,1) \to (2,1,3) \to (1,3,4)$	$(4,2,1) \to (2,1,3) \to (1,3,4) \to (3,4,2)$
$(4,1,2) \to (1,2,3) \to (2,3,4) \to (3,4,1)$	$(1,2,3) \to (2,3,4) \to (3,4,1) \to (4,1,2)$
$(4,2,3) \to (2,3,1) \to (3,1,4) \to (1,4,2)$	$(2,3,1) \to (3,1,4) \to (1,4,2) \to (4,2,3)$
$(1,4,3) \to (4,3,2) \to (3,2,1) \to (2,1,4)$	$(4,3,2) \to (3,2,1) \to (2,1,4) \to (1,4,3)$
$(1,2,4) \to (2,4,3) \to (4,3,1) \to (3,1,2)$	$(2,4,3) \to (4,3,1) \to (3,1,2) \to (1,2,4)$
$(1,3,2) \to (3,2,4) \to (2,4,1) \to (4,1,3)$	$(3,2,4) \to (2,4,1) \to (4,1,3) \to (1,3,2)$
$(2,1,3) \to (1,3,4) \to (3,4,2) \to (4,2,1)$	$(1,3,4) \to (3,4,2) \to (4,2,1) \to (2,1,3)$
$(2,3,4) \to (3,4,1) \to (4,1,2) \to (1,2,3)$	$(3,4,1) \to (4,1,2) \to (1,2,3) \to (2,3,4)$
$(3,1,4) \to (1,4,2) \to (4,2,3) \to (2,3,1)$	$(1,4,2) \to (4,2,3) \to (2,3,1) \to (3,1,4)$

TABLE 3. Admissible triplets defining the basic periodic tetrahedral chains.

For L_*-sequence of period for U (and D resp.) periodic chain we associate the corresponding operator $M_U = \bar{R}_{i_4} \dots \bar{R}_{i_1}$ (M_D resp.) such that the U-chain is an infinite composition of M_U (M_D resp.). Here $M_F = \bar{R}_{i_2}\bar{R}_{i_1}$.

As the any two current indexes i_k, i_{k+1} determine uniquely the third one i_{k+2} for the element of the chain of type U, D, F and so on for subsequent pair i_{k+1}, i_{k+2}, then we naturally define the corresponding maps

$$\mathcal{L}_U, \mathcal{L}_D, \mathcal{L}_F : \mathbb{I} \to \mathbb{I}, \tag{7.1}$$

$\mathcal{L}_*(i_1, i_2) = (i_2, L_*(i_1, i_2))$, $* = U, D, F$.

$\#\mathbb{I} = 12$ and on the basis of Table 1 $\mathcal{L}_*$ are bijections of $\mathbb{I}$.

For each period W_n we define the mapping $\mathcal{L}_{W_n} : \mathbb{I} \to \mathbb{I}$, $\quad \mathcal{L}_{W_n}(i_1, i_2) = (i_{n+1}, i_{n+2})$ which is composition of the bijections, $\mathcal{L}_U, \mathcal{L}_D, \mathcal{L}_F$ in the order of the adjoint word $\bar{W}_n$.

$$\mathcal{L}_* \dots \mathcal{L}_*(i_1, i_2) = (i_{n+1}, i_{n+2}),$$

This composition of bijections will be called the $\mathcal{L}-$sequence. Now there is a question if any periodic tetrahedral chain is represented by some "master" matrix M, i.e. a finite composition of twist-morphisms with cyclic property? Such that any part of the chain can be reconstructed by some power of . The answer of this question gives the following

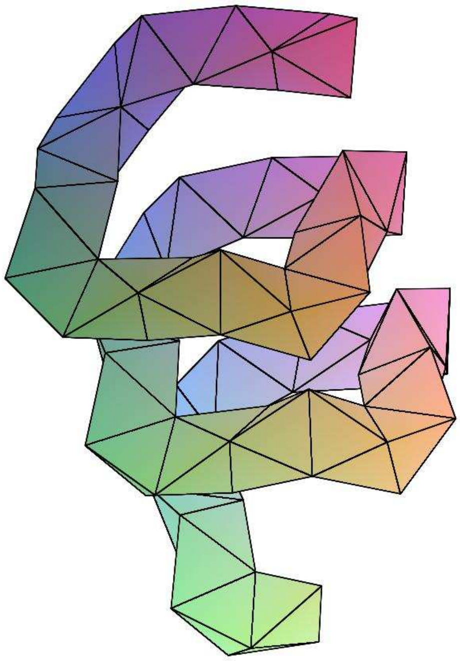

FIGURE 9. Periodic tetrahedral chain

Theorem 7.3. *Any periodic tetrahedral chain is represented by an operator* $M : \mathcal{T} \to \mathcal{T}$ *which is a finite cycling composition of twist-morphisms* $\bar{R}_i$, $i = 1 \ldots 4$.

