Discrete Painlevé equations, satisfied by the recurrence coefficients of orthogonal polynomials on a bi-lattice

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Joint work with Walter Van Assche

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Overview

- Generalized Charlier and Meixner polynomials

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- The bi-lattice
- Asymptotic behaviour

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- Charlier:

$$\sum_{k=0}^{\infty} C_n(k;a)C_m(k;a)w_k = a^{-n}e^a n!\delta_{n,m}$$

with

$$w_k = \frac{a^k}{k!}, \qquad a > 0.$$

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- Both are orthogonal with respect to a weight function on $\mathbb{N}$:
- Meixner:

$$\sum_{k=0}^{\infty} M_n(k;\gamma,c)M_m(k;\gamma,c)w_k = \frac{c^{-n}n!}{(\gamma)_n(1-c)^\gamma}\delta_{n,m}$$

with

$$w_k = \frac{(\gamma)_k c^k}{k!}, \qquad \gamma > 0, 0 < c < 1.$$

- Notation: the Pochhammer symbol $(\gamma)_k = \prod_{j=0}^{k-1}(\gamma + j) = \gamma(\gamma+1)\cdots(\gamma+k-1)$.

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- We are interested in these sequences a_n and b_n.

Recurrence coefficients in the classical case

For Charlier and Meixner polynomials, the a_n and b_n are explicitly known:

- Charlier:

$$a_n^2 = na, \qquad b_n = n + a.$$

- Meixner:

$$a_n^2 = \frac{n(n+\gamma-1)c}{(1-c)^2}, \qquad b_n = \frac{n + (n+\gamma)c}{1-c}.$$

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- With the potential

$$u(x) = \frac{w(x-1) - w(x)}{w(x)}$$

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- $$A_n(x) = a_n \sum_{\ell=0}^{\infty} p_n(\ell) p_n(\ell - 1) \frac{u(x+1) - u(\ell)}{x + 1 - \ell} w(\ell),$$

$$B_n(x) = a_n \sum_{\ell=0}^{\infty} p_n(\ell) p_{n-1}(\ell - 1) \frac{u(x+1) - u(\ell)}{x + 1 - \ell} w(\ell).$$

Compatibility relations (Meixner)

The structure relation

$$\Delta p_n(x) := p_n(x+1) - p_n(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x)$$

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$$B_n(x) + B_{n+1}(x) = \frac{x - b_n}{a_n} A_n(x) - u(x+1) + \sum_{j=0}^{n} \frac{A_j(x)}{a_j}$$

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and

$$a_{n+1}A_{n+1}(x) - a_n^2 \frac{A_{n-1}(x)}{a_{n-1}}$$

$$= (x - b_n)B_{n+1}(x) - (x + 1 - b_n)B_n(x) + 1.$$

Compatibility relations (Meixner)

With

$$A_n(x) = \frac{a_n}{a} R_n + \frac{a_n}{a} \frac{x+\beta}{x+\gamma} T_n$$

and

$$B_n(x) = \frac{1}{a} r_n + \frac{1}{a} \frac{x+\beta}{x+\gamma} t_n$$

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we find 6 relations connecting r_n, R_n, t_n, T_n, a_n, b_n.

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we find 6 relations connecting r_n, R_n, t_n, T_n, a_n, b_n. We immediately find that $r_n = t_n$ and $R_n = 1 - T_n$, so 4 equations connect t_n, T_n, a_n and b_n.

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\begin{aligned}
(u_n + v_n)(u_{n+1} + v_n) &= \frac{\gamma - 1}{a^2} v_n (v_n - a) \left(v_n - a \frac{\gamma - \beta}{\gamma - 1} \right) \\
(u_n + v_n)(u_n + v_{n-1}) &= \frac{u_n}{u_n - \frac{an}{\gamma-1}} (u_n + a) \left(u_n + a \frac{\gamma - \beta}{\gamma - 1} \right)
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with

$$a_n^2 = na - (\gamma - 1) u_n$$

and

$$b_n = n + \gamma - \beta + a - (\gamma - 1) v_n / a.$$

System of difference equations (Meixner)

Initial conditions: $u_0 = a_0 = 0$ and

$$b_0 = \frac{m_1}{m_0}, \text{ hence } v_0 = \frac{a}{\gamma - 1}\left(\gamma - \beta + a - \frac{m_1}{m_0}\right).$$

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m_0 and m_1 can be expressed using confluent hypergeometric functions:

$$b_0 = \frac{\gamma a}{\beta} \frac{M(\gamma+1, \beta+1, a)}{M(\gamma, \beta, a)}$$

where

$$M(a, b, z) = {}_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k k!}.$$

Discrete Painlevé (Meixner)

