Universität Bielefeld

Heat kernel bounds for symmetric Markov processes with singular jump measures

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The main result of this work can be seen as a step forward towards the following conjecture

Conjecture: Let $K(t,x;s,y)$ denote the fundamental solution of the operator

$$u \mapsto \partial_t u - \mathrm{p.\,v.} \int_{\mathbb{R}^d} \big(u(y) - u(x)\big) \boldsymbol{J(x, \mathrm{d}y)}\,, \tag{1}$$

where $J(x, \mathrm{d}y)$ is symmetric and satisfies $J(x,\mathrm{d}y) \asymp \boldsymbol{\nu(x - \mathrm{d}y)}$ for some Lévy measure ν. Then, $K(t,x;s,y)$ is comparable to $\widetilde{K}(t-s, x-y)$, where $\widetilde{K}$ is the heat kernel of the Lévy process corresponding to ν.

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Questions:

1. Does one need to restrict the class of admissible Lévy measures?
2. Do we know any counterexample?
3. How rich is the class of confirming examples?

Heat kernel bounds/Aronson bounds:

Given uniformly elliptic coefficients (a_{ij}), it is shown in [Aronson 68] that the fundamental solution $\Gamma(t,x;s,y)$ of the operator $u \mapsto \partial_t u - \partial_i(a_{ij}\partial_j u)$ satisfies for all $t, s > 0$ and $x, y \in \mathbb{R}^d$ the two-sided estimate holds:

$$g_1(t-s, x-y) \leq \Gamma(t,x;s,y) \leq g_2(t-s, x-y)\,, \tag{2}$$

where $g_j(t,x) = a_j t^{-d/2} \exp(-b_j \frac{|x-y|^2}{t})$ and a_j, b_j are some positive constants.

No further regularity of (a_{ij}) as a function on $\mathbb{R}^d$ is required.

In probabilistic language: The heat kernel of a non-degenerate diffusion is controlled from above and below by the heat kernel of the Brownian Motion.

Question: For which class of Markov processes/operators does such a universality result hold true?

Isotropic α-stable processes: [Bass/Levin 2002], [Chen/Kumagai 2003]

Let $K(t,x;s,y)$ denote the fundamental solution of the operator

$$u \mapsto \partial_t u - \mathrm{p.\,v.} \int_{\mathbb{R}^d} \big(u(y) - u(x)\big) J(x,y)\, \mathrm{d}y\,, \tag{3}$$

where $J(x,y)$ is symmetric and satisfies $J(x,y) \asymp |x-y|^{-d-\alpha}$ for some $\alpha \in (0,2)$. Then, for all $t,s>0$ and $x,y \in \mathbb{R}^d$ the two-sided estimate holds:

$$k_1(t-s, x-y) \leq K(t,x;s,y) \leq k_2(t-s,x-y)\,, \tag{4}$$

where $k_j(t,x) = c_j t^{-d/\alpha}\big(1 \wedge \frac{|x|^\alpha}{t}\big)^{\frac{-d-\alpha}{\alpha}}$ and $c_j > 0$.

Theorem (Chen/Kumagai): *For $0 < \alpha \leq 2$, the fundamental solution of the (fractional) Laplacian $(-\Delta)^{\alpha/2}$ bounds the fundamental solution of corresponding non-degenerate differential resp. integro-differential operator.*

Aim of this talk: Show that this theorem extends to other jump processes.

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$$u \mapsto \partial_t u - \mathrm{p.v.} \int_{\mathbb{R}^d} \big(u(y) - u(x)\big) \boldsymbol{J(x, \mathrm{d}y)} \,, \tag{5}$$

where $J(x,\mathrm{d}y)$ is symmetric and satisfies $J(x,\mathrm{d}y) \asymp \boldsymbol{\nu(x - \mathrm{d}y)}$ for some Lévy measure ν. Then, $K(t,x;s,y)$ is comparable to $\widetilde{K}(t-s, x-y)$, where $\widetilde{K}$ is the heat kernel of the Lévy process corresponding to ν.

