On Gevrey orders of power expansions of solutions to the fifth Painlevé equation

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The fifth Painlevé equation

We consider the fifth Painlevé equation (P5):

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + $$

$$+\frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1},$$

where $\alpha, \beta, \gamma, \delta$ are complex parameters, z and w are the respective independent and dependent variables. The P5 equation has two singular points $z = 0$ and $z = \infty$.

Types of expansions obtained

$$w = c_r(z)z^r + \sum_{s\in\mathbf{K}} c_s(z)z^s, \tag{1}$$

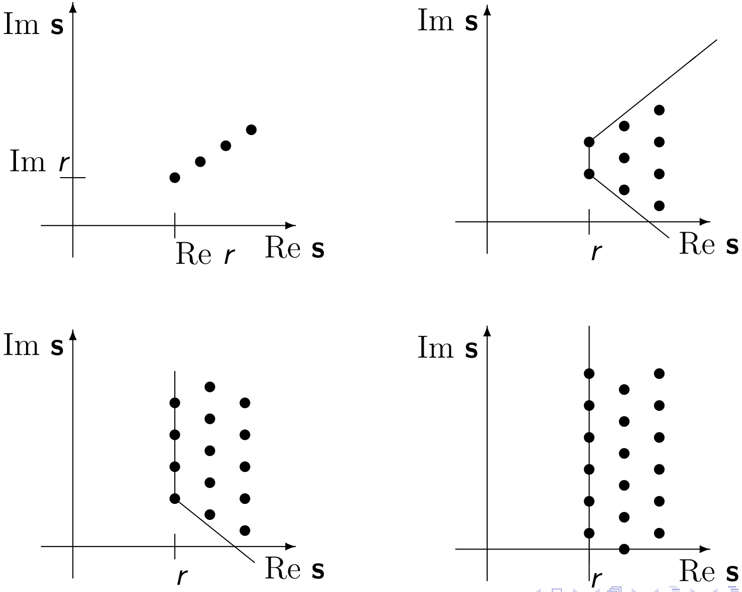
where $c_r(z), c_s(z), r, s \in \mathbb{C}$. The set $\mathbf{K}$ is a subset of a half-plane $\mathrm{Re}\,(s-r) > 0$ for the expansions as $z \to 0$ and is a subset of $\mathrm{Re}\,(s-r) < 0$ for the expansions as $z \to \infty$; it is a subset of some discrete grid in $\mathbb{C}$.

1. *Power expansions*: $c_r(z)$ and $c_s(z)$ are constants.
2. *Power-logarithmic*: $c_r(z)$ is constant, $c_s(z)$ are polinomials in $\log z$.
3. *Complicated*: $c_r(z)$, $c_s(z)$ are series in decreasing powers of $\log z$.
4. *Exotic*: $r, s \in \mathbb{R}$, $c_r(z)$ and $c_s(z)$ are series in z^i, c_r is a sum of no-more-than-countable number of terms, the set of power exponents of z^i in c_r is bounded either from above or from below.
5. Also we are looking for *exponential expansions* of solutions

$$w = \sum_{k=0}^{\infty} b_k(z) C^k \exp(k\varphi(z)),$$

where $b_k(z)$, $\varphi'(z)$ are power expansions with rational powers (for them the set $\mathbf{K} \subset \dfrac{\mathbb{Z}}{m}$, $m \in \mathbb{N}$).

Typical examples of set $\mathbf{K}$ in case $z \to 0$



Construction of the polygon

We consider an ordinary differential equation of order n of the form

$$P(z, w, w', \dots, w^{(n)}) = 0, \qquad (2)$$

P is a polynomial of its variables.
The left part of this equation is called a *differential sum*. To each differential monomial $a(z, w)$ in the differential sum (2) we put in correspondence its *two-dimensional vector power exponent* $\mathbf{Q}(a(z, w)) = (q_1, q_2)$ according to the following rule:

$$\mathbf{Q}(cz^r w^s) = (r, s);$$

$$\mathbf{Q}\left(\frac{d^l w}{dz^l}\right) = (-l, 1);$$

$$\mathbf{Q}(a(z, w)b(z, w)) = \mathbf{Q}(a(z, w)) + \mathbf{Q}(b(z, w)).$$

The set of all vector power exponents of differential monomials in the differential sum $P(z, w)$ (2) is called *a support of the differential sum* $P(z, w)$ and is denoted by $S(P)$. The convex hull $\Gamma(P)$ of a support $S(P)$ is called *a polygon of the differential sum* $P(z, w)$, the boundary $\partial\Gamma(P)$ of $\Gamma(P)$ consists of vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$.

