The monodromy of parametrized linear differential systems

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$\mathbb{C}(x)(Y_0) = \mathbb{C}(x, y_{11}, \ldots, y_{nn})$, is a differential field with derivation induced by the formula $Y' = AY$.

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The Picard-Vessiot group over $\mathbb{C}(x)$, or differential Galois group,

is the group of differential $\mathbb{C}(x)$-automorphisms of $\mathbb{C}(x)(Y_0)$.

It is a linear algebraic group, with a natural representation

$$\mathrm{Gal}_{\mathbb{C}(x)}((S)) \subset \mathrm{GL}(n, \mathbb{C}).$$

with respect to Y_0 and x_0 (it is Zariski-closed).

- Analytic continuation of Y_0 along a loop γ yields the monodromy representation :

$$\begin{aligned} \pi_1(\mathrm{U}_\Sigma; x_0) &\xrightarrow{\rho} \mathrm{GL}(n, \mathbb{C}) \\ [\gamma] &\longmapsto M_\gamma \end{aligned}$$

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- The monodromy matrices M_γ belong to the Picard-Vessiot group :

$$\mathrm{Im}\, \rho \subset \mathrm{Gal}_{\mathbb{C}(x)}((S))$$

Density theorems and inverse problems

- **Theorem 1** (Schlesinger, 1897) *If all singularities of (S) are regular singular, the monodromy group is Zariski-dense in the Picard-Vessiot group over $\mathbb{C}(x)$.*

  Schlesinger's original formulation: *Wenn die Differentialgleichung der Fuchs'schen Classe angehört, so ist ihre Transformationsgruppe die engste algebraische Gruppe linearer homogener Transformationen, die die Gruppe der Differentialgleichung als Untergruppe in sich schliesst.*

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- Ramis's density theorem for irregular singularities :

  Theorem 2 (Ramis, 1985) : *The local PV-group at* 0 *(PV-group over* $\mathbb{C}(\{x\})$*) is the Zariski closure in* $\mathrm{GL}(n, \mathbb{C})$ *of the subgroup generated by the formal monodromy, the exponential torus and the Stokes matrices.*

  The local PV- groups together generate a dense subgroup of the global PV-group over $\mathbb{C}(x)$.

- **Theorem 3** (Plemelj 1908, Treibich-Kohn 1983) *Any representation*

$$\rho : \pi_1(\mathrm{U}_\Sigma; x_0) \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

is realizable as the monodromy representation of some (S) with regular singularities, all Fuchsian but one, all Fuchsian if one of the elementary generators M_{γ_i} of $\mathrm{Im}\,\rho$ *is diagonalizable* .

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- **Theorem** (J. Hartmann, 2005) *Over any algebraically closed field C of char.0, any linear algebraic group is realizable as a Picard-Vessiot group over $C(x)$.*

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- $\mathcal{O}_{\mathcal{U}}((x - \alpha))$ the ring of formal Laurent series

$$f(x, t) = \sum_{i \geq m} a_i(t)(x - \alpha(t))^i$$

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- $\mathcal{O}_{\mathcal{U}}(\{x-\alpha\})$ the ring of convergent Laurent series $f \in \mathcal{O}_{\mathcal{U}}((x-\alpha))$, for $t \in \mathcal{U}$ and $0 < |x - \alpha(t)| < R_t$, for some $R_t > 0$.

 (For a given $f \in \mathcal{O}_{\mathcal{U}}(\{x-\alpha\})$, one may shrink $\mathcal{U}$ so that f converges for all $t \in \mathcal{U}$ and $0 < |x - \alpha(t)| < R$, independent of t).

- Consider a parametrized family of differential systems of order n

$$\frac{\partial Y}{\partial x} = A(x,t)Y$$

with coefficients in $\mathcal{O}_{\mathcal{U}}\left(\{x-\alpha(t)\}\right)$. Then for some $m \in \mathbb{N}$

$$A(x,t) = \frac{A_{-m}(t)}{(x-\alpha(t))^m} + \frac{A_{-m+1}(t)}{\left(x-\alpha(t)\right)^{m-1}} + \ldots = \sum_{i \geq -m} \left(x-\alpha(t)\right)^i A_i(t),$$

with $A_i \in \mathrm{gl}_n(\mathcal{O}_{\mathcal{U}})$ and $A_{-m} \not\equiv 0$.

