Gevrey solutions of threefold singular nonlinear PDEs

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Notation

$\mathbb{N} = \{0, 1, 2, ...\}$ $\quad$ $\mathbb{N}^\star = \mathbb{N} \setminus \{0\}$

$\mathcal{O}(S)$ holomorphic functions in S.

$\mathcal{O}_b(S)$ holomorphic and bounded functions in S.

$\mathcal{O}\{z_1, z_2, ..., z_n\}$ holomorphic functions on some neighborhood of $0 \in \mathbb{C}^n$ in the variables z_1,...,z_n.

$\mathbb{K}[z_1, z_2, ..., z_n]$ polynomials in z_1,...,z_n with coefficients in $\mathbb{K}$.

$D(0, R)$ disc at 0 with radius $R > 0$.

The main problem

We study a family of threefold singular nonlinear PDEs of the form

$$\left((z\partial_z+1)^{r_1}\epsilon^{r_3}(t^2\partial_t+t)^{r_2}+1\right)\partial_z^S X(t,z,\epsilon)$$
$$=\sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{s,\kappa_0,\kappa_1}(z,\epsilon)t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}X)(t,z,\epsilon)+P(t,z,\epsilon,X(t,z,\epsilon)),$$

for given initial conditions

$$(\partial_z^j X)(t,0,\epsilon)=\varphi_j(t,\epsilon)\in\mathcal{O}(\mathcal{T}\times\mathcal{E}),\quad 0\le j\le S-1.$$

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$r_1,r_3\in\mathbb{N}$ $\quad r_2,S\in\mathbb{N}^\star$ $\quad P(t,z,\epsilon,X)\in\left(\mathcal{O}\{z,\epsilon\}\right)[t,X]$

$\mathcal{S}$ finite subset of $\mathbb{N}^3$.
For every $(s,\kappa_0,\kappa_1)\in\mathcal{S}$
$\kappa_1<S$ $\quad b_{s,\kappa_0,\kappa_1}\in\mathcal{O}\{z,\epsilon\}$.

Introduction

The problem gives generalization to [1] in three respects:

- The fuchsian operator $(z\partial_z + 1)^{r_1}$ is added to the equation.
- The irregular operator $(t^2\partial_t + t)^{r_2}$ admits $r_2 \geq 1$ (not just $r_2 = 1$).
- More freedom on the choice of the powers of ϵ (not just $r_3 = 1$).

Fuchsian singularities for PDEs are widely studied (see [2]), and also the asymptotic properties of irregular singularities (see [3]).

[1] S. Malek, *On the summability of formal solutions for doubly nonlinear partial differential equations*, J. Dyn. Control Syst. 18 (2012), no. 1, 45-82.
[2] R. Gérard, H. Tahara, Singular nonlinear partial differential equations. Aspects of Mathematics. Friedr. Vieweg and Sohn, Braunschweig, 1996. viii+269 pp.
[3] H. Chen, Z. Luo, H. Tahara, *Formal solutions of nonlinear first order totally characteristic type PDE with irregular singularity*. Ann. Inst. Fourier (Grenoble) 51 (2001), no. 6, 1599–1620.

Auxiliary equation

We transform the main problem into an auxiliary problem by means of the linear map $t \mapsto t/\epsilon^{\frac{r_3}{r_2}}$. This strategy is adapted from [4].

For every ϵ, we consider the Cauchy problem

$$\left((z\partial_z + 1)^{r_1}(t^2\partial_t + t)^{r_2} + 1\right)\partial_z^S Y_\epsilon(t,z) = \sum_{(s,\kappa_0,\kappa_1)\in\mathcal{S}} b_{s,\kappa_0,\kappa_1}(z,\epsilon)\epsilon^{r(\kappa_0-s)}t^s(\partial_t^{\kappa_0}\partial_z^{\kappa_1}Y_\epsilon)(t,z) + \tilde{P}(t,z,\epsilon,X)$$

with initial conditions

$$(\partial_z^j Y_\epsilon)(t,0) = Y_{\epsilon,j}(t), \quad j = 0, ..., S-1.$$

- Regularly perturbed nonlinear PDE with irregular ularity at $t=0$ and fuchsian singularity at $z=0$.
- Poles in the coefficients with respect to $\epsilon = 0$ are induced.
- The domain of definition of the initial data depends on ϵ.

