Operators with large diagonal

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Overview

1. Operators with large diagonal
2. One parameter
3. Two parameters

Overview

A description of the problem class

- Let X be a Banach space and $T : X \to X$ a linear operator.

Find conditions on X and T such that the identity on X factors through T, i.e.

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}"] \arrow[d, "E"'] & X \\ X \arrow[r, "T"'] & X \arrow[u, "P"'] \end{tikzcd} \qquad \|E\|\|P\| \le C.$$

- The problem has finite dimensional (quantitative) and infinite dimensional (qualitative) aspects.
- Classical examples include: ℓ^p (Pelczynski), ℓ^∞ (Lindenstrauss), L^1 (Enflo-Starbird), L^p (Johnson-Maurey-Schechtman-Tzafriri), ℓ^p_n (Bourgain-Tzafriri).
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Operators with large diagonal

- Let X be a Banach space and $T : X \to X$ a linear operator.
- Suppose that X has an unconditional basis $(b_n)_{n\in\mathbb{N}}$, and let $b_n^* \in X^*$ be the n^{th} coordinate functional.
- We say that T has large diagonal (relative to (b_n)) if $\inf_{n\in\mathbb{N}} |\langle Tb_n, b_n^*\rangle| > 0$.
- For many Banach spaces X we know that the identity factors through operators T with large diagonal, i.e.

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Answer:

Theorem (N.J. Laustsen, P.F.X. Müller, R. L.)

There is an operator T on a Banach space X with an unconditional basis such that T has large diagonal, but the identity operator on X does not factor through T.

Main ingredients for the proof:

- X is the space X_G (Gowers) with an unconditional basis.
- Fredholm theory.

Theorem (N.J. Laustsen, P.F.X. Müller, R. L.)

The identity on mixed-norm Hardy spaces $H^p(H^q)$, $1 \leq p, q < \infty$ factors through any operator T with large diagonal relative to the bi-parameter Haar basis.

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Dyadic H^p

- $\mathscr{D} = \{[\frac{k-1}{2^n}, \frac{k}{2^n}[: \, k \geq 0, n \geq 0\}$ denotes the dyadic intervals on the unit interval,
- h_I the L^∞-normalized Haar function, $I \in \mathscr{D}$.
- Let $f = \sum_{I \in \mathscr{D}} a_I h_I$ be a finite linear combination,
- then the square function $\mathbb{S}(f)$ of f is given by

$$\mathbb{S}(f) = \Big(\sum_{I \in \mathscr{D}} a_I^2 h_I^2\Big)^{1/2}.$$

- The norm of the one-parameter Hardy space H^p, $1 \leq p < \infty$ is defined by

$$\|f\|_{H^p} = \|\,\mathbb{S}(f)\|_{L^p} = \Big(\int_0^1 \Big(\sum_{I \in \mathscr{D}} a_I^2 h_I^2(x)\Big)^{p/2} \,\mathrm{d}x\Big)^{1/p}.$$

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Andrew (1979)

Theorem

Let $1 < p < \infty$, $\delta > 0$ and $T : H^p \to H^p$ be a linear operator with large diagonal, i.e. $\langle Th_I, h_I \rangle \geq \delta |I|$. Then we have

$$\begin{tikzcd} H^p \arrow[r, "\mathrm{Id}"] \arrow[d, "E"'] & H^p \\ H^p \arrow[r, "T"'] & H^p \arrow[u, "P"'] \end{tikzcd} \qquad \|E\|\|P\| \leq C/\delta,$$

where the constant $C > 0$ is universal.

By Gamlen-Gaudet construction $(\mathscr{B}_I)$ and a random choice of signs ε_I there exists a block basis $b_I^{(\varepsilon)} = \sum_{K \in \mathscr{B}_I} \varepsilon_K h_K$ of the Haar system such that

$$Tb_I^{(\varepsilon)} = \alpha_I b_I^{(\varepsilon)} + \text{small error}, \qquad \alpha_I \geq \delta.$$

The orthogonal projection $Qf = \sum_{I \in \mathscr{D}} \frac{\langle f, b_I^{(\varepsilon)} \rangle}{\|b_I^{(\varepsilon)}\|_2^2} b_I^{(\varepsilon)}$ is bounded on H^p (Gamlen-Gaudet).

