RANDOM INTERVAL HOMEOMORPHISMS

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This is a joint work with 

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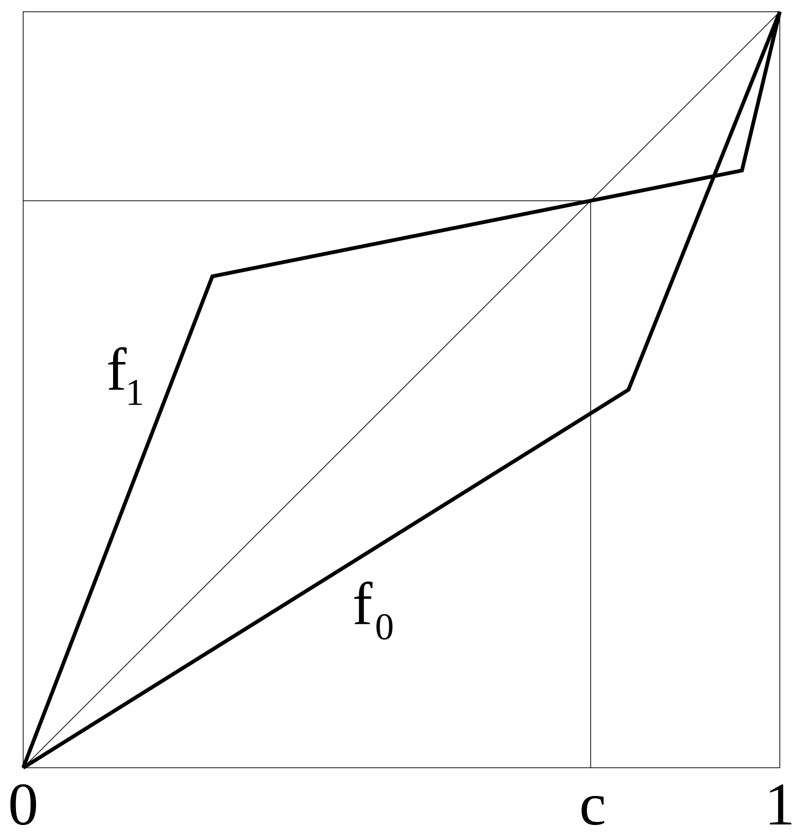
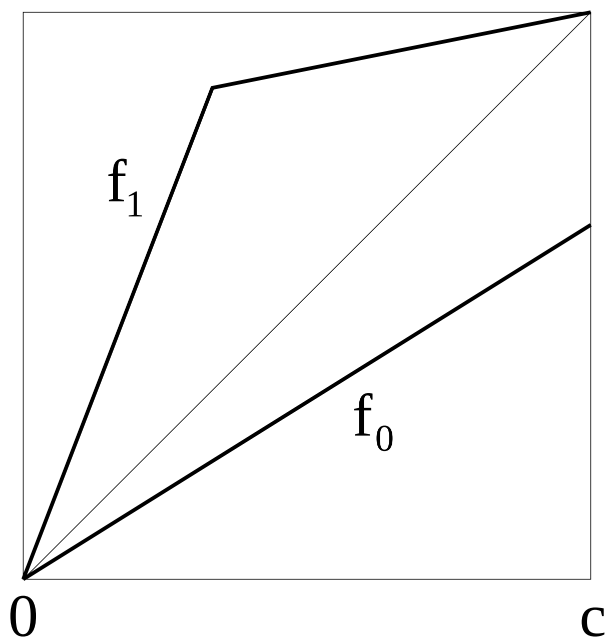
Motivation:

A talk by Yulij Ilyashenko. Two interval maps, applied randomly.