Proof. To each period W_n we prescribe uniquely a bijection $\mathcal{L}_{W_n}$ of the twelve-element set $\mathbb{I}$. $\mathcal{L}_{W_n} \in S_{12}$, and as an element of the symmetric group has a finite order, say k such that $\mathcal{L}^k_{W_n} = id_{\mathbb{I}}$. Thus for any pair $(i_1, i_2) \in \mathbb{I}$ after $nk + 2$ steps we get a cycle property. □

8. ALGEBRAIC STRUCTURE OF TETRAHEDRAL CHAINS.

The space of all finite regular tetrahedral chains, i.e. finite words build of U, D, F or equivalently $\bar{U}, \bar{D}, \bar{F}$ characters, we denote by $\Gamma_{\mathcal{T}}$. Space of the corresponding $\mathcal{L}$-sequences is denoted by $\Gamma_{\mathcal{L}}$. The duality in $\Gamma_{\mathcal{T}}$ defined in Section 7 is an inner automorphism of $\Gamma_{\mathcal{T}}$. To each element of $\Gamma_{\mathcal{T}}$ (which can be treated also as a dual word) we associate uniquely the composition of twist-morphisms $\bar{R}_{i_n} \ldots \bar{R}_{i_1}$ representing the corresponding configuration of $\bar{U}, \bar{D}, \bar{F}$ elements in the chain. The space of these compositions we identify with $\Gamma_{\mathcal{T}}$. By the construction of the sequence of numbers $i_1, \ldots, i_n$ (see Section 7) the elements of $\Gamma_{\mathcal{T}}$ are indexed uniquely by the finite $\mathcal{L}-$sequences belonging to $\Gamma_{\mathcal{L}}$. These sequences form the compositions of the three bijections $\mathcal{L}_{\bar{U}}, \mathcal{L}_{\bar{D}}, \mathcal{L}_{\bar{F}}$ which are elements of the permutation group S_{12}, i.e there is a well defined map $\Phi : \Gamma_{\mathcal{T}} \to \Gamma_{\mathcal{L}}$ of finite words of $\Gamma_{\mathcal{T}}$ into the space of finite configurations of these three elements.

Theorem 8.1. *The indexing space of $\Gamma_{\mathcal{T}}$, i.e. the image of the indexing map Φ is an infinite abstract group generated by three elements*

$$\mathcal{L}_{\bar{U}}, \quad \mathcal{L}_{\bar{D}}, \quad \mathcal{L}_{\bar{F}}$$

with the relations

$$\mathcal{L}_{\bar{U}}^3 = id, \quad \mathcal{L}_{\bar{D}}^3 = id, \quad \mathcal{L}_{\bar{F}}^2 = id, \quad (\mathcal{L}_{\bar{U}}\mathcal{L}_{\bar{D}})^2 = id \tag{8.1}$$

Elements $\mathcal{L}_{\bar{U}}, \mathcal{L}_{\bar{D}}$, with relations $\mathcal{L}_{\bar{U}}^3 = id, \quad \mathcal{L}_{\bar{D}}^3 = id, \quad (\mathcal{L}_{\bar{U}}\mathcal{L}_{\bar{D}})^2 = id$ generate the tetrahedral group of 12 elements. Thus the indexing sequence for a chain W_n is written in the form

$$\Phi(W_n) = a_{i_1}\mathcal{L}_F a_{i_2}\mathcal{L}_F a_{i_3}\mathcal{L}_F \dots \mathcal{L}_F a_{i_k}, \tag{8.2}$$

where a_{i_j} is an element of the tetrahedral group (cf. [1])

$$T = \{id, \mathcal{L}_{\bar{U}}, \mathcal{L}_{\bar{D}}, \mathcal{L}_{\bar{U}}^2, \mathcal{L}_{\bar{D}}^2, \mathcal{L}_{\bar{D}}\mathcal{L}_{\bar{U}}, \mathcal{L}_{\bar{U}}\mathcal{L}_{\bar{D}}, \mathcal{L}_{\bar{U}}^2\mathcal{L}_{\bar{D}}, \mathcal{L}_{\bar{U}}\mathcal{L}_{\bar{D}}^2, \mathcal{L}_{\bar{D}}^2\mathcal{L}_{\bar{U}}, \mathcal{L}_{\bar{D}}\mathcal{L}_{\bar{U}}^2, \mathcal{L}_{\bar{U}}\mathcal{L}_{\bar{D}}^2\mathcal{L}_{\bar{U}}\}$$

References

[1] H.S.M. Coxeter, Regular Polytopes, Dover, New York 1973.
[2] H.S.M. Coxeter, Close-packing and froth, Illinois J. Math. Vol. 2, Issue 4B (1958), 746-758.
[3] H.R. Farran, S.A. Robertson, Regular convex bodies, J. London Math. Soc. (2) 49 (1994), 371-384.
[4] Hans E. Debrunner, Tiling Euclidean d-space with congruent symplexes, Annals of the New Yourk Academy of Sci. Vol. 440, (1985), 230-261
[5] D.J. Hoylman, The densest lattice packing of tetrahedra, Bull. of the AMS, Vol 76, (1970), 135-137.
[6] A.C. Hurley, Some Helical Structures Generated by Reflections, Aust. J. Phys., Vol. 38, (1985), 299-310.
[7] J. H. Mason, Can regular tetrahedra be glued together face to face to form a ring?, Math. Gaz., 56 (1972) 194-197.
[8] S. K. Stein, The Planes Obtainable by Gluing Regular Tetrahedra, The American Mathematical Monthly, Vol. 85, No. 6 (1978),477-479
[9] A. F. Wells, Survey of Tetrahedral Structures, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 319, No. 1548 (Aug. 28, 1986), pp. 291-335
[10] Ch. Zheng,R. Hoffmann, D. R. Nelson, A Helical Face-Sharing Tetrahedron Chain with Irrational Twist, Stella Quadrangula, and Related Matters, J. Am. Chem. Soc. Vol. 112, (1990),3784-3791

(H. Babiker) Wydział Matematyki i Nauk Informacyjnych, Politechnika Warszawska, Pl. Politechniki 1, 00-661 Warszawa, Poland

E-mail address: babikerhm@gmail.pl

(S. Janeczko) Instytut Matematyczny PAN, ul. Śniadeckich 8, 00-950 Warszawa, Poland, and Wydział Matematyki i Nauk Informacyjnych, Politechnika Warszawska, Pl. Politechniki 1, 00-661 Warszawa, Poland

E-mail address: janeczko@mini.pw.edu.pl