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- Two parametrized systems

$$\frac{\partial Y}{\partial x} = AY, \qquad \frac{\partial Y}{\partial x} = BY,$$

are equivalent as such if

$$B = \frac{\partial P}{\partial x}P^{-1} + PAP^{-1}$$

for some invertible $P \in \mathrm{GL}_n\big(\mathcal{O}_{\mathcal{U}}(\{x-\alpha\})\big)$.

Definition: (parametric analogues of "Fuchsian" (= first kind) and "regular singular")

- The system (S) has simple singular points near 0 if $m = 1$ and $A_{-1} \not\equiv 0$ as a function $\in \mathrm{gl}(n, \mathcal{O}_{\mathcal{U}})$.

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- The system (S) has parametrized regular singular points near 0 (notation $p.r.s._0$) if it is equivalent to a system with simple singular points near 0.
- **Proposition** : Assume the system (S) has $p.r.s._0$. Then (S) is equivalent to a system

$$\frac{\partial Y}{\partial x} = \frac{\widetilde{A}(t)}{x} Y.$$

An example

- Consider the system $\partial Y/\partial x = AY$ where

$$A = \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix} \frac{1}{(x-t)^2} + \begin{pmatrix} t & 0 \\ 0 & t-2 \end{pmatrix} \frac{1}{x-t}$$

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- Note that B has the form announced in the Proposition.

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with $\tilde{A} \in \mathrm{gl}_n(\mathcal{O}_{\mathcal{U}'})$ and $Q_i \in \mathrm{gl}_n(\mathcal{O}_{\mathcal{U}'})$ for all $i \geq i_0$, As expected, solutions have moderate growth as "x tends to $\alpha(t)$".

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• for any r-tuple $(m_1,\ldots,m_r)$ of nonnegative integers there is an integer N such that for any $t \in \mathcal{U}'$ and any sector $\mathcal{S}_t$ from $\alpha(t)$ in the complex plane, of opening $< 2\pi$,

$$\lim_{\substack{x \to \alpha(t) \\ x \in \mathcal{S}_t}} (x-\alpha(t))^N \frac{\partial^{m_1+\ldots+m_r} Y(x,t)}{\partial^{m_1} t_1 \ldots \partial^{m_r} t_r} = 0.$$

Isomonodromy

Consider a parametrized differential system

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of order n, where

- $A(x, t)$ is analytic in $(x, t) \in \Omega \times \mathcal{U}$,

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  (for any base-point $x_0 \in \Omega$, the fundamental group $(\pi_1(\Omega; x_0)$ is generated by elementary loops $[\gamma_1], \ldots, [\gamma_m]$).

- for fixed $t \in \mathcal{U}$, there is one singularity $\alpha_i(t)$ in each D_i and none in Ω.

- The system

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is isomonodromic if for some solution $Y_t(x)$ of (S) at x_0 there are constant matrices $M_1, \ldots, M_m \in GL(n, \mathbb{C})$ such that for each fixed $t \in \mathcal{U}$ the M_i are the monodromy matrices of (S) with respect to $Y_t(x)$ along the elementary loops γ_i.

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- Classically, only Fuchsian systems

$$(F) \qquad \partial_x Y = \sum_{i=1}^{m} \frac{B_i(a)}{x - a_i}, \qquad \sum_{i=1}^{m} B_i(a) = 0$$

were considered, where the multi-parameter is the moving configuration $a = (a_1, \ldots, a_m)$ of the poles in a neighborhood $D(a^0)$ of their initial position a^0.

Schlesinger (1905) defined isomonodromy by requiring that the monodromy representation

$$\pi_1(\mathrm{U}_\Sigma; x_0) \xrightarrow{\rho_{\boldsymbol{a}}} \mathsf{GL}(n, \mathbb{C})$$

be independent of a for the particular solution Y_a with initial condition $\tilde{Y}_a(x_0) = I$ for each a.