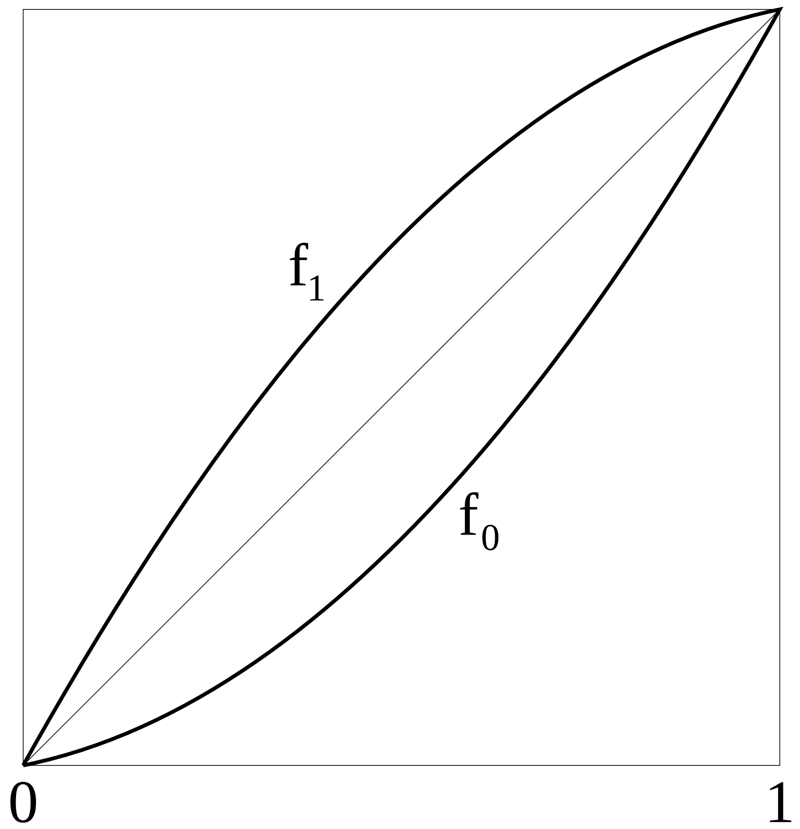
The interval $(c, 1]$ is inessential (all points except 1 leave it and never come back). Therefore really those two maps look like that:


f_1
f_0
0
c

An important thing is that *one of the maps is not a surjection.* This causes contraction on average and the existence of an attractor.

We want to get a similar effect for two homeomorphisms, f_0, f_1 of $[0,1]$ onto itself; f_0 should move all points of $(0,1)$ to the left, and f_1 to the right. To avoid 0 or 1 to be attracting, we require existence of derivatives at 0 and 1, with $f_0'(0)f_1'(0) > 1$ and $f_0'(1)f_1'(1) > 1$.

If f_0 and f_1 are to be smooth, it looks like a good idea to choose maps with negative Schwarzian derivative, for example quadratic functions:



The picture suggests that $f_0'(0)f_1'(0) < 1$ and $f_0'(1)f_1'(1) < 1$.

We apply a useful rule:

Before starting to investigate something, check whether it exists.

Let $f : [0,1] \to [0,1]$ be an increasing smooth function with negative Schwarzian derivative, with $f(0) = 0$ and $f(1) = 1$. Then f expands cross-ratios. In particular, if $0 < x < y < 1$ then