Next steps of exploration

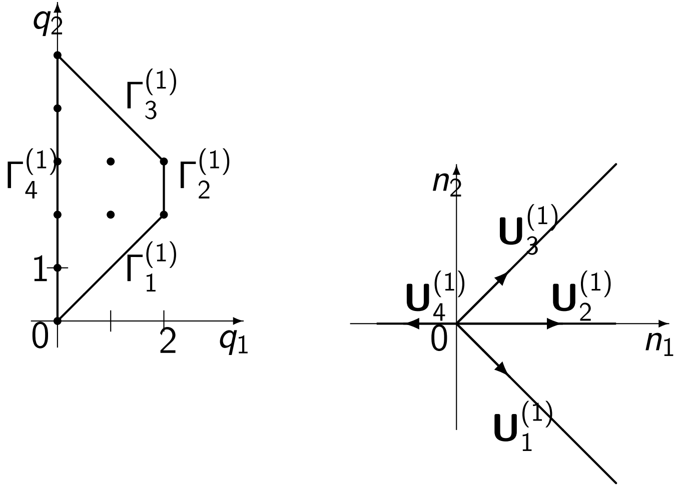
To find asymptotic forms of solutions to the equation we work with the *truncated equations*, corresponding to the vertices (and edges) of the polygon of the differential sum $P(z,w)$. These truncated equations contain all the terms of the equation, the power exponents of which belong to this vertex (or edge) of the polygon of the differential sum $P(z,w)$.

Let us consider an edge having an external normal (n_1, n_2). *The normal cone of the edge* is a ray $\lambda(n_1, n_2)$, where $\lambda > 0$. If we consider a vertex belonging to the edges having external normals (n_{11}, n_{21}) and (n_{21}, n_{22}) we define *the normal cone of the vertex* as $\lambda_1(n_{11}, n_{21}) + \lambda_2(n_{12}, n_{22})$, where $\lambda_1 > 0$ and $\lambda_2 > 0$. If a normal cone of a vertex or of an edge intersects with a part of a plane where $n_1 > 0$ solution to the truncated equation corresponding to it can give asymptotic form of solutions to the equation under consideration in the neighbourhood of infinity, if it intersects with $n_1 < 0$, it can give one in the neighbourhood of zero.

$$-z^2 w(w-1)w'' + z^2\left(\frac{3}{2}w - \frac{1}{2}\right)(w')^2 - zw(w-1)w' +$$

$$+(w-1)^3(\alpha w^2 + \beta) + \gamma z w^2(w-1) + \delta z^2 w^2 (w+1) = 0.$$

The polygon $\mathbf{\Gamma(P5)}$ as $\alpha\beta\gamma\delta \neq \mathbf{0}$



The results obtained

By means of power geometry we have obtained the following results: near a **nonsingular point** of the equation expansions of solutions to P_5 form 10 families: 7 Laurent and 3 Taylor sereies;

10 families of power expansions, 3 families of power-logarithmic, 6 families of complicated and 9 families of exotic expansions of solutions to P_5 **near zero**;

9 power expansions, each of which can be continued to a one-parameter family of exponential expansions of solutions to P_5, and 8 more families of exponential expansions **near infinity**.

Expansions corresponding to the edges $\mathbf{\Gamma_1^{(1)}}$, $\mathbf{\Gamma_2^{(1)}}$, $\mathbf{\Gamma_3^{(1)}}$ as $\alpha\beta\delta \neq \mathbf{0}$, $z \to \infty$

Power series:

$\Gamma_1^{(1)}$ (2 power expansions):

$$w = (-1)^k \sqrt{\frac{\beta}{\delta}} \frac{1}{z} + \left(-\frac{2\beta}{\delta} + (-1)^k \frac{\gamma}{2\delta} \sqrt{\frac{\beta}{\delta}} \right) \frac{1}{z^2} + \sum_{s=3}^{\infty} \frac{c_{s,\,k}}{z^s}.$$

$\Gamma_2^{(1)}$ (power expansion):

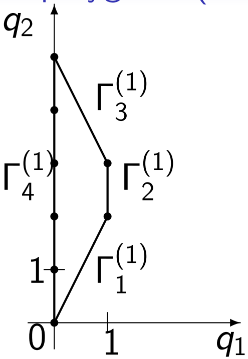
$$w = -1 + \frac{2\gamma}{\delta z} + \sum_{s=2}^{\infty} \frac{c_s}{z^s}.$$

$\Gamma_3^{(1)}$ (2 power expansions):

$$w = (-1)^k \sqrt{-\frac{\delta}{\alpha}} z + 2 + (-1)^k \frac{1}{2} \frac{\gamma}{\sqrt{-\alpha\delta}} + \sum_{s=1}^{\infty} \frac{c_{s,k}}{z^s},$$

where c_s, c_{sk} are uniquely defined constants, $k = 1, 2$.