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The Schlesinger isomonodromic deformations are characterized by the Pfaffian system (called the Schlesinger equation)

$$\mathrm{d}B_i(a) = -\sum_{j=1, j\neq i}^{m} \frac{[B_i(a), B_j(a)]}{a_i - a_j} \mathrm{d}(a_i - a_j), \quad i = 1, \ldots, m$$

which is the compatibility condition of the systems

$$\partial_{a_i} Y = -\frac{B_i(a)}{x - a_i} Y.$$

Bolibrukh (1995) extended Schlesinger's definition : Equation (S) is isomonodromic if there is a fundamental solution Y_a of (S) with initial value $Y_a(x_0) = C(a)$ holomorphic in a, such that ρ_a does not depend on a.

He proved (1997) that for Fuchsian equations this is equivalent to Definition 1 and gave examples of non-Schlesinger isomonodromic deformations.

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In the special case of order two Fuchsian systems with 4 singularities the Schlesinger equation translates into a Painlevé VI equation : a non-linear second order scalar equation with no moving essential singularities (the "Painlevé property"), satisfied by the additional apparent singularity of a linear scalar Fuchsian equation representing the system.

Theorem (Sibuya, 1990) *Let* (S) *be a parametrized system as before. The system (S) is isomonodromic if and only if* (S) *is part of an integrable system*

$$\left\{ \begin{array}{rcl} \partial_x Y & = & A(x,t)Y \\ \partial_{t_i} Y & = & B_i(x,t)Y, \quad i = 1, \ldots, r \end{array} \right.$$

where the matrices $B_i(x,t)$ *are analytic in* $\Omega \times \mathcal{U}$*.*
Assume (S) *has only parametrized regular singularities (p.r.s.) . Then if* A *is rational in* x*, so are the* B_i*.*

Monodromy evolving deformations

A special case of monodromy evolving deformation was studied in

/ / S. CHAKRAVARTY, M. J. ABLOWITZ, *Integrability, monodromy evolving deformations, and self-dual Bianchi IX systems*, Physical Review Letters 76, 6 (1996), 857–860./ /

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These authors studied the Darboux-Halphen equation and showed that it describes a certain type of m. e. d., in the same way as the Schlesinger equation accounts for the Schlesinger isomonodromy.

The Darboux-Halphen equation

The Darboux-Halphen V equation

$$(\mathrm{DH\ V})\quad \begin{cases} \omega_1' &= \omega_2\omega_3 - \omega_1(\omega_2+\omega_3) + \phi^2 \\ \omega_2' &= \omega_3\omega_1 - \omega_2(\omega_3+\omega_1) + \theta^2 \\ \omega_3' &= \omega_1\omega_2 - \omega_3(\omega_1+\omega_2) - \theta\phi \\ \phi' &= \omega_1(\theta-\phi) - \omega_3(\theta+\phi) \\ \theta' &= -\omega_2(\theta-\phi) - \omega_3(\theta+\phi), \end{cases}$$

plays an important rôle in physics.

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\omega_3' & = & & \omega_1\omega_2 & - & \omega_3(\omega_1+\omega_2) & - & \theta\phi \\
\phi' & = & & \omega_1(\theta-\phi) & - & \omega_3(\theta+\phi) & & \\
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plays an important rôle in physics.

It occurs as a reduction of the selfdual Yang-Mills equation (SDYM).

For $\theta = \phi$, (DH V) is equivalent to Einstein's selfdual vacuum equations.

For $\theta = \phi = 0$, it is Halphen's original equation (H II), solving Darboux's geometry problem about orthogonal surfaces.

History of the DH-equation

Equation (H II) goes back to Darboux's work on orthogonal systems of surfaces:

[1] Gaston DARBOUX : *Systèmes orthogonaux.* Ann. Sc. É.N.S. (**1866**)1e série, tome 3, pp. 97 -141.

[2] Gaston DARBOUX : *Mémoire sur la théorie des coordonnées curvilignes, et des systèmes orthogonaux.* Ann. Sc. É.N.S. (**1878**), 2e série, tome 7, pp. 101-150, 227-260, 275-348.