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where the constant $C > 0$ *is universal.*

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The Gamlen-Gaudet construction (1973)

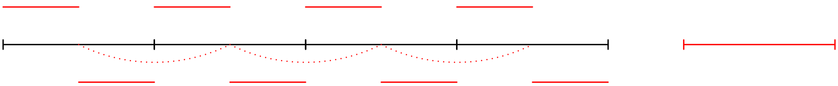

Figure: On the left side: construction of b_I. On the right side: the corresponding index intervals I.

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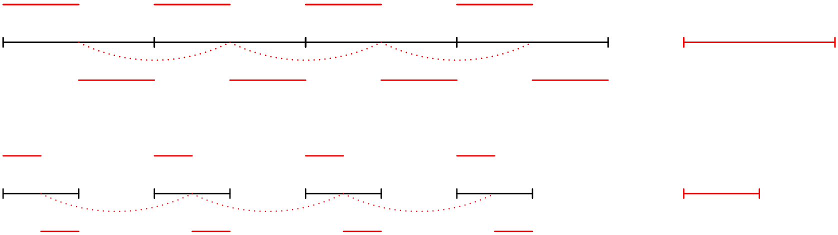

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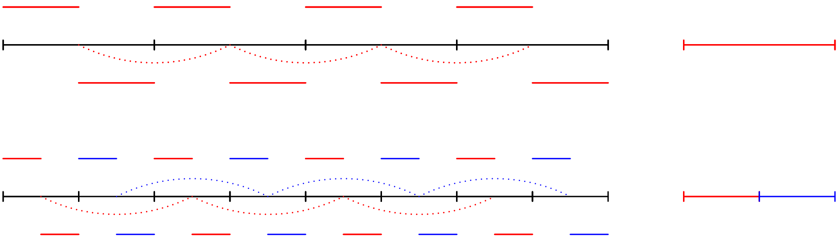

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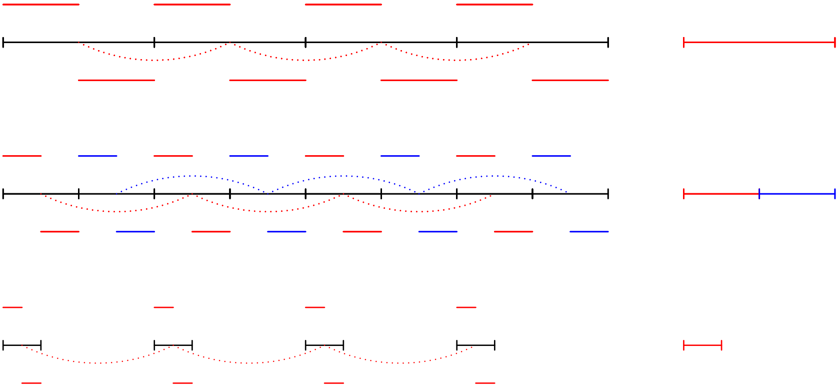

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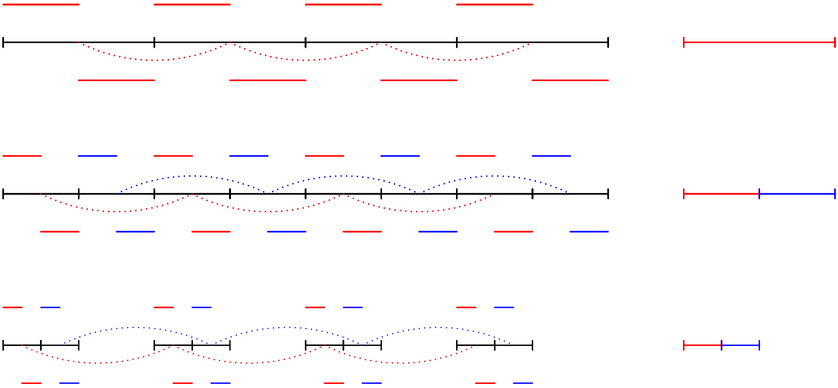