$$\frac{(1-0)(f(y)-f(x))}{(f(x)-0)(1-f(y))} > \frac{(1-0)(y-x)}{(x-0)(1-y)},$$

that is,

$$\frac{f(x)-0}{x-0} \cdot \frac{1-f(y)}{1-y} < \frac{f(y)-f(x)}{y-x}.$$

As $x \to 0$ and $y \to 1$, we get in the limit

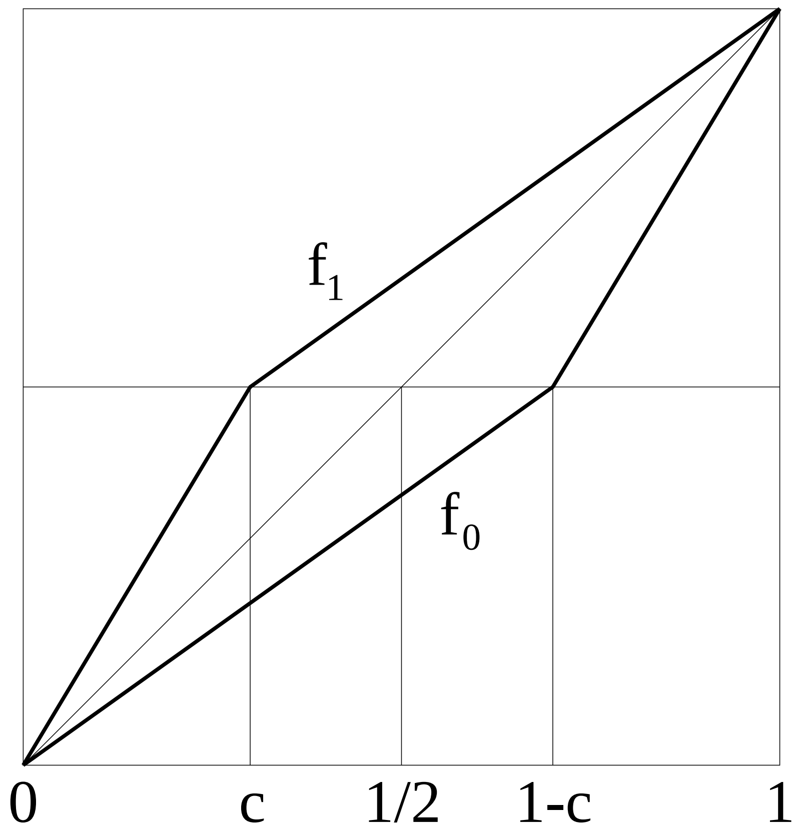
$$f'(0)f'(1) \leq 1.$$

Thus, if both f_0 and f_1 have negative Schwarzian derivative, then we cannot have $f_0'(0)f_1'(0) > 1$ and $f_0'(1)f_1'(1) > 1$.

In fact, we discovered later that the right choice of this type would be *positive* Schwarzian derivative. Such situation has been considered by Bonifant and Milnor [BM]. They investigated similar skew products, with expanding circle maps in the base.

Next best choice – piecewise linear maps. To simplify the situation, make the following choices:

- there are only two linear pieces for each map,
- the "critical value" for both maps is the same,
- the graph of f_1 is symmetric to the graph of f_0 with respect to the center of the square.



Denote the slopes by $a < 1$ and $b > 1$. We have $a = \dfrac{1/2}{1-c}$ and $b = \dfrac{1/2}{c}$, so $\dfrac{1}{a} + \dfrac{1}{b} = 2$. Thus, the harmonic mean of a and b is 1, so the geometric mean is larger than 1. Therefore, $ab > 1$.

Setup:

A skew product over the Bernoulli shift $(1/2, 1/2)$, one sided $(\Sigma_+, \sigma_+, \mu_+)$, or two-sided (Σ, σ, μ). In the one-sided case we have $F_+(\underline{\omega}, x) = (\sigma(\underline{\omega}), f_{\omega_0}(x))$, where $\underline{\omega} = (\omega_0, \omega_1, \omega_2, \dots) \in \Sigma_+$. In the two-sided case, $F(\underline{\omega}, x) = (\sigma(\underline{\omega}), f_{\omega_0} x))$, where $\underline{\omega} = (\dots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \dots) \in \Sigma$. In other words, we flip a coin and apply f_0 or f_1 depending on the results of the flip.

For a given $\underline{\omega}$ from Σ_+ or Σ, we will denote the projection to the second coordinate of $F_+^n(\underline{\omega}, x_0)$ or $F^n(\underline{\omega}, x_0)$ by x_n. The projection map will be denoted by π_2.

Main technical results:

Theorem 1. *For almost every* $\underline{\omega} \in \Sigma_+$ *and every* $x_0, y_0 \in (0,1)$ *we have*

$$\lim_{n\to\infty} |x_n - y_n| = 0.$$

Clearly, the same holds in the two-sided case.

I will indicate how to prove Theorem 1 later, if time allows. The basic idea is to introduce a new metric in $(0,1)$, in which the map is nonexpanding, and contracting (mildly) from time to time.

In the two-sided case, we get a measurable function with an invariant graph:

Theorem 2. *There exists a measurable function $\varphi : \Sigma \to (0,1)$, whose graph is invariant under F, such that for almost every $\underline{\omega} \in \Sigma$, if $x_0 < \varphi(\underline{\omega})$ then*

$$\lim_{n\to\infty} x_{-n} = 0$$

and if $x_0 > \varphi(\underline{\omega})$ then

$$\lim_{n\to\infty} x_{-n} = 1.$$

Proof. Let $G : \Sigma \times [0, \infty) \to \Sigma \times [0, \infty)$ be defined by the same (linear in x) formulas as F^{-1} close to 0. Set

$$\Gamma = \left\{ \underline{\omega} \in \Sigma : \lim_{n \to \infty} \tfrac{1}{n} \#\{k < n : \omega_k = 0\} = \tfrac{1}{2} \right\}.$$

By the Birkhoff Ergodic Theorem, $\mu(\Gamma) = 1$.