Problem 1 : On which condition on a given pair $(\mathcal{F}_1, \mathcal{F}_2)$ of orthogonal families of surfaces in $\mathbb{R}^3$ does there exist a family $\mathcal{F}_3$ such that $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is a triorthogonal system of pairwise orthogonal families?

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In [1], Darboux gave a necessary and sufficient condition on $(\mathcal{F}_1, \mathcal{F}_2)$ to solve the problem: that the intersection of any surfaces $S_1 \in \mathcal{F}_1$ and $S_2 \in \mathcal{F}_2$ be a line of curvature of both $\mathcal{F}_1$ and $\mathcal{F}_2$.

The necessary condition was Dupin's theorem (1813).

Problem 2 : On which condition on its parameter $u = \varphi(x, y, z)$ does a given one-parameter family $\mathcal{F}$ of surfaces belong to a triorthogonal system $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$, of three pairwise orthogonal families?

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In [2] Darboux found and solved an order three PDE satisfied by u. He obtained the general solution from a particular family of ruled helicoidal surfaces.

(based on previous work by Bonnet and Cayley)

Élie CARTAN later used his *exterior differential calculus* to prove that Problem 1 has a solution.

/ / Élie CARTAN : *Les systèmes différentiels extérieurs et leurs applications géométriques*, Exposés de géométrie XII, Hermann ed. (1945) / /

É. Cartan generalized the problem, replacing orthogonality by any given angle, or considering p pairwise orthogonal families of hypersurfaces in p-space.

Problem 3 (Darboux): Given two families $\mathcal{F}_1$ and $\mathcal{F}_2$ consisting each of parallel surfaces does there exist a family $\mathcal{F}$ orthogonal to both $\mathcal{F}_1$ and $\mathcal{F}_2$?

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It is easy to prove that a solution must either consist of planes, or of ruled quadrics.

If $\mathcal{F}$ consists of quadrics with a center, these have a simultaneously reduced equations:

$$\frac{x^2}{a(u)} + \frac{y^2}{b(u)} + \frac{z^2}{c(u)} = 1$$

which depends on the parameter u.

The family $\mathcal{F}$ is a solution iff

$$(D) \qquad c(a' + b') = b(a' + c') = a(b' + c').$$

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"*These equations do not seem to be integrable by known procedures*" (Darboux, 1878).

Darboux gives up. He limits his study to centerless quadrics. He finds a family $\mathcal{F}$ of paraboloids

$$\frac{y^2}{\alpha + u} + \frac{z^2}{\alpha - u} = 2x + \alpha \log u$$

solution of the problem, and claims there are surfaces of revolution as well.

Halphen's solution

Halphen (1881) solves Darboux's equation (D) in the following form :

$$(\mathrm{H\ I}) \quad \begin{cases} \omega_1' + \omega_2' = \omega_1\omega_2 \\ \omega_2' + \omega_3' = \omega_2\omega_3 \\ \omega_3' + \omega_1' = \omega_3\omega_1 \end{cases}$$

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(Halphen I equation)

He solves the more general QHDS (quadratic homogeneous differential system)

$$(\mathrm{H\ II})\quad \left\{\begin{array}{ccl} \omega_1' & = & a_1\omega_1^2 + (\lambda - a_1)(\omega_1\omega_2 + \omega_3\omega_1 - \omega_2\omega_3) \\ \omega_2' & = & a_1\omega_2^2 + (\lambda - a_2)(\omega_2\omega_3 + \omega_1\omega_2 - \omega_3\omega_1) \\ \omega_3' & = & a_1\omega_3^2 + (\lambda - a_3)(\omega_3\omega_1 + \omega_2\omega_3 - \omega_1\omega_2) \end{array}\right.$$

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$$

(Halphen II equation) by means of hypergeometric functions.

He also considers more general QHDS

$$
\{\omega_r' = \psi_r(\omega_1, \ldots, \omega_r)\}_{r=1,\ldots,l}
$$

where ψ_r are quadratic forms (with an extra symmetry condition) like equation(DH-V) above.