The conjecture has been confirmed for several isotropic cases of fixed order in the last 15 years and extended to cases with isotropic variable order. Some recent works in this direction.

Bae/Kang/Kim/Lee:*Heat kernel estimates for symmetric jump processes with mixed polynomial growths*, arXiv

Bae/Kang/Kim/Lee:*Heat kernel estimates and their stabilities for symmetric jump processes with general mixed polynomial growths on metric measure spaces*, arXiv

Chen/Chen/Wang:*Heat kernel for non-local operators with variable order*, arXiv

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Present project: We consider an α-stable measure ν, which is not isotropic

Theorem (Kim/MK/Kumagai): *For $0 < \alpha \leq 2$, the fundamental solution of the (fractional) Laplacian $\sum_{k=1}^{d} (-\partial_{kk})^{\alpha/2}$ bounds the fundamental solution of the corresponding non-degenerate differential resp. integro-differential operator.*

Let $\alpha \in (0,2)$. Let ν^α be the classical isotropic α-stable measure. Let ν be a measure on the Borel sets of $\mathbb{R}^d$ defined by

$$\nu(\mathrm{d}h) = c_{d,\alpha} \sum_{i=1}^{d} \Big[|h^i|^{-1-\alpha} \mathrm{d}h^i \prod_{j \neq i} \delta_{\{0\}}(\mathrm{d}h^j) \Big]$$

Both measures ν^α and ν are non-degenerate α-stable measures. They are limit cases in the family of non-degenerate α-stable measures.

Recall that a measure ν on $\mathcal{B}(\mathbb{R}^d)$ is a **non-degenerate α-stable measure**, if for some $\alpha \in (0,2)$

$$\nu(E) = (2-\alpha) \int_{S^{d-1}} \int_0^\infty \mathbb{1}_E(r\theta) r^{-1-\alpha} \, dr \, \pi(d\theta) \qquad (E \in \mathcal{B}(\mathbb{R}^d)), \qquad (6)$$

where π is some finite measure on S^{d-1} and $lin(\operatorname{supp} \pi) = \mathbb{R}^d$. The measure π is sometimes called spectral measure.

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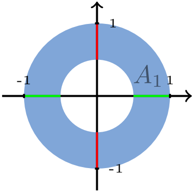
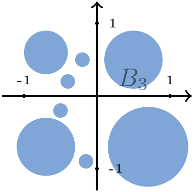
The Lévy process corresponding to ν is the process (Z_t). ν charges only sets that have a nonempty intersection with one of the coordinate axes.





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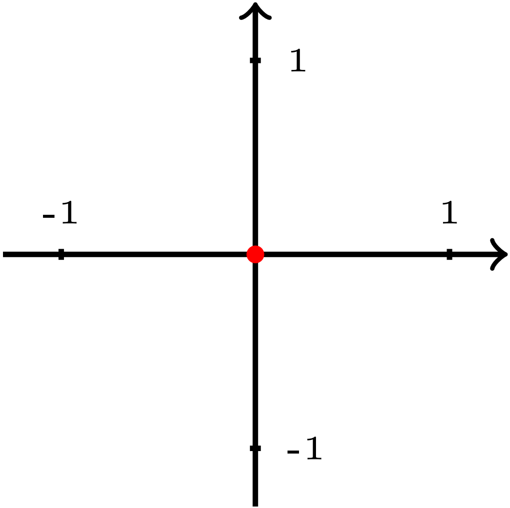

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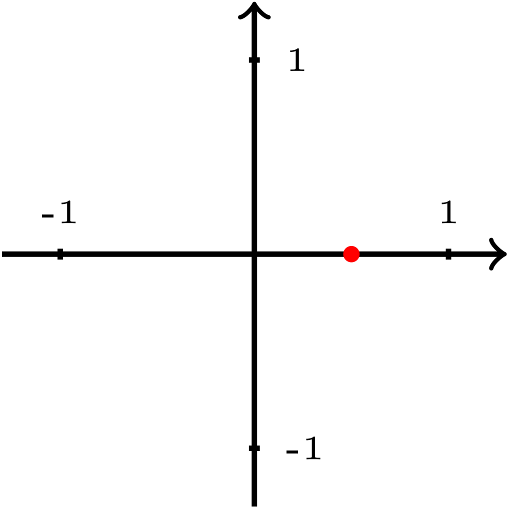