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(X_n + Y_n)(X_{n+1} + Y_n) &= \frac{(Y_n - A)(Y_n - B)(Y_n - C)(Y_n - D)}{(Y_n + \Gamma - Z_n)(Y_n - \Gamma - Z_n)}, \\
(X_n + Y_n)(X_n + Y_{n-1}) &= \frac{(X_n + A)(X_n + B)(X_n + C)(X_n + D)}{(X_n + \Delta - Z_{n+1/2})(Z_n - \Delta - Z_{n+1/2})}
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\end{aligned}
$$

with

$$
X_n = u_n - \frac{1}{\epsilon}, \qquad Y_n = v_n + \frac{1}{\epsilon}, \qquad Z_n = \frac{a}{\gamma - 1}\left(n - \frac{1}{2}\right) + \frac{1}{\epsilon},
$$

$$
A = \frac{1}{\epsilon}, \qquad B = -\frac{3}{\epsilon} - a - a\frac{\gamma - \beta}{\gamma - 1}, \qquad C = a + \frac{1}{\epsilon}, \qquad D = \frac{1}{\epsilon} + a\frac{\gamma - \beta}{\gamma - 1},
$$

$$
\Gamma^2 = \frac{-4a^2}{(\gamma - 1)\epsilon}, \qquad \Delta = \frac{2}{\epsilon}, \qquad \epsilon \to 0.
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- w vanishes at poles of the denominator: $x = -1, -2, \ldots$ and $x = -\beta, -\beta - 1, \ldots$.

Bi-lattice (Meixner)

- Hence we can also use the shifted lattice $\mathbb{N} + 1 - \beta$:

$$\sum_{k=0}^{\infty} q_n(k+1-\beta) q_m(k+1-\beta) w(k+1-\beta) = \delta_{m,n}$$

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$$\sum_{k=0}^{\infty} q_n(k+1-\beta)q_m(k+1-\beta)w(k+1-\beta)=\delta_{m,n}$$

- or even the bi-lattice $\mathbb{N}\cup\mathbb{N}+1-\beta$:

$$\sum_{k=0}^{\infty} r_n(k)r_m(k)w(k)$$

$$+t\sum_{k=0}^{\infty} r_n(k+1-\beta)r_m(k+1-\beta)w(k+1-\beta)=\delta_{m,n},$$

with $t>0$.

Bi-lattice (Meixner)

- Both the shifted lattice and the bi-lattice give a new family of orthogonal polynomials, hence new sequences $\hat{a}_n$ and $\hat{b}_n$, resp. $\tilde{a}_n$ and $\tilde{b}_n$.

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- Both the shifted lattice and the bi-lattice give a new family of orthogonal polynomials, hence new sequences $\hat{a}_n$ and $\hat{b}_n$, resp. $\tilde{a}_n$ and $\tilde{b}_n$.
- However, they still satisfy the same system of recurrence relations, related to α-dP_{IV}.
- The only difference: the initial condition

$$\hat{b}_0 = \frac{\hat{m}_1}{\hat{m}_0}$$

for the shifted lattice;

$$\tilde{b}_0 = \frac{m_1 + t\hat{m}_1}{m_0 + t\hat{m}_0}$$

for the bi-lattice.

Generalized Charlier

Generalized Charlier: the recurrence coefficients a_n, b_n are determined by

$$\begin{aligned} b_n + b_{n-1} - n + \beta &= \frac{an}{a_n^2} \\ (a_{n+1}^2 - a)(a_n^2 - a) &= a(b_n - n)(b_n - n + \beta - 1) \end{aligned}$$

with $a_0 = 0$ and

$$b_0 = \frac{m_1}{m_0} = \sqrt{a}\frac{I_\beta(2\sqrt{a})}{I_{\beta-1}(2\sqrt{a})},$$

where I_ν is the modified Bessel function

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}.$$

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- For $\beta = 1$ (considered by Van Assche, Foupouagnigni), this is a discrete Painlevé-II.
- For $\beta \neq 1$ it is the limiting case of dP_{IV} (D_4^c):

$$x_n x_{n+1} = \frac{(y_n - z_n)^2 - A}{y_n^2 - B}$$

$$y_n + y_{n-1} = \frac{z_{n-1/2} - C}{1 + D x_n} + \frac{z_{n-1/2} + C}{1 + x_n / D}$$