/ / Georges Henri HALPHEN : *Sur un sytème d'équations différentielles*, C. R. Acad. Sci. 92 (**1881**), pp. 1101-1103. / /

/ / Georges Henri HALPHEN : *Sur certains systèmes d'équations différentielles*, C. R. Acad. Sci. 92 (**1881**), pp. 1404-1406. / /

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where $Q(X) = X^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2$ (a, b, c, constants)

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- Equation (H-II) is the integrability condition of the Lax pair

$$\frac{\partial Y}{\partial x} = \left(\frac{\mu I}{(x-x_1)(x-x_2)(x-x_3)} + \sum_{i=1}^{3} \frac{\lambda_i K}{x - x_i} \right) Y \qquad (1)$$

$$\frac{\partial Y}{\partial t} = \left(\nu I + \sum_{i=1}^{3} \lambda_i x_i K \right) Y - Q(x) \frac{\partial Y}{\partial x} \qquad (2)$$

where

$x_i = x_i(t)$ are parametrized (simple) singularities

K is a constant traceless 2×2 matrix, I the identity matrix

$\mu, \ \lambda_i$ are constants, $\mu \neq 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$ ($\Rightarrow$ no singularity at ∞)

ν is solution of

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- Since ν is not rational in x, the system (1) is non-isomonodromic, by Sibuya's criterion.

Monodromy of Equation (1)

Fix Y, a fundamental solution of the Lax pair, at some x_0 not belonging to fixed disks D_i with centers $x_i(t)$, for all i. A computation shows that the monodromy of Equation (1) is

$$M_i(t) = c_i(t)\ M_i(t_0)$$

with

$$M_i(t_0) = e^{2\pi\sqrt{-1}L_i(t_0)}$$

$$c_i(t) = e^{-2\pi\sqrt{-1}\mu\int_{t_0}^{t}\beta_i(t)dt}$$

where

$$\frac{x + \sum_{i=1}^{3} x_i}{\prod_{i=1}^{3}(x - x_i(t))} = \sum_{i=1}^{3} \frac{\beta_i(t)}{x - x_i(t)}.$$

Parametrized Picard-Vessiot theory

The parametrized Picard-Vessiot theory was developed by

/ / Ellis R. KOLCHIN : *Differential algebraic groups*,
Academic Press, New York, 1985./ /

/ / Phyllis J. CASSIDY, Michael F. SINGER : *Galois theory of parameterized differential equations and linear differential algebraic groups*,
IRMA Lectures in Mathematics and Theoretical Physics 9 (2006), 113–157.
(Special volume in memory of A. A. Bolibrukh)/ /

/ / Peter LANDESMANN : *Generalized differential Galois theory*
Trans. Amer. Math. Soc. 360, 8 (2008), 4441–4495./ /

- Let $\Delta = \{\partial_0, \partial_1, \ldots, \partial_r\}$ be a set of commuting derivations on a field L,

  $L\{y_1, y_2, \ldots\}_\Delta$ the L-algebra of Δ-differential polynomials:
  polynomials in the differential indeterminates $\{\partial_j^{(k)} y_i\}_{i,j \geq 1, k \geq 0}$.

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- Closed subsets for the Kolchin topology in the affine space L^p are the zero-sets of systems

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- Linear differential algebraic groups are the subgroups of $\mathrm{GL}(n, L)$ which are Kolchin-closed.

Differentially closed fields

- **Definition:** A Δ-differential field L is differentially closed if for any differential polynomials $P_1, \ldots, P_s, Q \in L\{y_1, y_2 ...\}_\Delta$, the system

$$\begin{cases} P_1 = \ldots = P_s = 0 \\ Q \neq 0 \end{cases}$$

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- Differentially closed fields are (almost) analogues of algebraically closed fields.

- ROBINSON (1959), BLUM (1968), SHELAH (1972), KOLCHIN (1974), gave different (equivalent) definitions.

  They proved the existence, for any differential field k, of a unique *differential closure*, that is, a differential, differentially closed extension of k that can be embedded in any other differentially closed extension of k.

PPV-extensions

Consider an (ordinary) differential system of order n

$$(S) \qquad \partial_0 Y = AY,$$

where A has entries in the Δ-differential field k, $\Delta = \{\partial_0, \ldots, \partial_r\}$.