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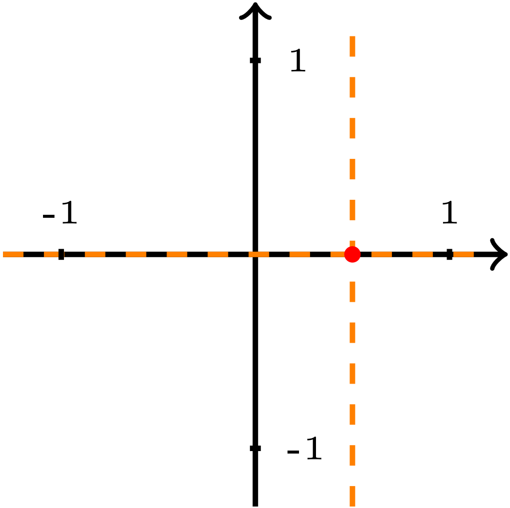

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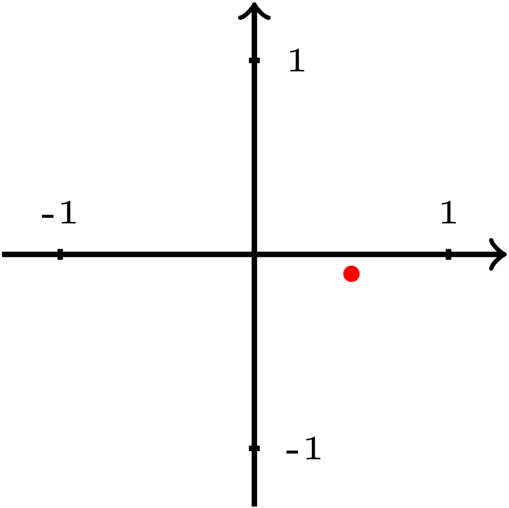

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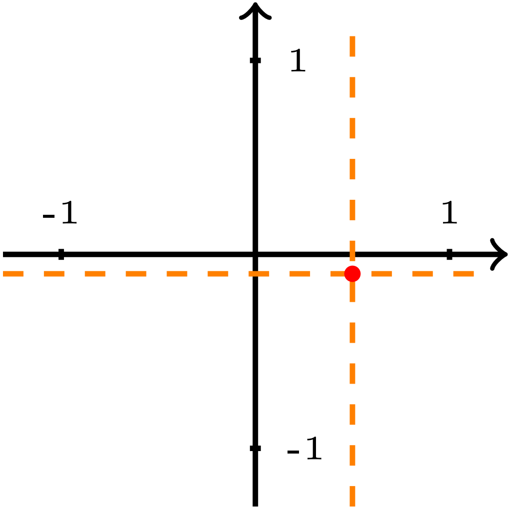

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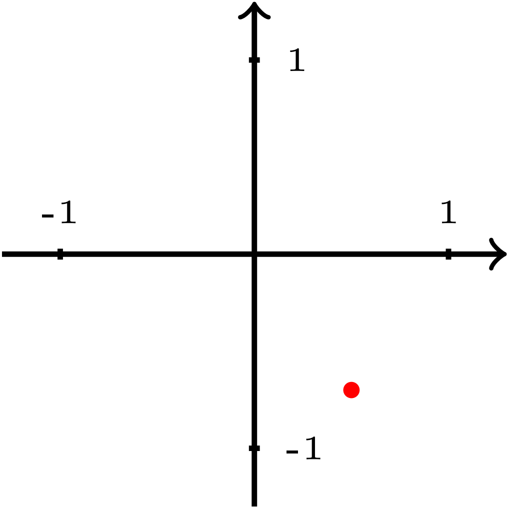