Let $\xi_n(\underline{\omega}) = \pi_2(G^n(\underline{\omega}, 1))$. Since $ab > 1$, if $\underline{\omega} \in \Gamma$, then $\xi_n(\underline{\omega}) \to 0$ as $n \to \infty$. In particular, $\xi(\underline{\omega}) = \max\{\xi_n(\underline{\omega}) : n = 0, 1, 2, \dots\}$ is finite and positive. All functions ξ_n are measurable, so ξ is also measurable. Set $\zeta(\underline{\omega}) = c/\xi(\underline{\omega})$. The function ζ is positive and measurable. Moreover, for every $x_0 \in [0, \zeta(\underline{\omega})]$ we have $x_{-n} = \xi_n(\underline{\omega}) \cdot x_0$, so $x_{-n} \to 0$ as $n \to \infty$.

If we set $y_0 = x_k$, then $y_{-n} = x_{k-n}$, so also $y_{-n} \to 0$ as $n \to \infty$. Thus, if we set

$$\varphi(\underline{\omega}) = \sup\{\pi_2\big(F^n\big(\sigma^{-n}(\underline{\omega}), \zeta(\sigma^{-n}(\underline{\omega}))\big)\big) : n \in \mathbb{Z}\},$$

then φ is measurable and positive, its graph is invariant for F, and for any $x_0 < \varphi(\underline{\omega})$ we have $x_{-n} \to 0$ as $n \to \infty$.

In a similar way we construct a measurable function $\widehat{\varphi} : \Sigma \to [0,1)$, with an invariant graph, such that for any $x_0 > \widehat{\varphi}(\underline{\omega})$ we have $x_{-n} \to 1$ as $n \to \infty$. Clearly, $\varphi(\underline{\omega}) \leq \widehat{\varphi}(\underline{\omega})$, so $0 < \varphi(\underline{\omega}) \leq \widehat{\varphi}(\underline{\omega}) < 1$. Now, using Theorem 1 and a simple lemma from [AM] we get $\widehat{\varphi} = \varphi$ a.e. $\square$

Observe that $\varphi(\underline{\omega})$ depends only on terms with negative indices of $\underline{\omega}$ (its "past").

Pullback attractor:

The graph of φ is a pullback attractor in the following sense:

Theorem 3. *For almost every $\underline{\omega} \in \Sigma$ and for every compact set $A \subset (0,1)$ and $\varepsilon > 0$ there exists N such that for every $n \geq N$*

$$F^n(\{\sigma^{-n}(\underline{\omega})\} \times A) \subset (\varphi(\underline{\omega}) - \varepsilon, \varphi(\underline{\omega}) + \varepsilon).$$

Proof. Take $x_0 < \varphi(\underline{\omega}) < y_0$ such that $(x_0, y_0) \subset (\varphi(\underline{\omega}) - \varepsilon, \varphi(\underline{\omega}) + \varepsilon)$ (if $\varphi(\underline{\omega}) = 0$ then we take $x_0 = 0$; if $\varphi(\underline{\omega}) = 1$ then we take $y_0 = 1$). Then by Theorem 2 (and monotonicity of our maps) $\lim_{n\to\infty} x_{-n} = 0$ and $\lim_{n\to\infty} y_{-n} = 1$. Therefore for n sufficiently large we have $A \subset (x_{-n}, y_{-n})$. $\square$

The graph of φ is a forward attractor in the following sense:

Theorem 4. *For almost every* $\underline{\omega} \in \Sigma$ *and for every* $x_0 \in (0,1)$ *we have*

$$\lim_{n \to \infty} |x_n - \varphi(\sigma^n(\underline{\omega}))| = 0.$$

Proof. This follows from Theorem 1, the fact that the values of φ are in $(0,1)$ and the F-invariance of the graph of φ. $\square$

Invariant measures:

The relevant invariant measures for F and F_+ are those that project to μ and μ_+. There are two trivial ergodic ones: $\mu \times \delta_0$ and $\mu \times \delta_1$ (in the one-sided case, $\mu_+ \times \delta_0$ and $\mu_+ \times \delta_1$).

