Self-similar solutions of kinetic-type equations

Kamil Bogus

Wrocław University of Science and Technology (POLAND)

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Article

The talk is based on joint work with Dariusz Buraczewski and Alexander Marynych:

K. Bogus, D. Buraczewski, A. Marynych,
Self-similar solutions of kinetic-type equations: The boundary case,
Stochastic Processes and their Applications, p. 18 (2019)
https://doi.org/10.1016/j.spa.2019.03.005.

Fourier–Stieltjes transform

$(\rho_t)_{t\geq 0}$ – time dependent family of probability measures,

$\phi(t,\xi)$ – Fourier–Stieltjes transform (the characteristic function) of ρ_t,

$$\phi(t,\xi) = \int_{\mathbb{R}} e^{i\xi v} \rho_t(\mathrm{d}v), \quad t \geqslant 0, \quad \xi \in \mathbb{R},$$

Introduction - Smoothing transform

$\widehat{Q}$ – smoothing transform

$$\widehat{Q}(\phi_1, \ldots, \phi_N)(\xi) := \mathbf{E}(\phi_1(A_1\xi) \cdot \ldots \cdot \phi_N(A_N\xi)), \quad \xi \in \mathbb{R},$$

where

$$\begin{aligned} \phi_1, \ldots, \phi_N &- \text{characteristic functions,} \\ N &- \text{ fixed positive integer,} \\ \mathcal{A} = (A_1, \ldots, A_N) &- \text{ vector of real-valued random variables.} \end{aligned}$$

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Example:
$N = 2$ and $\mathcal{A} = (\sin\theta, \cos\theta)$, where θ is a random angle uniformly distributed on $[0, 2\pi)$

Cauchy problem

We consider the following Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\phi(t,\xi) + \phi(t,\xi) & = & \widehat{Q}(\phi(t,\cdot),\dots,\phi(t,\cdot))(\xi), \quad t > 0, \\ \phi(0,\xi) & = & \phi_0(\xi), \quad \xi \in \mathbb{R}, \end{cases}$$

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Main Goal:
Study asymptotic behavior of the solution ϕ.

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(Φ) For the function $\Phi : [0,\infty) \mapsto \mathbb{R} \cup \{+\infty\}$ defined via

$$\Phi(s) = \mathbf{E}\left[\sum_{i=1}^{N} A_i^s\right] - 1, \quad s \geqslant 0,$$

we assume that $s_\infty > 0$ where $s_\infty := \sup\{s \geqslant 0 : \Phi(s) < \infty\}$

Spectral function

The function

$$\mu(s) = \frac{\Phi(s)}{s}, \quad s > 0,$$

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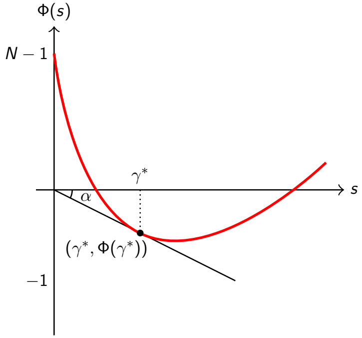

Figure: Plot of the function $s \mapsto \Phi(s)$ (solid red) with $\tan\alpha = \mu(\gamma^*) = \Phi'(\gamma^*) = \Phi(\gamma^*)/\gamma^*$.

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- Results on probabilistic interpretation of the solution can be found in works of Gabetta, Regazzini, Carlen and Carvalho,
- Based on McKean's ideas, Bassetti, Ladelli and Matthes expressed the solution in a convenient probabilistic way,
- Assuming that (H_γ) holds for some $\gamma \in (0, 2]$ and there exists $\delta > \gamma$ such that $\mu(\delta) < \mu(\gamma) < \infty$, they showed that $\phi(t, e^{-\mu(\gamma)t}\xi)$ converges to a nondegenerate limit being the characteristic function of the law of the limit of some positive martingale related to a family of random labelled trees.

Main result - existence of the limit

Theorem (KB, DB, AM (SPA 2019))

Assume that $(\mathbb{A}), (\Phi)$ *hold, the hypothesis* (H_γ) *is satisfied for some* $\gamma \in (0, 2]$ *and*

$$\gamma = \underset{s\in(0,s_\infty)}{\arg\min}\, \mu(s) = \gamma^* \in (0, s_\infty).$$

Then there exists a probability measure ρ_∞ *such that the function* ϕ*, the unique solution to* $\partial\phi_t/\partial t + \phi_t = \widehat{Q}\phi_t$ *with the initial condition* ϕ_0*, satisfies*

$$\lim_{t\to\infty} \phi\big(t, t^{\frac{1}{2\gamma}} e^{-\mu(\gamma)t}\xi\big) = w_\infty(\xi), \quad \xi \in \mathbb{R},$$

where w_∞ *is the Fourier-Stieltjes transform of* ρ_∞*.*

Main result - representation of the limit

Theorem (KB, DB, AM (SPA 2019))

Characteristic function w_∞ has the following representation

$$w_\infty(\xi) = \mathbf{E}\widehat{g}_\gamma(\xi c_\gamma D_\infty^{1/\gamma}),$$

where $c_\gamma := \left(\frac{2}{\pi\gamma^2\Phi''(\gamma)}\right)^{\frac{1}{2\gamma}}$ and D_∞ is a.s. positive random variable which satisfies the following stochastic fixed-point equation

$$D_\infty \stackrel{d}{=} \mathcal{U}^{\Phi(\gamma)} \sum_{k=1}^{N} A_k^\gamma D_\infty^{(k)},$$

where $(D_\infty^{(k)})_{k=1}^N$ are independent copies of D_∞; $\mathcal{U}$ has a uniform distribution on $(0,1)$ and $(D_\infty^{(k)})_{k=1}^N$, $\mathcal{U}$ and $(A_1, \ldots, A_N)$ are independent.

Thank you very much for attention.