The Dirichlet form corresponding to ν

For $u \in C_c^\infty(\mathbb{R}^d)$, the corresponding generator L evaluates as follows

$$Lu(x) = \text{p. v.} \int_{\mathbb{R}^d} \big(u(x+h) - u(x)\big)\nu(\mathrm{d}h) \qquad (x \in \mathbb{R}^d)$$

$$\mathcal{F}(-Lu)(\xi) = \Big(\sum_{i=1}^{d} |\xi^i|^\alpha\Big)\mathcal{F}(u)(\xi) = c_\alpha \mathcal{F}\Big((-\partial_{11})^{\alpha/2} + \ldots + (-\partial_{dd})^{\alpha/2}\Big)(\xi)\,.$$

The corresponding Dirichlet form on $L^2(\mathbb{R}^d)$ is given by $(\mathcal{E}^\alpha, D^\alpha)$, where

$$D^\alpha = \{u \in L^2(\mathbb{R}^d) |\ \mathcal{E}^\alpha(u,u) < \infty\}$$

$$\mathcal{E}^\alpha(u,v) = \int_{\mathbb{R}^d} \Big(\sum_{i=1}^{d} \int_{\mathbb{R}} \big(u(x+e^i\tau) - u(x)\big)\big(v(x+e^i\tau)-v(x)\big)\frac{\mathrm{d}\tau}{|\tau|^{1+\alpha}}\Big)\mathrm{d}x$$

$$= \int_{\mathbb{R}^d} \Big(\sum_{i=1}^{d} \int_{\mathbb{R}} \big(u(x+e^i\tau)-u(x)\big)\big(v(x+e^i\tau)-v(x)\big) J^\alpha(x, x+e^i\tau)\mathrm{d}\tau\Big)\mathrm{d}x\,.$$

Here, $J^\alpha(x,y) = |y^i - x^i|^{-1-\alpha}$ if, for some i, $x^i \neq y^i$ and $x^j = y^j$ for every $j \neq i$. There is no need to specify values $J^\alpha(x,y)$ for $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}$. For simplicity, we set $J^\alpha(x,y) = 0$ if $x^i \neq y^i$ for more than one index i.

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$(\mathcal{E}^\alpha, D^\alpha)$ is a regular (symmetric) Dirichlet form on $L^2(\mathbb{R}^d)$. For the heat kernel p_t^α one has $p_t^\alpha(x,y) \asymp t^{-d/\alpha} \prod_{i=1}^d \Big(1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|}\Big)^{1+\alpha}$

Probabilistic interpretation of the theorem

Consider a Markov jump process Z in $\mathbb{R}^d$ defined by $Z_t = (Z_t^1, \ldots, Z_t^d)$, where the coordinate processes $Z_t^1, \ldots, Z_t^d$ are independent one-dimensional symmetric stable processes of index $\alpha \in (0, 2)$.

The infinitesimal generator of the corresponding semigroup of the process Z is the integro-differential operator $L = (-\partial_{11})^{\alpha/2} + (-\partial_{22})^{\alpha/2} + \ldots + (-\partial_{dd})^{\alpha/2}$, whose symbol resp. multiplier is given by $\sum_{i=1}^{d} |\xi^i|^\alpha$.

The stated theorem shows the following: If X is a d-dimensional pure jump Markov process in $\mathbb{R}^d$ whose jump kernel is comparable to that one of the process Z, then the heat kernels of Z and X satisfy the same sharp two-sided estimates.

Within the framework of quadratic forms we can set up the result.

Assumptions and main result

Assume $J : \mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag} \to [0, \infty]$ satisfies for all $x \neq y$

$$\Lambda^{-1} J^\alpha(x, y) \leq J(x, y) \leq \Lambda J^\alpha(x, y) \tag{6}$$

Set

$$D = \{u \in L^2(\mathbb{R}^d) | \ \mathcal{E}(u, u) < \infty\}$$

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \Big(\sum_{i=1}^{d} \int_{\mathbb{R}} \big(u(x+e^i\tau) - u(x)\big)\big(v(x+e^i\tau) - v(x)\big) J(x, x+e^i\tau) \mathrm{d}\tau \Big) \mathrm{d}x \,,$$

$(\mathcal{E}, \mathcal{F})$ is a regular (symmetric) Dirichlet form on $L^2(\mathbb{R}^d)$ where $\mathcal{F} = \overline{C_c^1(\mathbb{R}^d)}^{\mathcal{E}_1}$. Let p_t be transition density of the corresponding conservative Hunt process X.

