# Weighted boundedness of multilinear maximal function using Dirac deltas

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# Motivation

- For a locally integrable function $f$ on $\mathbb{R}^n$, Hardy-Littlewood Maximal function is defined as

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

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- It maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Classical proof depends on Covering lemmas(Vittali). By interpolation, it maps $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$.

# Introduction

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$K^*$ is weak type $(1,1)$ if and only if $\exists\, C > 0$ such that, for any set of distinct points $a_1, \dots, a_N$ and for each $\lambda > 0$

$$\left| \{x : \sup_j \left| \sum_{i=1}^{N} K_j(x - a_i) \right| > \lambda\} \right| \le C \frac{N}{\lambda}. \qquad \text{(Guzmán)}$$

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> Let $\phi = \sum_{i=1}^{N} b_i \delta_{a_i}$ and for any $\lambda > 0$, define $E_\lambda := \{x : M\phi(x) > \lambda\}$. Then
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> $$|E_\lambda| \leq \frac{2^n}{\lambda} \sum_{i=1}^{N} |b_i|.$$

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This provides another proof for weak $(1,1)$ boundedness.

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- M. Trinidad Menárguez and F. Soria(1992):

If $K_j \geq 0$, then the constant in the weak $(1,1)$ inequality is same as the constant in the following inequality

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- This method played a crucial role in obtaining the best constant in the weak type $(1,1)$ inequality for the centred Hardy-Littlewood maximal operator on $\mathbb{R}$ in the works of Menárguez and Soria(1992), Manfredi and Soria, J.M. Aldaz(1998) and finally it is settled by A. D. Melas(2003).

# Weighted Ineq.

Theorem (B. Muckenhoupt,1972)

$M$ is of weighted weak-type $(1, 1)$ if and only if $w \in A_1$ and when $1 < p < \infty$, $M$ is weighted strong type $(p, p)$ if and only if $w \in A_p$.

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- Classical proofs depend on Calderón-Zygmund decomposition.

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$$w(\{x : M\phi(x) > \lambda\}) \lesssim \frac{C_{w,n}}{\lambda} \sum_{i=1}^{N} |b_i| w(a_i). \qquad (*)$$

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Given $w \in C(\mathbb{R}^n)$, $M$ satisfies (*) on linear combination of Dirac deltas if and only if $w \in A_1$.

# Contd.

**Theorem (Termini and Vitanza, 1989)**

Hardy-Littlewood maximal function is weak type $(1, 1)$ with respect to $w$ if and only if $w \in A_1$.

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M. Trinidad Menárguez extended this for more general class of maximal convolution operators.

# Main Result

Let $\{K_i^1\}$ and $\{K_i^2\}$ be families of locally integrable kernels defined on $\mathbb{R}^n$,

$$T_i(f_1, f_2)(x) = \left(K_i^1 * f_1\right)(x)\left(K_i^2 * f_2\right)(x).$$

Consider the bilinear maximal operator defined by

$$T^*(f_1, f_2)(x) = \sup_{i \in \mathbb{N}} |T_i(f_1, f_2)(x)|.$$

# Contd.

Let $w_1, w_2$, and $v$ be continuous weight functions and $q > 0$. Assume that

1. $K_i^1, K_i^2 \in L^\infty(\mathbb{R}^n)$ and $K_i^1(. - y), K_i^2(. - y) \in L^1(v)$ for all $y \in \mathbb{R}^n$.
2. Given $\epsilon > 0$, a ball $B \subseteq \mathbb{R}^n$ and $i \in \mathbb{N}$, there are $\gamma_1 = \gamma(\epsilon, i, B)$ and $\gamma_2 = \gamma(\epsilon, i, B)$ such that

$$\int_B |K_i^j(x - y_1) - K_i^j(x - y_2)| v(x) dx < \epsilon$$

whenever $|y_1 - y_2| < \gamma_j$ for $j = 1, 2$.

# Main Result

Theorem (-, Shrivastava, Shuin)

Under the hypothesis the following are equivalent

- $T^*$ is bounded from $L^1(w_1) \times L^1(w_2)$ to $L^{q,\infty}(v)$.
- For any set of distinct points $\{a_l\}_{l=1}^N$ and $\{b_k\}_{k=1}^L$ and for any $\lambda > 0$, $\exists\, C > 0$ such that

$$v\Big\{x \in \mathbb{R}^n : T^*(\sum_{l=1}^N \delta_{a_l}, \sum_{k=1}^L \delta_{b_k})(x) > \lambda\Big\} \leq \frac{C_{\vec{w},n}}{\lambda^q}\Big(\sum_{l=1}^N w_1(a_l)\Big)^q \Big(\sum_{k=1}^L w_2(b_k)\Big)^q.$$

# Applications

- Multilinear fractional maximal function(Kabe Moen, 2009):

For $0 \leq \alpha < 2n$, the multilinear fractional maximal function is defined as follows

$$\mathscr{M}_\alpha(f_1, f_2)(x) = \sup_{r>0} \prod_{i=1}^{2} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{2n}}} \int_{B(x,r)} |f_i(y)| dy.$$

- For $\alpha = 0$, the corresponding operator is the multilinear Hardy-Littlewood maximal operator defined by Lerner *et al*(2009).