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Definition A parametrized Picard-Vessiot extension (PPV-extension) of k for (S) is a Δ-differential extension K of k such that

- $K = k\langle Z\rangle_\Delta$ for some fundamental solution Z of (S) in K

(= the Δ- extension generated by the entries of Z)

- $K^{\partial_0} = k^{\partial_0}$ (no new ∂_0-constants).

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The corresponding parametrized Picard-Vessiot group (PPV-group), or parametrized differential Galois group, is

$$\mathrm{Gal}_\Delta(S) = \mathrm{Aut}_{\Delta-diff}(K|k)$$

Existence of PPV-extensions

In analogy with classical Picard-Vessiot theory, the key condition here is that k^{∂_0}, the field of ∂_0-constants of k, be Δ-differentially closed.

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(2) *its PPV-group is a linear differential algebraic group*

$$\mathrm{Gal}_\Delta(S) \subset \mathrm{GL}(n, k^{\delta_0})$$

PPV-Galois correspondence

Let K be a PPV-extension of k for (S), and G the corresponding PPV-group.

In PPV-theory, PPV-Galois correspondence holds between
$\{$intermediate differential extensions $k \subset L \subset K\}$ and
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Note that differentially closed $\Rightarrow$ algebraically closed.

Let $\tilde{K}$ be the (usual) PV extension of k for (S) . Then $\tilde{K} \subset K$ and

$$G^{PV}(S) = \overline{G^{PPV}(S)}$$

(the PPV-group is Zariski-dense in the PV-group)

For second order equations, parametrized analogues of the Kovacic algorithm were given by:

/ / Thomas DREYFUS : *Computing the parameterized differential Galois group of some parameterized linear differential equation of order two*, arXiv:1110.1053 (2011), to appear in Proceedings of the AMS. / /

/ / Carlos E. ARRECHE : *Computing the differential Galois group of a one-parameter family of second order linear differential equations* arXiv:1208.2226 (2012)./ /

The basic example

Consider the parametrized differential equation

$$(E) \qquad \partial_x y = \frac{t}{x} y, \quad t \in \mathbb{C} \qquad (\partial_x = \frac{d}{dx})$$

over the differential base-field $\mathbb{C}(x, t)$.

- (E) has simple singularities near 0 and ∞.

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- (E) has simple singularities near 0 and ∞.
- Let $C = \overline{\mathbb{C}(t)}^{\Delta}$ (differential closure). The PPV-extension of $C(x)$ is

$$K = C(x)\langle x^t \rangle = C(x, x^t, \log x).$$

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- The PPV-group over $C(x)$ is

$$G = \{a \in C^*, \ (\partial_t{}^2 a)a - (\partial_t a)^2 = 0\},$$

(Kolchin-closed in $C^* = \mathrm{GL}(1, C^*)$).

Parametrized monodromy

- **Definition** : Let $Y(x,t)$ be a fundamental solution of the parametrized differential system

$$(S) \qquad \partial_x Y = A(x,t)Y.$$

The parametrized monodromy matrix of (S) around $\alpha_i(t)$ is $M_i(t)$, where for each fixed $t \in \mathcal{U}$, $M_i(t)$ is the monodromy matrix for Y around $\alpha_i(t)$.

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- **Question**: Do the parametrized monodromy matrices $M_i(t)$ belong to the PPV-group ? in which sense? over which differential field?

- Assume the coefficients of

$$(S) \qquad \partial_x Y = A(x, t) Y$$

are rational in x.

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- Let $\partial_x = \frac{d}{dx}$, $\partial_{t_i} = \frac{d}{dt_i}$, $\Delta = \{\partial_x, \partial_{t_1}, \ldots, \partial_{t_r}\}$, $\Delta_t = \{\partial_{t_1}, \ldots, \partial_{t_r}\}$

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- **Theorem 1** (M. F. Singer & C. M.) *Let C be a differentially closed Δ_t-field containing $\mathbb{C}$, such that the entries of A, in*

$$(S) \qquad \partial_x Y = A(x, t) Y$$

belong to $C(x)$.