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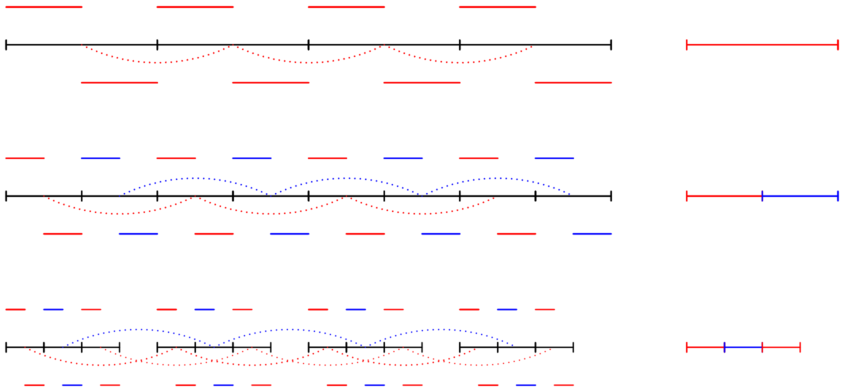

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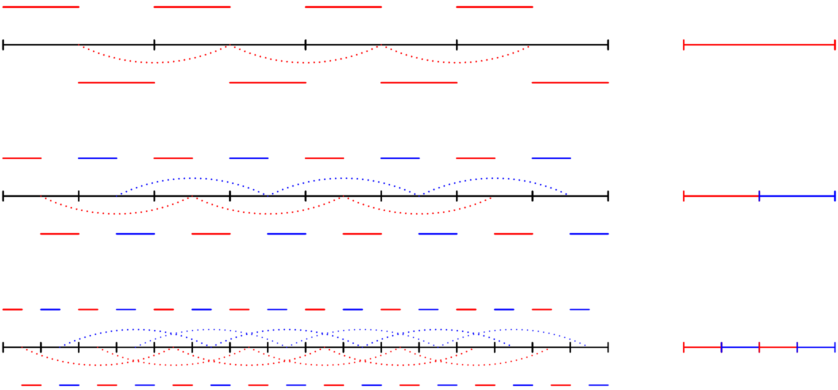

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Comments on Andrew (1979)

- The Rademacher system converges weakly to 0 in H^p.
- First, the operator T is preconditioned by multiplying the Haar system with highly oscillating Rademacher functions.
- This gives that $\langle Tb_I^{(\varepsilon)}, b_J^{(\varepsilon)} \rangle \approx 0$, if $I \neq J$.
- The second part consists of choosing signs ε_I such that $\langle Tb_I^{(\varepsilon)}, b_I^{(\varepsilon)} \rangle \geq \delta \|b_I^{(\varepsilon)}\|_2^2$.
- The block basis $\{b_I^{(\varepsilon)} : I \in \mathscr{D}\}$ is equivalent to the Haar system.
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Overview

Mixed-norm Hardy spaces $H^p(H^q)$

- $\mathscr{R} = \{I \times J \,:\, I, J \in \mathscr{D}\}$ denotes the dyadic rectangles on the unit square,
- $h_{I\times J}(x, y) = h_I(x)h_J(y)$ the L^∞-normalized tensor product Haar function, $I \times J \in \mathscr{R}$.
- Let $f = \sum_{I\times J\in\mathscr{R}} a_{I\times J} h_{I\times J}$ be a finite linear combination,
- then the square function $\mathbb{S}(f)$ of f is given by

$$\mathbb{S}(f) = \Big(\sum_{I\times J\in\mathscr{R}} a_{I\times J}^2 h_{I\times J}^2\Big)^{1/2}.$$

- we define the norm of the mixed-norm Hardy spaces $H^p(H^q)$, $1 \leq p, q < \infty$ by

$$\|f\|_{H^p(H^q)} = \Big(\int_0^1 \Big(\int_0^1 \big(\sum_{I\in\mathscr{D}} a_{I\times J}^2 h_{I\times J}^2(x, y)\big)^{q/2} \,\mathrm{d}y\Big)^{p/q} \,\mathrm{d}x\Big)^{1/p}.$$

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Capon (1982): $H^p(H^q)$ is primary (augmented)

The first result on factorization in mixed-norm spaces is the following theorem of Capon.