The proof of the following theorem is basically taken from [BMS]:

Theorem 5. *There is at most one nontrivial ergodic measure invariant for F (resp. F_+) that projects to μ (resp. μ_+).*

Proof. If there are two such measures, say ν_1 and ν_2, there is an $\underline{\omega}$ for which Theorem 1 applies and two points x_0, y_0, such that $(\underline{\omega}, x_0)$ is generic for ν_1 and $(\underline{\omega}, y_0)$ is generic for ν_2. Then in the weak-* topology, the averages of the images of the Dirac delta measure at $(\underline{\omega}, x_0)$ converge to ν_1 and the averages of the images of the Dirac delta measure at $(\underline{\omega}, y_0)$ converge to ν_2. However, $|x_n - y_n|$ goes to 0, so we get $\nu_1 = \nu_2$. $\square$

We can easily identify those measures for F and F_+. For F, since μ is σ-invariant and the graph of φ is F-invariant, the measure μ lifted to this graph is F-invariant. Since μ is ergodic, this measure is also ergodic. It is nontrivial because φ is nontrivial.

Let λ be the Lebesgue measure on $[0, 1]$.

Theorem 6. *The measure $\mu_+ \times \lambda$ is invariant for F_+.*

Proof. It is enough to check that the measure of the preimage of a set is equal to the measure of the set itself for sets of the form $C \times J$, where C is a cylinder in Σ_+ and J is an interval not containing $1/2$ in the interior. The preimage of such a set is the disjoint union of two sets of the form $C_0 \times f_0^{-1}(J)$ and $C_1 \times f_1^{-1}(J)$, where C_0 and C_1 are cylinders of measure $\mu_+(C)/2$ each. The Lebesgue measures of the intervals $f_0^{-1}(J)$ and $f_1^{-1}(J)$ are $\lambda(J)/a$ and $\lambda(J)/b$ (not necessarily in that order). Since $\left(\frac{1}{a} + \frac{1}{b}\right)/2 = 1$, we get $(\mu_+ \times \lambda)(F_+^{-1}(C \times J)) = (\mu_+ \times \lambda)(C \times J)$. $\square$

One-sided vs. two-sided case:

As we saw in Theorem 4, the graph of φ is an attractor for the invertible system. However, a similar theorem does not hold for the noninvertible system.

Theorem 7. *There is no measurable function* $\varphi_+ : \Sigma_+ \to (0,1)$ *whose graph is* F_+*-invariant.*

Proof. If such a function exists, the measure μ_+ lifted to its graph would be a nontrivial F_+-invariant ergodic measure, so by Theorems 5 and 6 it would be equal to $\mu_+ \times \lambda$, a contradiction. $\square$

In fact, we can even drop the assumption that an attractor is invariant.

Theorem 8. *There is no measurable function* $\varphi_+ : \Sigma_+ \to (0,1)$ *whose graph is an attractor in the sense that for almost every* $\underline{\omega} \in \Sigma_+$ *and every* $x_0 \in (0,1)$ *we have*

$$\lim_{n\to\infty} |x_n - \varphi_+(\sigma_+^n(\underline{\omega}))| = 0.$$

Proof. Assume that such φ_+ exists. Let $\Pi : \Sigma \to \Sigma_+$ be the natural projection. Then the graph of $\varphi_+ \circ \Pi$ is an attractor for F, so by a lemma from [AM] $\varphi_+ \circ \Pi = \varphi$ a. e. Invariance of the graph of φ can be written as $\varphi(\sigma(\underline{\omega})) = \pi_2(F(\underline{\omega}, \varphi(\underline{\omega})))$ for a. e. $\underline{\omega} \in \Sigma$. For a. e. $\underline{\omega}_+ \in \Sigma_+$ there is such $\underline{\omega} \in \Sigma$ for which additionally $\Pi(\underline{\omega}) = \underline{\omega}_+$, and then

$$\begin{aligned}\varphi_+(\sigma_+(\underline{\omega}_+)) &= \varphi_+(\Pi(\sigma(\underline{\omega}))) = \varphi(\sigma(\underline{\omega})) = \pi_2(F(\underline{\omega}, \varphi(\underline{\omega}))) \\ &= \pi_2(F(\underline{\omega}, \varphi_+(\underline{\omega}_+))) = \pi_2(F_+(\underline{\omega}_+, \varphi_+(\underline{\omega}_+))).\end{aligned}$$

This shows that the graph of φ_+ is F_+-invariant, which is impossible by Theorem 7. $\square$

Mystery of the vanishing attractor:

We get a paradox. For an invertible system an attractor exists, but it vanishes when we pass to the noninvertible system. This happens in spite of the fact that in the definition of an attractor we only look at forward orbits, and that in the base the future is completely independent of the past.

On a philosophical level, we may conclude that even if the future is independent of the past, knowledge of the history may essentially simplify the description of the predictions for the future.