[4] M. Canalis-Durand, J. Mozo-Fernandez, R. Schäfke, *Monomial summability and doubly singular differential equations*, J. Differential Equations 233 (2007), no. 2, 485-511.

Auxiliary equation

We assume each initial conditions $(Y_{\epsilon,j}(t))_{0\le j\le S-1}$ are chosen

$$Y_{\epsilon,j}(t) = \mathcal{L}_{\tau}^{d}(V_j(\tau,\epsilon))(t)$$

with $V_j(\tau,\epsilon) \in \mathcal{O}((S_d \cup D(0,\rho_j)) \times \mathcal{E})$ and with

$$|V_j(\tau,\epsilon)| \le \delta \left(1 + \frac{|\tau|^2}{|\epsilon|^{2r_3/r_2}}\right)^{-1} \exp\left(\frac{\sigma|\tau|}{|\epsilon|^{r_3/r_2}} \sum_{n=0}^{j} \frac{1}{(n+1)^2}\right),$$

for some $\delta, \sigma > 0$. This is valid for every $\epsilon \in \mathcal{E}$, $\tau \in S_d \cup D(0,\rho_j)$.

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for some $\delta, \sigma > 0$. This is valid for every $\epsilon \in \mathcal{E}$, $\tau \in S_d \cup D(0,\rho_j)$.
Then, one can construct $(V_\beta)_{\beta\geq 0}$ and

$$Y_{\epsilon,\beta}(t) = \mathcal{L}_{\tau}^{d}(V_\beta(\tau,\epsilon))(t),$$

where

$$Y_\epsilon(t,z) = \sum_{\beta\geq 0} Y_{\epsilon,\beta}(t)\frac{z^\beta}{\beta!}$$

turns out to be a formal solution to the auxiliary Cauchy problem.

Auxiliary equation

The formal power series $V_\epsilon(\tau,z)=\sum_{\beta\ge 0}V_\beta(\tau,\epsilon)\frac{z^\beta}{\beta!}$ is a formal solution to an auxiliary convolution integro-differential Cauchy problem.

$$((z\partial_z+1)^{r_1}\tau^{r_2}+1)\partial_z^S V_\epsilon(\tau,z)=\sum_{(\kappa_0,\kappa_1)\in\mathcal{A}_1}a_{(\kappa_0,\kappa_1)}(\tau,z,\epsilon)\partial_\tau^{-\kappa_0}\partial_z^{\kappa_1}V_\epsilon(\tau,z)$$

$$+\sum_{(\ell_0,\ell_1)\in\mathcal{A}_2}\alpha_{(\ell_0,\ell_1)}(\tau,z,\epsilon)\partial_\tau^{-\ell_0}(V_\epsilon(\tau,z))^{\star\ell_1},$$

with initial contitions

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with initial contitions

$$(\partial_z^j V_\epsilon)(\tau,0)=V_j(\tau,\epsilon),\quad j=0,...,S-1.$$

$\mathcal{A}_1$, $\mathcal{A}_2$ are finite subsets of $\mathbb{N}_0^2$,
$a_{(\kappa_0,\kappa_1)}\in\mathcal{O}_b((S_d\cup D(0,\rho_0))\times\mathcal{E})[[z]]$,
$\alpha_{(\ell_0,\ell_1)}$ is a formal power series in z with coefficients being holomorphic functions for $\tau\in S_d\cup D(0,\rho_0)$ and meromorphic for $\epsilon\in\mathcal{E}$.

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$V_\beta \in \mathcal{O}((S_d \cup D(0, \rho_\beta)) \times \mathcal{E})$, with $\rho_\beta \to 0$, when $\beta \to \infty$.