Theorem

Let $1 < p, q < \infty$ or $p = q = 1$. For any operator T the identity on $H^p(H^q)$ factors through $H = T$ or $H = \mathrm{Id} - T$, i.e.

$$\begin{tikzcd} H^p(H^q) \arrow[r, "\mathrm{Id}"] \arrow[d, "E"'] & H^p(H^q) \\ H^p(H^q) \arrow[r, "H"'] & H^p(H^q) \arrow[u, "P"'] \end{tikzcd} \qquad \|E\|\|P\| \le C.$$

Mixed-norm spaces require specific bi-parameter techniques.

Capon invents a specific bi-parameter Gamlen-Gaudet selection process. It gives a block basis $b_{I\times J} = \sum_{K\times L\in\mathscr{B}_{I\times J}} h_{K\times L}$ such that

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Large diagonal relative to the bi-parameter Haar system

Theorem (N.J. Laustsen, P.F.X. Müller, R. L.)

Let $1 \leq p, q < \infty$, $\delta > 0$ and $T : H^p(H^q) \to H^p(H^q)$ be a linear operator with large diagonal, i.e. $\langle T h_{I\times J}, h_{I\times J}\rangle \geq \delta |I \times J|$. Then we have

$$\begin{tikzcd} H^p(H^q) \arrow[r, "\mathrm{Id}"] \arrow[d, "E"'] & H^p(H^q) \\ H^p(H^q) \arrow[r, "T"'] & H^p(H^q) \arrow[u, "P"'] \end{tikzcd} \qquad \|E\|\|P\| \leq C/\delta,$$

where the constant $C > 0$ is universal.

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Let $1 \leq p, q < \infty$, $\delta > 0$ and $T : H^p(H^q) \to H^p(H^q)$ be a linear operator with large diagonal, i.e. $\langle Th_{I\times J}, h_{I\times J}\rangle \geq \delta|I \times J|$. Then we have

$$\begin{tikzcd} H^p(H^q) \arrow[r, "\mathrm{Id}"] \arrow[d, "E"'] & H^p(H^q) \\ H^p(H^q) \arrow[r, "T"'] & H^p(H^q) \arrow[u, "P"'] \end{tikzcd} \qquad \|E\|\|P\| \leq C/\delta,$$

where the constant $C > 0$ is universal.

By Capon's bi-parameter construction $(\mathscr{B}_{I\times J})$ and a random choice of signs $\varepsilon_{I\times J}$, there exists a block basis $b^{(\varepsilon)}_{I\times J} = \sum_{K\in\mathscr{B}_{I\times J}} \varepsilon_{K\times L} h_{K\times L}$ of the Haar system such that

$$Tb^{(\varepsilon)}_{I\times J} = \alpha_{I\times J} b^{(\varepsilon)}_{I\times J} + \text{small error}, \qquad \alpha_{I\times J} \geq \delta.$$

The orthogonal projection $Qf = \sum_{I\in\mathscr{D}} \frac{\langle f, b^{(\varepsilon)}_{I\times J}\rangle}{\|b^{(\varepsilon)}_{I\times J}\|_2^2} b_{I\times J}$ is bounded on $H^p(H^q)$.

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Proof – ordering of $\mathscr{R}$

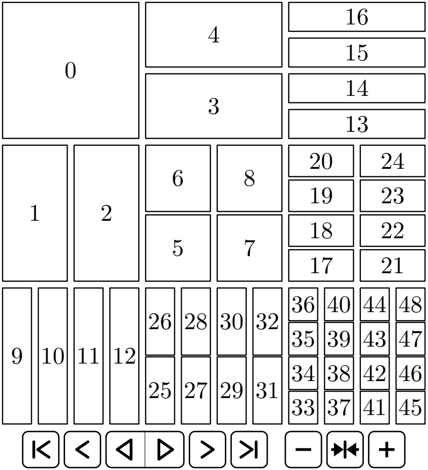

Figure: Order of the first 49 rectangles.