with

$$x_n = i(a_n^2 - a)/\sqrt{aB}, \;\; y_n = b_n, \;\; z_n = n - (\beta - 1)/2,$$

$$A = (\beta - 1)^2/4, \; C = -\beta/2, \; D = -i\sqrt{B/a}, \; B \to \infty.$$

Asymptotics: generalized Charlier

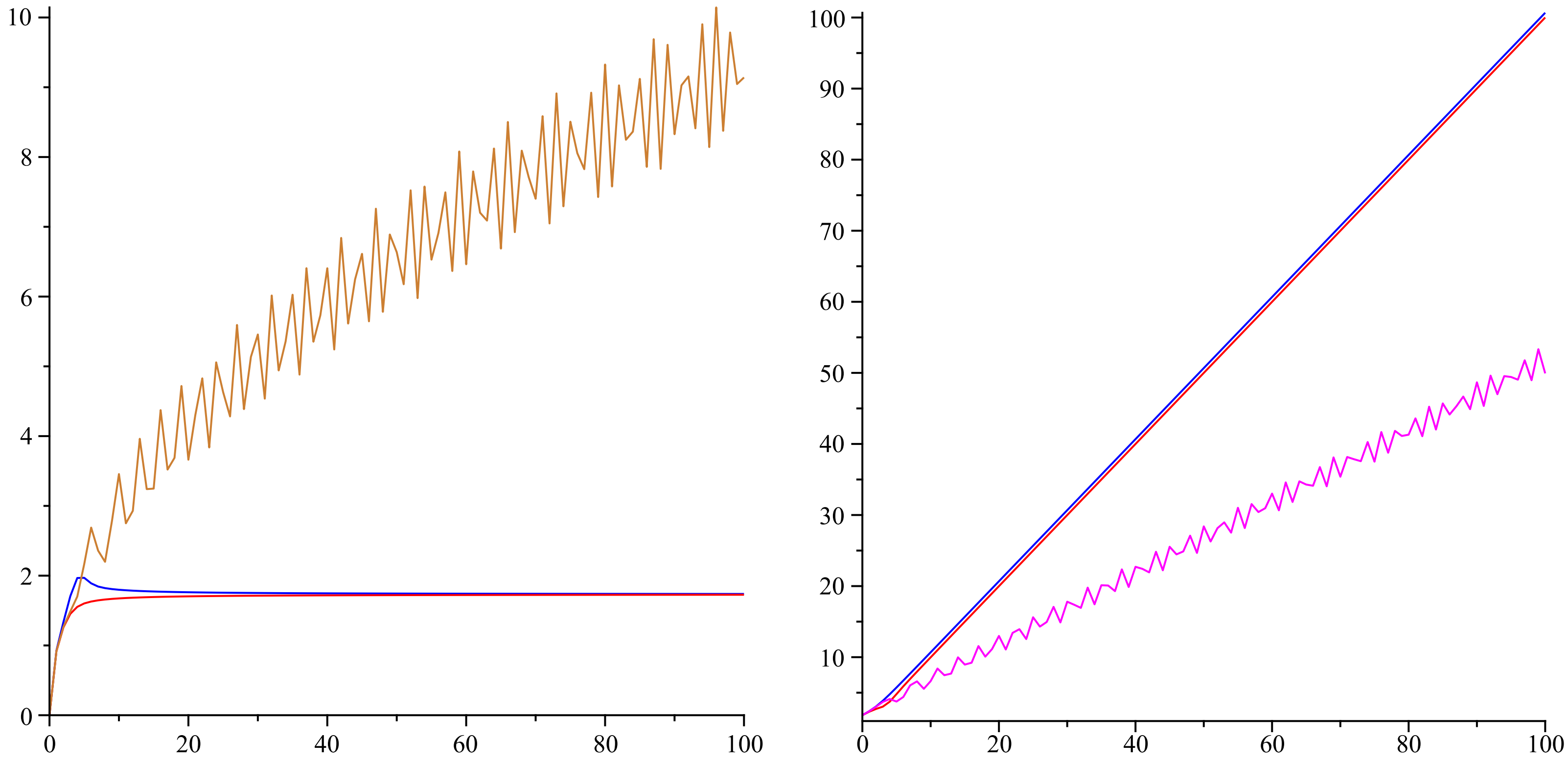

Figure 1: Recurrence coefficients for generalized Charlier polynomials ($a = 3$, $\beta = 1/3$, $t = 10$). Left: a_n, right: b_n

