Abstract of the PhD thesis

Some applications of set theory in Banach spaces and operator algebras

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The dissertation is devoted to selected problems in functional analysis whose solutions rely on set-theoretic and topological methods. We discuss four topics involving issues such as invariants of Banach spaces, convergence of Radon measures or the existence of embeddings of various C*-algebras into the Calkin algebra.

In the first part we study the σ-ideals of subsets of Banach spaces generated by hyperplanes and investigate their standard cardinal characteristics: the additivity, the covering number, the uniformity and the cofinality. We determine their values for separable Banach spaces, and we show that it is consistent that they depend only on the density for all Banach spaces. The remaining questions can be reduced to deciding if the following can be proved in ZFC for every nonseparable Banach space X:

- X can be covered by ω_1-many of its hyperplanes,
- all subsets of X of cardinalities less than $\mathrm{cf}([\mathrm{dens}(X)]^\omega)$ can be covered by countably many hyperplanes.

We also answer these questions in the affirmative in many well-investigated classes of Banach spaces. The first question is related to the problem whether every compact Hausdorff space which has a small diagonal is metrizable and the second to large cardinals.

The second topic concerns Banach spaces of continuous functions on compact spaces. We show that if K is a separable connected compact space, $C(K)$ has few operators (i.e. every bounded linear operator $T\colon C(K) \to C(K)$ is of the form $T(f) = fg + S(f)$, where S is weakly compact and $g \in C(K)$) and $C(K)$ is isomorphic to $C(L)$ for some compact space L, then K and L are homeomorphic modulo finitely many points. Next, for every natural number $n > 0$ we construct, assuming Jensen's diamond principle ($\diamondsuit$), a compact space K that has the covering dimension equal to n and possesses the above mentioned properties. We conclude that if L is a compact space such that $C(L)$ is isomorphic to $C(K)$, then $\dim L = n$.

The third topic concerns measure-theoretic properties of Boolean algebras and related Banach spaces. We define a σ-centered notion of forcing that forces the existence of a Boolean algebra with the Grothendieck property and without the Nikodym property. In particular, we prove that the existence of such an algebra is consistent with the negation of the continuum hypothesis. The algebra we construct consists of Borel subsets of the Cantor set and has cardinality ω_1. We also show how to apply our method to streamline Talagrand's construction of such an algebra under the continuum hypothesis.

The last part of the dissertation is devoted to the Calkin algebra $\mathcal{Q}(\ell_2)$ i.e. the C*-algebra of bounded operators on ℓ_2 divided by the ideal of compact operators. We show that in the Cohen model there is no *-embedding of ℓ_∞-sum of Calkin algebras into $\mathcal{Q}(\ell_2)$. We conclude that in the Cohen model the corona of the stabilization of $\mathcal{Q}(\ell_2)$ is not isomorphic to $\mathcal{Q}(\ell_2)$.