- If $I \in \mathscr{D}$, then $\widetilde{I} \in \mathscr{D}$ is such that $\widetilde{I} \supset I$ and $|\widetilde{I}| = 2|I|$.
- $\mathcal{O}_\triangleleft(I \times J)$ is the order number of $I \times J$.
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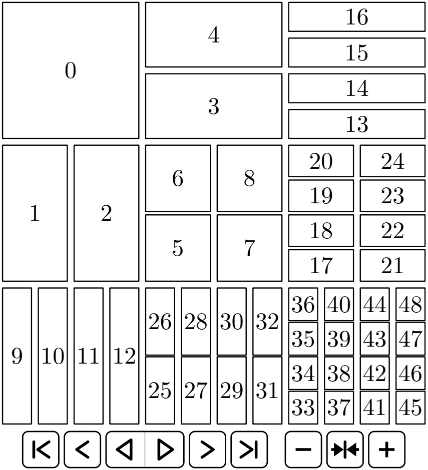

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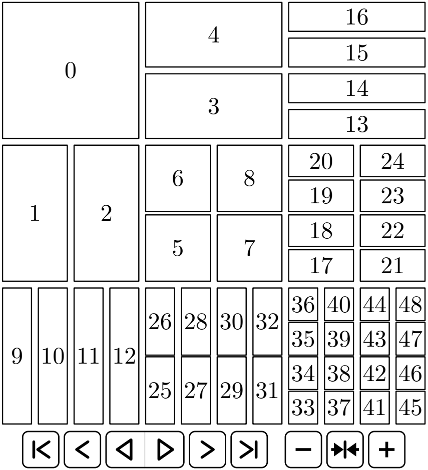

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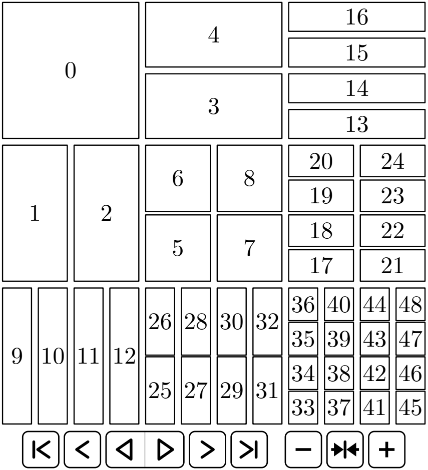

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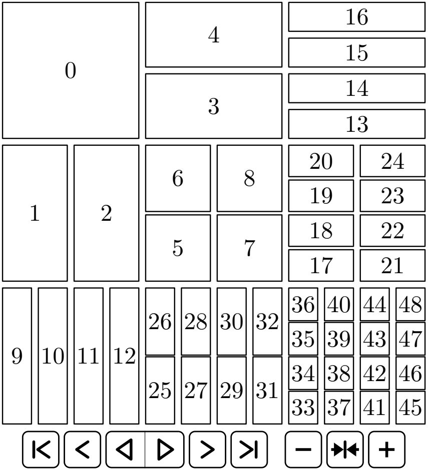

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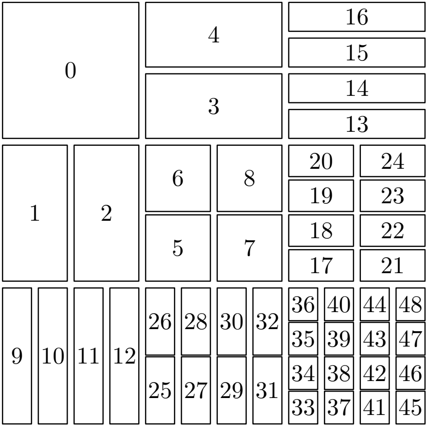

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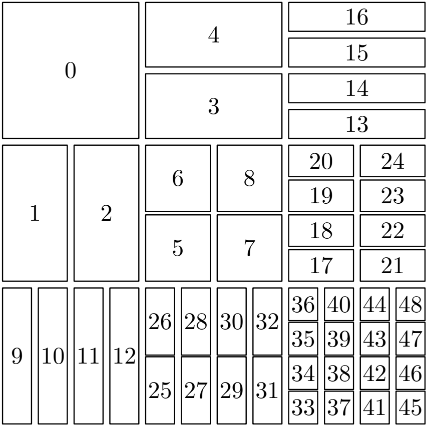

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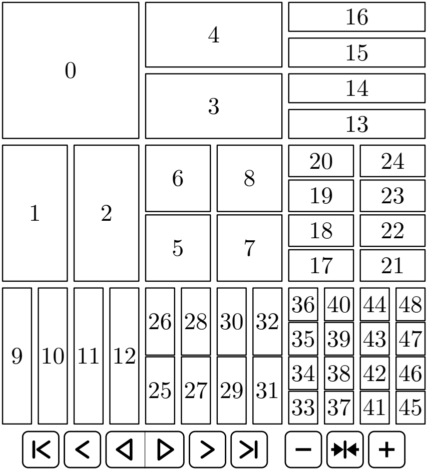