If C_1 is any differentially closed Δ_t-extension of C containing the coefficients of the parametrized monodromy matrices $M_i(t)$, then

$$M_i(t) \in G(C_1)$$

where $G = \mathrm{Gal}_{C(x)}((S))$ is the PPV-group of (S) over $C(x)$.

The proof in particular relies on

Theorem (Seidenberg, 1969) *Let K and K_1, with $K \subset K_1$, be finitely generated differential extensions of $\mathbb{Q}$. Assume that K consists of meromorphic functions on some open subset $\Omega \in \mathbb{C}^r$. Then K_1 is differentially isomorphic to a field $\widetilde{K_1}$ of functions meromorphic on an open subset $\Omega_1 \subset \Omega$, s. t. the restrictions of functions of K to Ω_1 belong to $\widetilde{K_1}$.*

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Note that the asumption on K is always satisfied (once the theorem holds). The important information here is Ω.

Basic example

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$$\mathrm{Gal}_{C(x)}((E)) = \{a \in C^*,\ (\partial_t{}^2 a)a - (\partial_t a)^2 = 0.\}$$
- Note that the equation is obviously non-isomonodromic since $m_0(t) = e^{2\pi i t}$, since also it extends to an integrable system with the non-rational equation $\partial_t y = \log(x) y$ (Sibuya's criterion).

Parametric version of Schlesinger's density theorem

The following extends Schlesinger's theorem:

Theorem 2 (M. F. Singer & C. M.) *Consider an order n system*

$$(S) \qquad \partial_x Y = A(x, t)Y$$

with $A \in \mathrm{gl}(n, \mathcal{O}_{\mathcal{U}}(x))$, $(x, t) \in \Omega \times \mathcal{U}$ *with asumptions as before.*

Assume (S) has parametrized regular singularities near each $\alpha_i(0)$ *and let a differentially closed* Δ_t*-field* C *contain :*

- all coefficients of powers of x *fo the entries of* A

- all entries of the parametrized monodromy matrices $M_i(t)$ *for each* i*.*

Then the $M_i(t)$ *generate a Kolchin-dense subgroup of* $G(C)$*, where* G *is the PPV-group of (S) over* $C(x)$*.*

The proof uses Galois correspondence and

Lemma *Let $\mathcal{F}$ be a differential field of meromorphic functions in (x, t) on $\mathcal{U} \times \mathcal{V}$, $\mathcal{U} \subset \mathbb{C}$, $\mathcal{V} \subset \mathbb{C}^r$ (assume $x \in \mathcal{F}$) and let C denote the field of ∂_x-constants of $\mathcal{F}$.*

If a function $f \in \mathcal{F}$ is such that $f(x, t) \in \mathbb{C}(x)$ for all fixed $t \in \mathcal{V}$, then $f(x, t) \in C(x)$.

(adapted from a result of R. Palais, 1978)

Parametric version of the weak Riemann-Hilbert problem

Theorem 3 (M. F. Singer & C. M.) *Let* $\Sigma = \{a_1, \ldots, a_s\} \subset \overline{\mathbb{C}}$ *(distinct) and* $\mathcal{U} \subset \mathbb{C}^r$ *an open polydisk. Let* $M_i(t) \in \mathrm{GL}(n, \mathcal{O}_{\mathcal{U}})$*,* $i = 1, \ldots, s$*, be matrices such that*

$$M_1(t) \ldots M_s(t) = \mathrm{I}\,.$$

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Then there is a parametrized system

$$(S) \qquad \partial_x Y = A(x, t) Y$$

with $A \in \mathrm{gl}(n, \mathcal{O}_{\mathcal{U}}{}'(x))$, $\mathcal{U}' \subset \mathcal{U}$, *such that*

- *the set of singular points of (S) is* Σ
- *the parametrized monodromy matrix of (S) around each* a_i *is* $M_i(t)$ *(with respect to some fund. sol. and arbitrary fixed base-point* $x_0 \notin \Sigma$*).*

Moreover, the $M_i(t)$ *can be realized by a system (S) with all singularities simple, but one.*