Theorem (Kim/MK/Kumagai): *There exists $C \geq 1$ such that for any $t > 0, x, y \in \mathbb{R}^d$*

$$C^{-1} t^{-d/\alpha} \prod_{i=1}^{d} \left(1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|} \right)^{1+\alpha} \leq p_t(x, y) \leq C t^{-d/\alpha} \prod_{i=1}^{d} \left(1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|} \right)^{1+\alpha}.$$

Some results on the cylindrical stable process Z and related processes

Theorem (Bass/Chen, 2006): *The system*

$$dY_t^i = \sum_{j=1}^{d} A_{ij}(Y_{t-})\mathrm{d}Z_t^j$$

admits a unique weak solution if A is continuous and non-degenerate.

Theorem (Bass/Chen, 2010):
(1) The process Z resp. its generator does not satisfy a Harnack inequality.
(2) Bounded functions that are harmonic with respect to Y are Hölder continuous.

Theorem (Kulczycki/Ryznar/Sztonyk, 2018): *The semigroup corresponding to Y is strong Feller if A is non-degenerate, bounded and Lipschitz continuous.*

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Remark: Existence of weak solutions has been extended by Jamil Chaker (arxiv) to the case, where the coordinate processes Z_t^i are stable processes of index $\alpha^i \in (0,2)$. Regularity of harmonic functions is proved separately if A is diagonal.

Some results on the nonlocal Dirichlet form generating X

Theorem (Dyda/MK, to appear in Anal. & PDE):
Assume

$$\mathcal{E}(u,v) = \iint \big(u(y)-u(x)\big)\big(v(y)-v(x)\big)\mu(x,\mathrm{d}y)\mathrm{d}x\,,$$

*and $\mu(x,\mathrm{d}y)$ is uniformly (w.r.t. the variable x) comparable on small scales to $\widehat{\nu}(\mathrm{d}y-\{x\})$ for **some** non-degenerate α-stable measure $\widehat{\nu}$, $\alpha \in (0,2)$. Then solutions $u:\mathbb{R}^d \to \mathbb{R}$ to*

$$\mathcal{E}(u,\phi) = 0 \qquad (\phi \in C_c^\infty(B_1)),$$

satisfy uniform Hölder regularity estimates in the interior of B_1.

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and $\mu(x,\mathrm{d}y)$ *is uniformly (w.r.t. the variable* x*) comparable on small scales to* $\widehat{\nu}(\mathrm{d}y-\{x\})$ *for* ***some*** *non-degenerate* α*-stable measure* $\widehat{\nu}$*,* $\alpha \in (0,2)$*. Then solutions* $u:\mathbb{R}^d \to \mathbb{R}$ *to*

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satisfy uniform Hölder regularity estimates in the interior of B_1*.*

Remark: This theorem has been extended by Jamil Chaker/MK (arxiv) to the case, where μ is singular and comparable to a Lévy measure corresponding to a process consisting of coordinate processes Z_t^i that are stable processes of index $\alpha^i \in (0,2)$.

Some results on the process X generated by our nonlocal Dirichlet form

What is known about the heat kernel for X ?

Some results on the process X generated by our nonlocal Dirichlet form

Theorem (Xu, 2013):
(1) The corresponding Hunt process X is conservative and X has a Hölder continuous transition density $p(t,x,y)$ on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$.
(2) There exists $C \geq 1$ such that for any $t>0, x, y \in \mathbb{R}^d$

$$p_t(x,y) \geq C^{-1} t^{-d/\alpha} \prod_{i=1}^{d} \left(1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|}\right)^{\boldsymbol{1+\alpha}} .$$

(3) There exists $C \geq 1$ such that for any $t>0, x, y \in \mathbb{R}^d$

$$p_t(x,y) \leq C t^{-d/\alpha} \prod_{i=1}^{d} \left(1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|}\right)^{\boldsymbol{\alpha/3}} .$$

Remark: The exponent $\alpha/3$ appears in this result because the Davies' method does not seem to allow for a better exponent.