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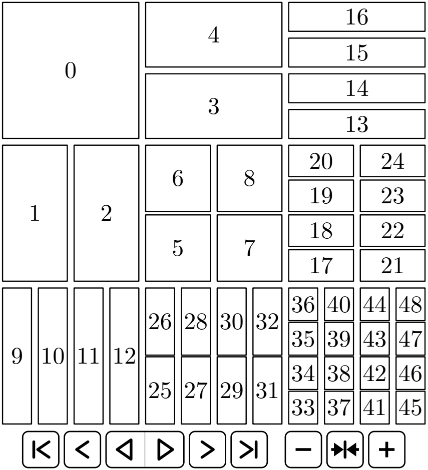

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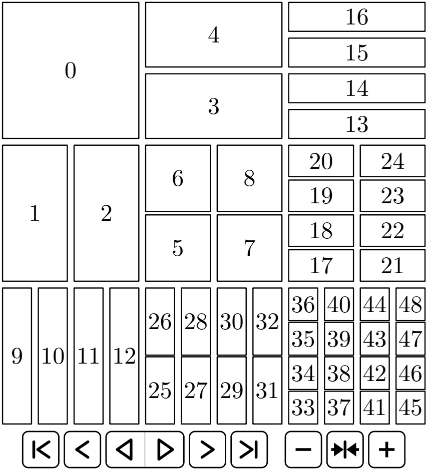

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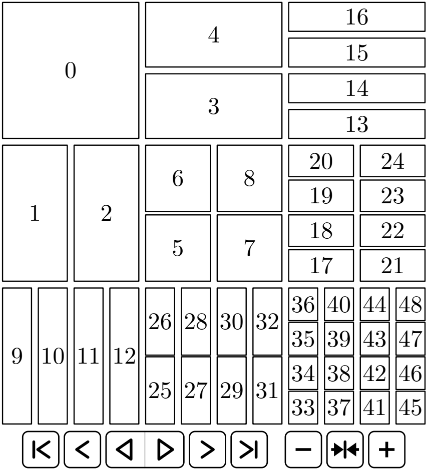

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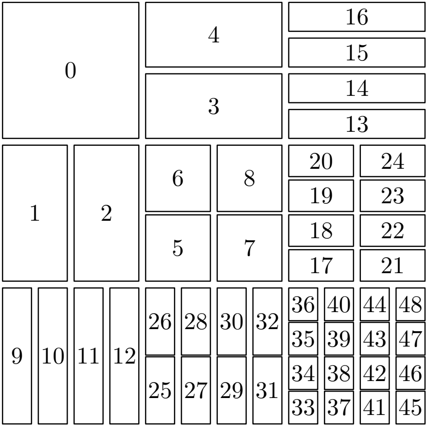

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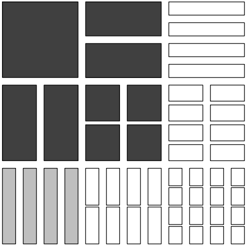

Figure: Darkgray=past, lightgray=present, white=future.

- $K_0 \times [0,1] \in \mathscr{B}_{\widetilde{I} \times [0,1]}$
- $K \times [0,1] \in \mathscr{F}_m$

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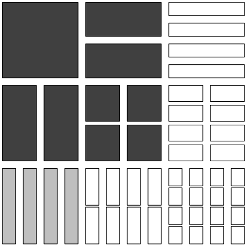

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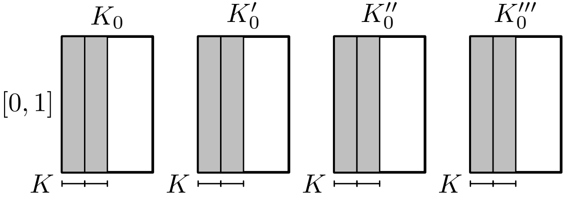
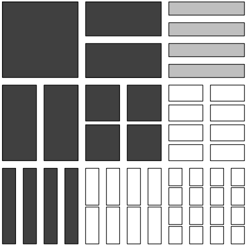