On a mathematical level, we see that the idea of a pullback attractor seems to be useful only for invertible systems (we mean essentially invertible, so we include also noninvertible ones with zero entropy).

Let us stress that we are talking about measurable functions. A nonmeasurable function with an invariant graph can be easily constructed. Using Axiom of Choice we can choose one point from every full orbit of σ_+ and assign to our function the value $1/2$ at such point. Since the maps f_0 and f_1 are bijections, this function can be extended uniquely to a function on Σ_+ with an invariant graph. By Theorem 1, this graph will be an attractor.

Note that such a function will have no connections with the function φ constructed earlier.

Ideas for the proof of Theorem 1:

The main idea is to introduce a better metric on $(0,1)$. Let $h:(0,1)\to\mathbb{R}$ be a homeomorphism given by the formula

$$h(x)=\begin{cases}\log x-\log\frac{1}{2} & \text{if } x\le\frac{1}{2},\\ \log\frac{1}{2}-\log(1-x) & \text{if } x>\frac{1}{2}.\end{cases}$$

Then we use the metric $d(x,y)=|h(x)-h(y)|$.

Lemma 9. *If either* $x, y \in (0, 1/2]$ *or* $x, y \in [1-c, 1)$ *then* $d(f_0(x), f_0(y)) = d(x, y)$*. If* $x, y \in [1/2, 1-c]$ *then*

$$d(f_0(x), f_0(y)) \leq \left(1 - \frac{2c}{3} d(x, y)\right) d(x, y). \tag{1}$$

If $x, y \in (0, c]$ *or* $x, y \in [1/2, 1)$ *then* $d(f_1(x), f_1(y)) = d(x, y)$*. If* $x, y \in [c, 1/2]$ *then* (1) *holds with* f_1 *instead of* f_0*.*

Set

$$\Gamma_+ = \left\{ \underline{\omega} \in \Sigma_+ : \lim_{n \to \infty} \tfrac{1}{n} \#\{k < n : \omega_k = 0\} = \tfrac{1}{2} \right\}.$$

By the Birkhoff Ergodic Theorem, $\mu_+(\Gamma_+) = 1$.

Lemma 10. *Let* $\underline{\omega} \in \Gamma_+$ *and* $x_0 \in (0, 1)$*. Then there are infinitely many values of* n *such that* $x_n \in (0, 1/2]$ *and infinitely many values of* n *such that* $x_n \in [1/2, 1)$.

Lemma 11. *For every* $x, y \in (0,1)$ *we have*

$$d(f_0(x), f_0(y)) \le d(x,y) \qquad \text{and} \qquad d(f_1(x), f_1(y)) \le d(x,y).$$

Lemma 12. *There exists* $\eta > 0$ *such that if* $x \le 1/2 \le y$ *and* $d(x,y) < \eta$ *then* $f_0(x) < f_0(y) < 1/2$ *and* $1/2 < f_1(x) < f_1(y)$*.*

Lemma 13. *Let* $1/2 \le x_0 < y_0$ *and* $x_n < y_n \le 1/2$ *for some* $n \ge 1$*. Assume also that* $d(x_0, y_0) < \eta$*, where* η *is the constant from the preceding lemma. Then*

$$d(x_n, y_n) \le \frac{2 + \frac{c}{3} d(x_0, y_0)}{2 + \frac{2c}{3} d(x_0, y_0)} d(x_0, y_0).$$

Define a function $\chi : [0, \infty) \to \mathbb{R}$ by

$$\chi(t) = \begin{cases} \dfrac{2 + \frac{c}{3}t}{2 + \frac{2c}{3}t}t & \text{if } 0 \le t \le \frac{\eta}{2}, \\ \\ \dfrac{2 + \frac{c\eta}{6}}{2 + \frac{c\eta}{3}}t & \text{if } t > \frac{\eta}{2}, \end{cases}$$

where η is the constant from Lemma 12. It is easy to see that χ is continuous, $\chi(0) = 0$ and $\chi(t) < t$ if $t > 0$. Moreover, for every $t \ge 0$ we have

$$\lim_{n\to\infty} \chi^n(t) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \chi^{-n}(t) = \infty. \tag{2}$$

Lemma 14. *Let $1/2 \le x_0 < y_0$ and $x_n < y_n \le 1/2$ for some $n \ge 1$. Then*

$$d(x_n, y_n) \le \chi(d(x_0, y_0)).$$

Now, Theorem 1 follows from Lemmas 14 and 11 and the first equality of (2).


Happy Birthday
Felix