Inverse problem

Corollary (M. F. Singer & C. M.) *Let* $G \subset \mathrm{GL}(n, C)$ *be a* Δ_t*-linear differential algebraic group, where* C *is a* Δ_t*-universal field* C *and* $\Delta_t = \{\partial_{t_1}, \dots, \partial_{t_r}\}$*. If* G *contains a finitely generated, Kolchin-dense subgroup, then* G *is realizable as the PPV-group over* $C(x)$ *of some* $\partial_x Y = AY$ *with coefficients in* $C(x)$*.*

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Theorem (Singer, 2012) *Let (C, ∂) be a universal field and let G be a linear algebraic group defined over C. Then $G(C)$ is realizable as the PPV-group over $(C(x), \partial, \partial_x)$ iff the identity component G^0 of G has no quotient (as an algebraic group) isomorphic to the $\mathbb{G}_{\mathrm{a}}$ or $\mathbb{G}_{\mathrm{m}}$.*

Isomonodromy (PPV-criterion)

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Theorem (Cassidy & Singer, 2006) *Assume the coefficients of*

$$(S) \qquad \partial_x Y = A(x, t) Y$$

are rational in x, and that (S) has p.r.s. only.
Let C be a Δ_t-differentially closed extension of $\mathcal{O}_{\mathcal{U}}$, with $\Delta_t = \{\partial_{t_1}, \ldots, \partial_{t_r}\}$.
Then (S) is isomonodromic if and only if the PPV-group is conjugate in $\mathrm{GL}(n, C)$ to a constant linear algebraic group (that is, a subgroup of $\mathrm{GL}(n, \mathbb{C})$).

Projective isomonodromy

- **Definition**: With notation as before, a parametrized system (S) with singularities $\alpha_1(t), \ldots, \alpha_s(t)$ is projectively isomonodromic if for all i there are

  - constant matrices $G_i \in \mathrm{GL}(n, \mathbb{C})$
  - analytic functions $c_i : \mathcal{U} \to \mathbb{C}^*$

  such that for each fixed $t \in \mathcal{U}$, some fundamental solution $Y_t(x)$ of (S) has the parametrized monodromy matrix

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- **Remark**: $Y_t(x)$ is not necessarily analytic in t but it is possible to find such a solution which is analytic (proof similar to Bolibrukh's proof in the isomonodromic case).

Projective isomonodromy of Fuchsian systems

Proposition: A Fuchsian (analytic) parametrized system

$$(S) \qquad \partial_x Y = \sum_{i=1}^{m} \frac{A_i(t)}{x - x_i(t)}$$

is projectively isomonodromic if and only if for each i

$$A_i = B_i + b_i \, \mathrm{I}$$

where $b_i : \mathcal{D} \to \mathbb{C}, \quad B_i : \mathcal{D} \to \mathrm{gl}(n, \mathbb{C})$

are analytic functions such that

$$\partial_x Y = \sum_{i=1}^{m} \frac{B_i(t)}{x - x_i(t)}$$

is isomonodromic.

In the (DH V) example, Equation (1) of the Lax pair meets this condition:

$$\partial_x Y = \left(\frac{\mu \, \mathrm{I}}{(x-x_1)(x-x_2)(x-x_3)} + \sum_{i=1}^{3} \frac{\lambda_i K}{x - x_i} \right) Y$$

Here

$$b_i = \frac{\mu}{(x-x_1)(x-x_2)(x-x_3)}, \quad B_i = \frac{\lambda_i K}{x - x_i}$$

and

$$\partial_x Y = \left(\sum_{i=1}^{3} \frac{\lambda_i K}{x - x_i} \right) Y$$

is clearly isomonodromic since K is a constant matrix.

Theorem (Singer & M.) : *if a system* (S) *is absolutely irreducible over* $C(x)$*, then it is projectively isomonodromic if and only if the commutator subgroup* (G, G) *of the PPV-group* G *is conjugate in* $\mathrm{GL}(n, C)$ *to a constant subgroup (* $=$ *subgroup of* $\mathrm{GL}(n, \mathbb{C})$*).*

/ / M. F. Singer, C. M. : *Projective isomonodromy and Galois groups*, Bull. London Math. Soc. 44 (5), 913-930 (2012)./ /

Thank you for your attention

Dziękuję za uwagę

Podziękowania dla organizatorów