Asymptotics: generalized Meixner

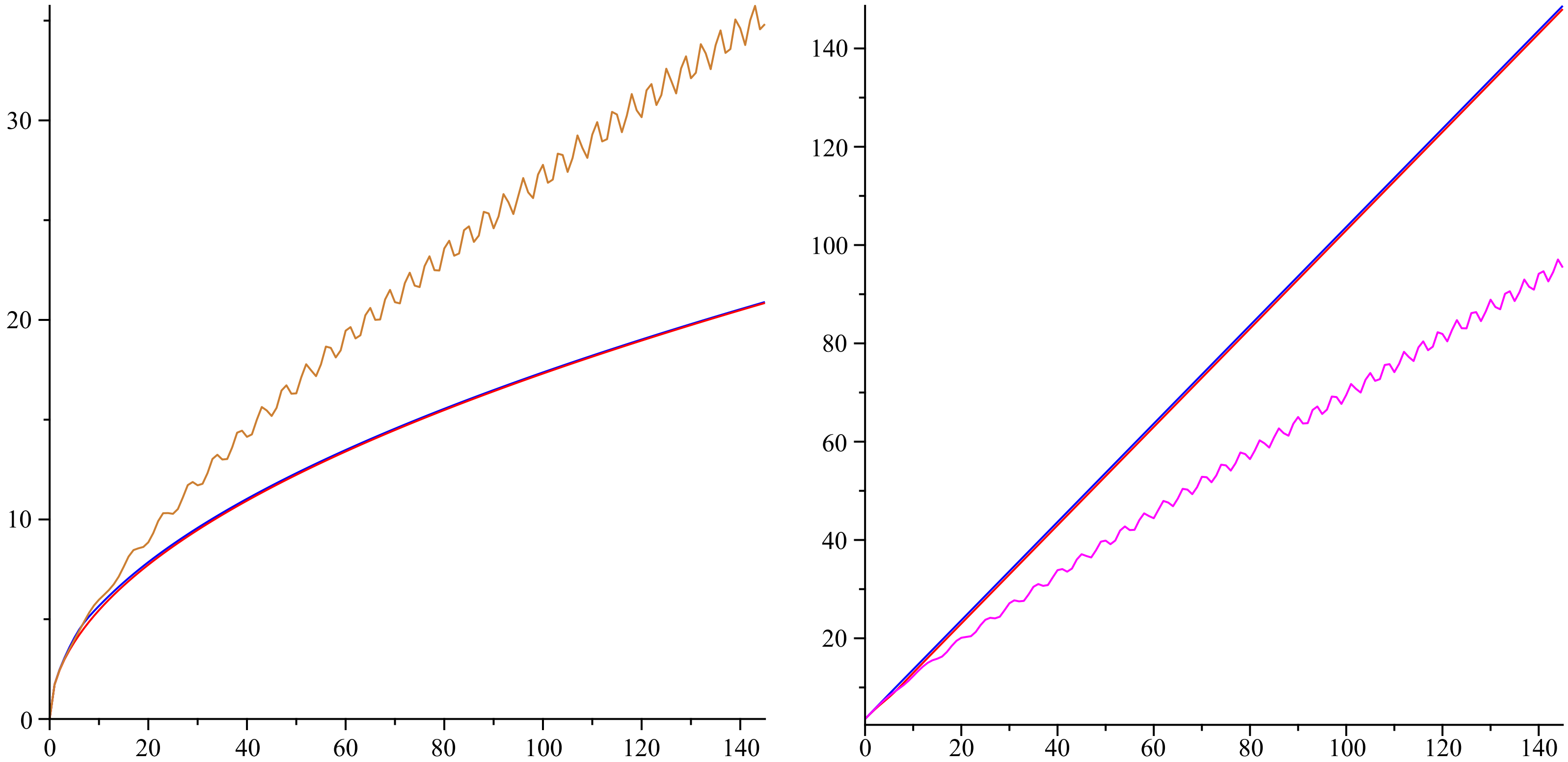

Figure 2: Recurrence coefficients for generalized Meixner polynomials ($a = 3$, $\beta = 2/3$, $\gamma = 9/10$, $t = 2$). Left: a_n, right: b_n

Asymptotics: conjectures

- Generalized Charlier on the simple lattices:

$$\lim_{n\to\infty} a_n^2 = a, \lim_{n\to\infty} (b_n - n) = \begin{cases} 0, & \text{on } \mathbb{N}; \\ 1-\beta, & \text{on } \mathbb{N}+1-\beta. \end{cases}$$

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- Generalized Charlier on the bi-lattice:

$$a_n^2 = n\sqrt{a}/2 + O(1),$$

$$b_n = n/2 + O(1),$$

with the $O(1)$ terms oscillating.

Asymptotics: conjectures

- Generalized Meixner on the simple lattices:

$$\lim_{n\to\infty}(a_n^2 - an) = \begin{cases} (\gamma-\beta)a, & \text{on } \mathbb{N}; \\ (\gamma-1)a, & \text{on } \mathbb{N}+1-\beta, \end{cases}$$

$$\lim_{n\to\infty}(b_n - n) = \begin{cases} a, & \text{on } \mathbb{N}; \\ a+1-\beta, & \text{on } \mathbb{N}+1-\beta. \end{cases}$$

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- Generalized Meixner on the bi-lattice:

$$a_n^2/n^{3/2} = O(1),$$

$$b_n/n = O(1),$$

with the $O(1)$ terms oscillating.

Arxiv:

CS, Walter Van Assche: Orthogonal polynomials on a bi-lattice.
http://arxiv.org/abs/1101.1817