# Contd.

$$\mathscr{M}_\alpha(f_1, f_2)(x) = \sup_{r_i \in \mathbb{Q}^+} |K_{r_i} * f_1(x)||K_{r_i} * f_2(x)|$$
$$= \sup_{i \geq 1} |K_i * f_1(x)||K_i * f_2(x)|$$

where $K_i = K_{r_i}(x) = \frac{\chi_{B(0,r_i)}(x)}{|B(0,r_i)|^{1-\frac{\alpha}{2n}}}$.

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where $K_i = K_{r_i}(x) = \frac{\chi_{B(0,r_i)}(x)}{|B(0,r_i)|^{1-\frac{\alpha}{2n}}}$.

$$|K_i(x - y_2) - K_i(x - y_1)| = \frac{1}{|B(0, r_i)|^{1-\frac{\alpha}{2n}}} \chi_{B_i^*}$$

where $B_i^* = B(y_1, r_i) \cup B(y_2, r_i) \setminus B(y_1, r_i) \cap B(y_2, r_i)$.

**Lemma (-, Shrivastava, Shuin)**

Let $(w_1, w_2, v)$ be continuous weight functions. Suppose that for any set of distinct points $\{a_l\}_{l=1}^N$ and $\{b_k\}_{k=1}^L$ and for any $\lambda > 0$, $\exists\, C > 0$ such that

$$v\Big\{x \in \mathbb{R}^n : \mathscr{M}_\alpha(\sum_{l=1}^N \delta_{a_l}, \sum_{k=1}^L \delta_{b_k})(x) > \lambda\Big\} \leq \frac{C}{\lambda^q}\left(\sum_{l=1}^N w_1(a_l)\right)^q \left(\sum_{k=1}^L w_2(b_k)\right)^q$$

holds, then the bilinear fractional maximal function $\mathscr{M}_\alpha$ is bounded from $L^1(w_1) \times L^1(w_2)$ to $L^{q,\infty}(v)$.

# Contd.

**Lemma (-, Shrivastava, Shuin)**

Let $(w_1, w_2, v)$ be continuous weight functions. Then $\mathscr{M}_\alpha$ is bounded from $L^1(w_1) \times L^1(w_2)$ to $L^{q,\infty}(v)$ on finite sum of Dirac deltas if and only if $(w_1, w_2, v) \in A_{\vec{1},\alpha}$.

We say $(w_1, w_2, v) \in A_{\vec{1},\alpha}$ if and only if

$$\frac{1}{|B|}\int_B v(x)dx \leq C(ess\inf_B w_1(x))^q(ess\inf_B w_2(x))^q$$

for all balls $B \subseteq \mathbb{R}^n$ and $q = \frac{n}{2n-\alpha}$.

# Contd.

For continuous weighs, $\mathscr{M}_\alpha$ is bounded from $L^1(w_1) \times L^1(w_2)$ to $L^{q,\infty}(v)$ if and only if $(w_1, w_2, v) \in A_{\vec{1},\alpha}$.

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From continuous to arbitrary weights:

- $(w_1, w_2) \in A_{\vec{1},\alpha}$ if and only if $v_{\vec{w},\alpha}, w_1^q, w_2^q \in A_1$, where $v_{\vec{w},\alpha} = w_1^q w_2^q$ and $q = \frac{n}{2n-\alpha}$.
- Take $\vec{w} = (w_1, w_2) \in A_{\vec{1},\alpha}$. For $\epsilon > 0$, define $\vec{w_\epsilon} = (w_{1,\epsilon}, w_{2,\epsilon})$ and $v_{\vec{w},\epsilon}$ by

$$w_{i,\epsilon}^q(x) := \frac{1}{|B(x,\epsilon)|} \int_{B(x,\epsilon)} w_i^q(y)dy \quad \text{for } \ i = 1, 2$$

and $v_{\vec{w},\epsilon}(x) = w_{1,\epsilon}^q(x) w_{2,\epsilon}^q(x)$.

# Contd.

**Lemma (-, Shrivastava, Shuin)**

Let $\vec{w} = (w_1, w_2) \in A_{1,\vec{\alpha}}$ and $\epsilon > 0$. Then $\vec{w_\epsilon} = (w_{1,\epsilon}, w_{2,\epsilon}) \in A_{1,\vec{\alpha}}$ with a constant independent of $\epsilon$.

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Let $\vec{w} = (w_1, w_2) \in A_{1,\vec{\alpha}}$ and $\epsilon > 0$. Then $\vec{w_\epsilon} = (w_{1,\epsilon}, w_{2,\epsilon}) \in A_{1,\vec{\alpha}}$ with a constant independent of $\epsilon$.

Theorem (-, Shrivastava, Shuin)

Let $(w_1, w_2, v)$ be weights. The bilinear maximal function $\mathscr{M}_\alpha$ is bounded from $L^1(w_1) \times L^1(w_2)$ to $L^{q,\infty}(v)$ if and only if

$$\frac{1}{|B|}\int_B v(x)dx \leq C(ess\inf_B w_1(x))^q(ess\inf_B w_2(x))^q$$

for all balls $B \subseteq \mathbb{R}^n$.

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*Thank You!!!*