Remark: It remains to prove the sharp upper bound.

The main idea is to use

(A) ... self-improving properties of upper heat kernel bounds and to apply them in the right order

Technically, the proof consists of ...

(B) ... several reduction tricks due to symmetry and on-diagonal bounds

(C) ... tracing the process when it leaves a ball

(D) ... smart upper bounds on $(P_t f, g)$ for non-negative localized functions f, g

(A) Self-improving properties of upper heat kernel bounds

For any $q > 0$ and $l \in \{1, \dots, d-1\}$, consider the following conditions.

(H_q^0) There exists $C_0(q) > 0$ such that for all $t > 0$, $x, y \in \mathbb{R}^d$,

$$p_t(x,y) \le C_0 t^{-d/\alpha} \prod_{i=1}^{d} \left(\frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^q . \tag{7}$$

(H_q^l) There exists $C_l(q) > 0$ such that for all $t > 0$ and all $x, y \in \mathbb{R}^d$ and every permutation σ satisfying $|x^{\sigma(1)} - y^{\sigma(1)}| \le \ldots \le |x^{\sigma(d)} - y^{\sigma(d)}|$:

$$p_t(x,y) \le C_l t^{-d/\alpha} \prod_{i=1}^{d-l} \left(\frac{t}{|x^{\sigma(i)} - y^{\sigma(i)}|^\alpha} \wedge 1 \right)^q \prod_{i=d-l+1}^{d} \left(\frac{t}{|x^{\sigma(i)} - y^{\sigma(i)}|^\alpha} \wedge 1 \right)^{1+\alpha^{-1}} \tag{8}$$

Lemma: *Condition (H_q^l) implies the following condition:*

$(H_q^l)'$ *There exists $C_l(q) > 0$ such that for all $t > 0$ and all $x, y \in \mathbb{R}^d$*

$$p_t(x,y) \le C_l t^{-d/\alpha} \prod_{i=1}^{d-l} \left(\frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^q \prod_{i=d-l+1}^{d} \left(\frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^{1+\alpha^{-1}} \tag{9}$$

Set

$$\lambda_l := \frac{1}{2}\Big(\sum_{i=1}^{d-l-1}(1+\alpha^{-1})^i\Big)^{-1} \quad \text{for} \quad l \in \{0,1,\dots,d-2\}, \quad \text{and} \quad \lambda_{d-1} := 1.$$

Note $(\alpha/3)^{d-l-1} \le \lambda_l \le \frac{\alpha}{2(1+\alpha)}$ for $l \in \{0,1,\dots,d-2\}$. Our aim is to prove assertion $\big(H^{d-1}_{1+\alpha^{-1}}\big)$. It will be the last assertion in a sequence of assertions which are proved subsequently in the followings order:

$$\begin{array}{llll} (H^0_0) & \hookrightarrow (H^0_{\lambda_0}) & \hookrightarrow (H^0_{2\lambda_0}) \dots & \hookrightarrow (H^0_{N_0\lambda_0}) \\ \hookrightarrow (H^1_0) & \hookrightarrow (H^1_{\lambda_1}) & \hookrightarrow \dots\dots\dots & \hookrightarrow (H^1_{N_1\lambda_1}) \\ \vdots & & & \vdots \\ \hookrightarrow (H^{d-1}_0) & \hookrightarrow (H^{d-1}_1) & \hookrightarrow \dots\dots\dots & \hookrightarrow (H^{d-1}_{N_{d-1}}) \end{array}$$

where $N_l := \lfloor 1 + \frac{\alpha^{-1}}{\lambda_l} \rfloor$ for $l \in \{0,\dots,d-1\}$. Note $N_{d-2} > \frac{2}{\alpha} + \frac{2}{\alpha^2} > \frac{3}{2}$ and $N_0 > \frac{2}{\alpha}\sum_{i=1}^{d-1}(1+\alpha^{-1})^i > 2\left(\left(\frac{3}{2}\right)^{d-1} - 1\right)$.

