Extrapolation for multilinear Muckenhoupt weight classes

Delft University of Technology

Bas Nieraeth

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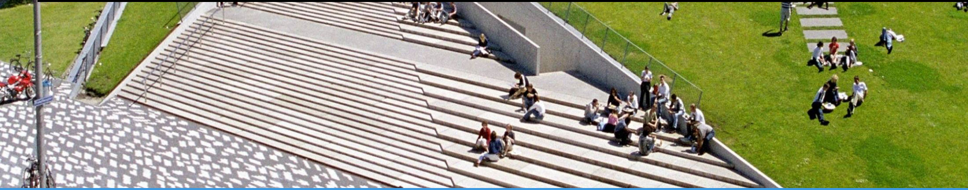

Outline

1. Motivation: Multilinear operators
2. Weight classes
3. Extrapolation: the Rubio de Francia algorithm
4. Multilinear extrapolation

The bilinear Hilbert transform

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BHT *is bounded* $L^{p_1} \times L^{p_2} \to L^p$ *for all* $p_1, p_2 \in (1,\infty]$ *and* $p \in (\frac{2}{3},\infty)$. *Here* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

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- Note: As opposed to the linear setting, for bilinear singular integrals the cases $p_j=\infty$, $p<1$ need to be considered.

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- Warning: The condition $[w]_\infty$ does not characterize $A_\infty = \bigcup_{p\in[1,\infty)} A_p$.

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1. $M(W^{-1}) \leq 2\|M\|_{L^p_w \to L^p_w} W^{-1}$. Thus $[W]_\infty \leq 2\|M\|_{L^p_w \to L^p_w}$.
2. $\|W^{-1}\|_{L^p_w} \leq \sum_{k=0}^\infty \frac{\|f\|_{L^p_w}}{2^k}$

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- Their proof relies on sparse domination techniques.
- [N., 2018] There are operators $N_j : L^{p_j}_{w_j} \to L^{p_j}_{w_j}$ such that $\|N_j\|_{L^{p_j}_{w_j}\to L^{p_j}_{w_j}} \lesssim 1$,

  $M_{1,1}(f_1,f_2)(x) \lesssim [w_1,w_2]_{p_1,p_2}^{\max(p_1',p_2')} N_1 f_1(x) N_2 f_2(x)$.

  $N_j f_j = M^{w^p}_{p_j-1}\big(M^{w_j^{-p_j'}}(f_j w_j^{p_j'}) w_j^{1-p_j'} w^{-\frac{p}{p_j}}\big) w^{\frac{p}{p_j}} w_j^{-1}$.
- Setting $W_j^{-1} = \sum_{k=0}^\infty \frac{N_j^k f}{(2\|N_j\|_{L^{p_j}_{w_j}\to L^{p_j}_{w_j}})^k}$ does the job!

$$
\begin{aligned}
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Multilinear extrapolation

Theorem (N. 2018)

Let $(f_1, \ldots, f_m, h)$ be an $m+1$-tuple of measurable functions. Suppose that for some $q_1, \ldots, q_m \in [1, \infty]$, there is an increasing function $\phi_{\vec{q}}$ such that for all $\vec{w} = (w_1, \ldots, w_m) \in A_{\vec{q}}$, $w = \prod_{j=1}^{m} w_j$,

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Then for all $p_1 \ldots, p_m \in (1, \infty]$ with $p < \infty$ and all $\vec{w} \in A_{\vec{p}}$

$$\|h\|_{L^p_w} \leq 2^{m^3} \phi_{\vec{q}}\left(C_{\vec{p},\vec{q}} [\vec{w}]_{\vec{p}}^{\frac{1}{m} \max\left(\frac{p_1'}{q_1'}, \ldots, \frac{p_m'}{q_m'}, \frac{p}{q}\right)}\right)^m \prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}}.$$

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Multilinear extrapolation: a short overview

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Applications

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- Suppose a bilinear operator T satisfies the *sparse domination*

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- Moreover, extrapolation yields vector-valued extensions

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for $p_1, p_2, q_1, q_2 \in (1, \infty]$, $p, q \in (\frac{1}{2}, \infty)$.

Applications: The bilinear Hilbert transform

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whenever $\max(\frac{1}{r_1}, \frac{1}{2}) + \max(\frac{1}{r_2}, \frac{1}{2}) + \max(\frac{1}{s'}, \frac{1}{2}) < 2$.

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- The fact that they allow ℓ^∞ served as a motivation to consider the endpoints $p_j = \infty$ in the extrapolation theorem.
- Using the weighted estimates obtained from (1), multilinear limited range extrapolation completely recovers (2).