Main problem: holomorphic solution

The elements in $(V_\beta(\tau,\epsilon))_{\beta\ge 0}$ are constructed by means of a fixed point argument in a Banach space of holomorphic functions attained to appropriate bounds. V_β satisfies exponential bounds for $\beta\ge 0$, in the shape of V_j, $j=0,...,S-1$.

$$|V_\beta(\tau,\epsilon)|\le C\beta!\left(\frac{1}{\rho}\right)^{\beta}\left(1+\frac{|\tau|^2}{|\epsilon|^{2r_3/r_2}}\right)^{-1}\exp\left(\frac{\sigma|\tau|}{|\epsilon|^{r_3/r_2}}\sum_{n=0}^{\beta}\frac{1}{(n+1)^2}\right),$$

for some $C,\rho>0$, for every $(\tau,\epsilon)\in(S_d\cup D(0,\rho_\beta))\times\mathcal{E}$.

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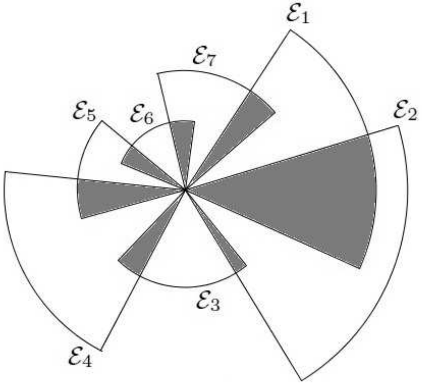
for some $C,\rho>0$, for every $(\tau,\epsilon)\in(S_d\cup D(0,\rho_\beta))\times\mathcal{E}$.

The solution of the main problem is given by

$$X(t,z,\epsilon):=Y_{\epsilon^{r_3/r_2}}(t,z)\in\mathcal{O}((\mathcal{T}\cap D(0,h))\times D(0,\rho)\times\mathcal{E}),$$

for some $h,\rho>0$.

Let $(\mathcal{E}_i)_{1\leq i\leq n}$ be a good covering at the origin in $\mathbb{C}$.


$\mathcal{E}_1$
$\mathcal{E}_7$
$\mathcal{E}_2$
$\mathcal{E}_5$
$\mathcal{E}_6$
$\mathcal{E}_3$
$\mathcal{E}_4$

Formal solution and asymptotics

For every $1 \leq i \leq n$, we have

$$\sup_{\substack{t \in \mathcal{T} \cap D(0,h') \\ z \in D(0,\rho)}} |X_{i+1}(t,z,\epsilon) - X_i(t,z,\epsilon)| \leq K_i \exp\left(-\frac{M_i}{|\epsilon|^{\frac{r_3}{r_1+r_2}}}\right), \quad \epsilon \in \mathcal{E}_i \cap \mathcal{E}_{i+1},$$

for some $0 < h' < h$, and some $K_i, M_i > 0$.

- This result rests on a careful estimation of a Dirichlet-like series of the form $\sum_{\kappa \geq 0} e^{-\frac{1}{(\kappa+1)^{\alpha}}\frac{1}{\epsilon}} a^{\kappa}$, for some $0 < a < 1$, $\alpha > 0$ and $0 < \epsilon < \epsilon_0$.
- It leans on Malgrange-Sibuya theorem (see [5]).

[5] W. Balser, Formal power series and linear systems of meromorphic ordinary differential equations, Springer-Verlag, New-York, 2000.

Formal solution and asymptotics

For a small enough $h' > 0$, there exists a formal power series

$$\hat{X}(t,z,\epsilon) = \sum_{\kappa \geq 0} H_{\kappa}(t,z) \frac{\epsilon^{\kappa}}{\kappa!} \in \mathcal{O}_b((\mathcal{T} \cap D(0,h')) \times D(0,\rho))[[\epsilon]].$$

Moreover, for every $0 \leq i \leq n$ and all $T_i \prec \mathcal{E}_i$ there exist $K, M > 0$ such that

$$\sup_{\substack{t \in \mathcal{T} \cap D(0,h') \\ z \in D(0,\rho)}} \left| X_i(t,z,\epsilon) - \sum_{n=0}^{N-1} H_n(t,z) \frac{\epsilon^n}{n!} \right| \leq K M^N N!^{\frac{r_1+r_3}{r_2}} |\epsilon|^N,$$

for $\epsilon \in T_i$.