The above scheme will be established with the help of several implications.

Lemma 1:
Assume $\left(H_q^l\right)$ holds for some $l \in \{0, \ldots, d-2\}$, $q < \alpha^{-1}$. Then $\left(H_{q+\lambda_l}^l\right)$ holds.

Lemma 2:
Assume $\left(H_q^l\right)$ holds for some $l \in \{0, \ldots, d-2\}$, $q > \alpha^{-1}$. Then $\left(H_0^{l+1}\right)$ holds.

Lemma 3:
(i) Assume $\left(H_q^{d-1}\right)$ holds for some $q < \alpha^{-1}$. Then $\left(H_{q+\lambda_{d-1}}^{d-1}\right)$ holds true.
(ii) Assume $\left(H_q^{d-1}\right)$ holds for some $q > \alpha^{-1}$. Then $\left(H_{1+\alpha^{-1}}^{d-1}\right)$ holds true.

Note that assertion (i) of Lemma 3 and assertion of Lemma 1 can be seen as one implication $\left(H_q^l\right) \Rightarrow \left(H_{q+\lambda_l}^l\right)$ being true for every $l \in \{0, \ldots, d-1\}$. However, we decide to split the assertion into two cases. The proof of Lemma 3 is much simpler than the one of Lemma 1. However, both rely on our main technical result.

(B) Reduction tricks due to symmetry and on-diagonal bounds

Definition: Let $x_0, y_0 \in \mathbb{R}^d$ and $t > 0$. Let $\sigma : \{1, \dots, d\} \to \{1, \dots, d\}$ denote a permutation such that $|x_0^{\sigma(i)} - y_0^{\sigma(i)}| \leq |x_0^{\sigma(i+1)} - y_0^{\sigma(i+1)}|$ for every $i \in \{1, \dots, d-1\}$. Set $\rho := t^{1/\alpha}$. For $i \in \{1, \dots, d\}$ define $\theta_i \in \mathbb{Z}$ and $R_i \in \mathbb{R}$ such that

$$\tfrac{5}{4} 2^{\theta_i} \rho \leq |x_0^{\sigma(i)} - y_0^{\sigma(i)}| < \tfrac{10}{4} 2^{\theta_i} \rho \qquad \text{and} \qquad R_i = 2^{\theta_i} \rho \,. \tag{10}$$

Then $\theta_i \leq \theta_{i+1}$ and $R_i \leq R_{i+1}$. We say that a condition $\mathcal{R}(i_0)$ holds if

$$\theta_1 \leq \ldots \leq \theta_{i_0 - 1} \leq 0 < 1 \leq \theta_{i_0} \leq \ldots \leq \theta_d \,. \tag{$\mathcal{R}(i_0)$}$$

We say that condition $\mathcal{R}(d+1)$ holds if $\theta_1 \leq \ldots \leq \theta_d \leq 0 < 1$.

Lemma: *Let $t > 0$ and x_0, y_0 be such that condition $\mathcal{R}(i_0)$ holds for some $i_0 \in \{d - l + 1, \dots, d+1\}$. Assume* (8) *holds for some $l \in \{0, \dots, d-1\}$ and $q \geq 0$. Then*

$$p_t(x_0, y_0) \leq C t^{-d/\alpha} \prod_{i=1}^{d} \left(\frac{t}{|x_0^i - y_0^i|^\alpha} \wedge 1 \right)^{1 + \alpha^{-1}} \tag{11}$$

for some constant $C > 0$ independent of t and x_0, y_0.