Proof – case $J \neq [0,1]$, $I = [0,1]$



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- $[0,1] \times L_0 \in \mathscr{B}_{[0,1]\times\widetilde{J}}$
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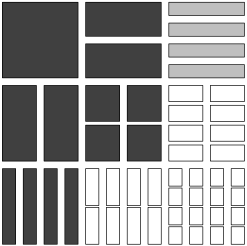

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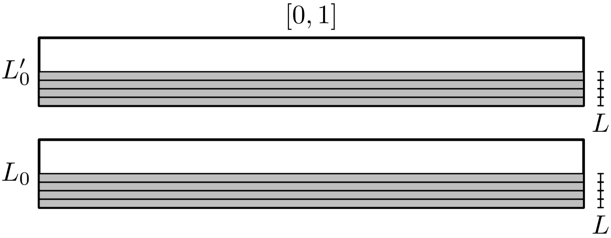

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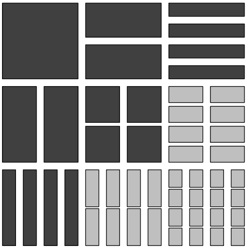

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Proof – case $J \neq [0,1]$, $I \neq [0,1]$

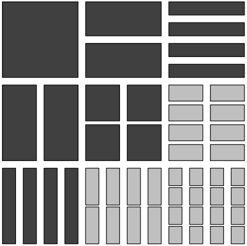

Figure: Darkgray=past, lightgray=present.

- $K_0 \times L_0 \in \mathscr{B}_{I \times \tilde{J}}$
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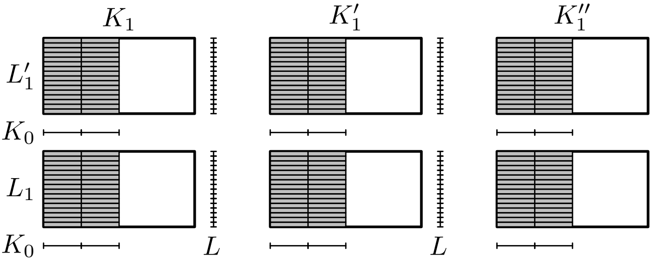

Proof – the block basis $b^{(\varepsilon)}_{I\times J}$

- $\mathscr{F}_m$ is determined by the inductive construction.
- We define $f^{(\varepsilon)}_m = \sum_{K\times L\in\mathscr{F}_m} \varepsilon_{K\times L} h_{K\times L}$.
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- Thus, $\sum_{j=1}^{i-1} |\langle T b^{(\varepsilon)}_j, f^{(\varepsilon)}_m\rangle| + |\langle f^{(\varepsilon)}_m, T^* b^{(\varepsilon)}_j\rangle| \leq \eta 4^{-i} \|f^{(\varepsilon)}_m\|_2^2$.
- What we need is $\langle T f^{(\varepsilon)}_m, f^{(\varepsilon)}_m\rangle \geq (\delta - \eta)\|f^{(\varepsilon)}_m\|_2^2$.
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Putting it together

- Define $Bh_{I\times J} = b^{(\varepsilon)}_{I\times J}$, then $\|B\|\|B^{-1}\| \le 1$.
- $Tb^{(\varepsilon)}_{I\times J} = \alpha_{I\times J} b^{(\varepsilon)}_{I\times J} +$ small error, $\alpha_{I\times J} \ge \delta$.
- T is almost diagonal relative to $(b^{(\varepsilon)}_{I\times J})$ with large diagonal entries
- The projection $Qf = \sum_i \frac{\langle f, b_i^{(\varepsilon)}\rangle}{\|b_i^{(\varepsilon)}\|_2^2} b_i^{(\varepsilon)}$ is bounded on $H^p(H^q)$.

$$\begin{tikzcd} H^p(H^q) \arrow[r, "I"] \arrow[d, "B"'] & H^p(H^q) \\ \operatorname{span}\{b_i^{(\varepsilon)}\} \arrow[r, "I"'] & \operatorname{span}\{b_i^{(\varepsilon)}\} \arrow[u, "B^{-1}"'] \end{tikzcd}
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$\|J\|\|V\| \le C/\delta$, and $C > 0$ is a universal constant.

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ArXiv e-prints, September 2015.

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