(C) Tracing the process when it leaves a ball

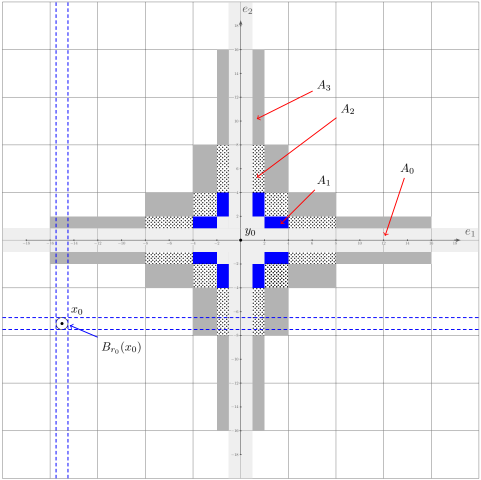

Figure: The sets A_k and D_k^m

(D) Upper bounds on $(P_t f, g)$ for non-negative localized functions f, g

For $f, g \geq 0$ on $\mathbb{R}^d$, set $(f,g) = \int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x$. For any non-negative Borel functions f on $\mathbb{R}^d$ and for any $t > 0$, $x \in \mathbb{R}^d$, let $\{P_t\}_{\{t\geq 0\}}$ be the transition semigroup of X defined by

$$P_t f(x) = \mathbb{E}^x \left[f(X_t) \right] = \int_{\mathbb{R}^d} f(y) p(t,x,y) dy.$$

Lemma (Barlow/Grigoryan/Kumagai, 2009): *Let $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^d$ be disjoint, non-empty and open. Let f, g be non-negative Borel functions on $\mathbb{R}^d$. Let $\tau = \tau_U$ and $\tau^{'} = \tau_V$ be the first exit times from U and V, respectively. Then, for all $a, b, t > 0$ with $a + b = t$, we have*

$$\Big(P_t f, g\Big) \leq \Big(\mathbb{E}^x \left[\mathbb{1}_{\{\tau \leq a\}} P_{t-\tau} f(X_\tau) \right], g\Big) + \Big(\mathbb{E}^x \left[\mathbb{1}_{\{\tau' \leq b\}} P_{t-\tau} g(X_{\tau'}) \right], f\Big). \tag{12}$$

Proposition: *Let* $\alpha \in (0,2)$ *and* $t > 0$*. Assume that* $\left(H_q^l\right)$ *holds true for some* $l \in \{0,1,\dots,d-1\}$ *and* $q \in [0, 1+\alpha^{-1}]$*. Let* $x_0, y_0 \in \mathbb{R}^d$ *satisfy the condition* $\mathcal{R}(i_0)$ *for some* $i_0 \in \{1,\dots,d-l\}$*. Set* $\rho = t^{1/\alpha}$ *and* $R_j = 2^{\theta_j}\rho$ *as defined above. For each* $j_0 \in \{i_0,\dots,d-l\}$*, let* $\tau := \tau_{B(x_0, R_{j_0}/8)}$*. Then there exists* $C > 0$ *independent of* x_0, y_0 *and* t *such that for every* $x \in B(x_0, \rho/8)$*,*

$$\mathbb{E}^x\left[\mathbb{1}_{\{\tau \le t/2\}} P_{t-\tau} f(X_\tau)\right]$$
$$\le C t^{-d/\alpha} \|f\|_1 \prod_{j=j_0+1}^{d-l} \left(\frac{t}{|x_0^{\sigma(j)} - y_0^{\sigma(j)}|^\alpha} \wedge 1\right)^q \prod_{j=d-l+1}^{d} \left(\frac{t}{|x_0^{\sigma(j)} - y_0^{\sigma(j)}|^\alpha} \wedge 1\right)^{1+\alpha^{-1}}$$
$$\times \begin{cases} \left(\frac{t}{|x_0^{\sigma(j_0)} - y_0^{\sigma(j_0)}|^\alpha} \wedge 1\right)^{1+q} & \text{if } q < \alpha^{-1} \\ \left(\frac{t}{|x_0^{\sigma(j_0)} - y_0^{\sigma(j_0)}|^\alpha} \wedge 1\right)^{1+\alpha^{-1}} & \text{if } q > \alpha^{-1}. \end{cases} \tag{13}$$