The polygon $\mathbf{\Gamma(P5)}$ as $\alpha\beta\gamma \neq \mathbf{0},\ \delta = \mathbf{0}$



Expansions of solutions corresponding to the edges $\Gamma_1^{(1)}$, $\Gamma_2^{(1)}$, $\Gamma_3^{(1)}$ $\alpha\beta\gamma \neq \mathbf{0}$, $\delta = 0$, $z \to \infty$

Puiseaux series:
$\Gamma_1^{(1)}$ (2 power expansions):

$$w = (-1)^k \sqrt{-\frac{\beta}{\gamma}} \frac{1}{\sqrt{z}} + \frac{\beta}{\gamma} \frac{1}{z} + \sum_{s=3}^{\infty} \frac{c_{s,k}}{z^{s/2}}.$$

$\Gamma_2^{(1)}$ (power expansion): $w = 1$.
$\Gamma_3^{(1)}$ (2 power expansions):

$$w = (-1)^k \sqrt{-\frac{\gamma}{\alpha}} \sqrt{z} + 1 + \sum_{s=1}^{\infty} \frac{c_{s,k}}{z^{s/2}},$$

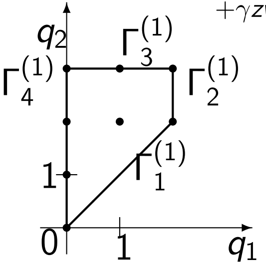
where c_s, c_{sk} are uniquely defined complex constants, $k = 3, 4$.

$\alpha = 0,\ \delta \neq 0$

If $\alpha = 0$ P5 has the following form:

$$-z^2 w(w-1)w'' + z^2\left(\frac{3}{2}w - \frac{1}{2}\right)(w')^2 - zw(w-1)w' + \beta(w-1)^3 +$$

$$+\gamma z w^2(w-1) + \delta z^2 w^2(w+1) = 0.$$



$\alpha = 0,\ z \to \infty$

If $\alpha = 0$, $\delta \neq 0$, $z \to \infty$ we obtain the following one-parameter families of exponential asymptotic expansions of the form

$$w_m = C\exp(\varphi_m(z)) + \sum_{k=0}^{\infty} b_{k,m}(z) C^{-k} \exp(-k\varphi(z)), \qquad (3)$$

where $b_{k,m}(z)$ are power series,

$$\exp(\varphi_m(z)) = z^{1 - \dfrac{(-1)^m\gamma}{\sqrt{-2\delta}}} \exp\{(-1)^m\sqrt{-2\delta}z + \sum_{s=1}^{\infty} d_{s,m} z^{-s}\};$$

are defined in the domains $(-1)^m \mathrm{Re}(\sqrt{-\delta}z) > 0$, $m = 1, 2$.
If $\alpha = 0$, $\delta = 0$ we obtain 2 one-parameter families of exponential asymptotic expansions $\mathcal{G}_{3,4}$ of the form (3), where

$$\exp(\varphi_m(z)) = \sqrt{z}\exp\{(-1)^m 2\sqrt{-2\gamma z} + \sum_{s=1}^{\infty} d_{s,m} z^{-s/2}\},\ m = 3, 4.$$

They are defined in the domain $(-1)^m \mathrm{Re}(\sqrt{-2\gamma z}) > 0$.

$\beta = 0,\ z \to \infty$

If $\beta = 0$ and $\delta \neq 0$ we obtain two more families of exponential expansions $\mathcal{G}_{5,6}$, and if $\beta = 0$, $\delta = 0$ two more families $\mathcal{G}_{7,8}$. These expansions can be obtained from ones obtained when $\alpha = 0$ with the help of the symmetry

$$(z, w, \alpha, \beta, \gamma, \delta) = (z, \frac{1}{\tilde{w}}, -\tilde{\beta}, -\tilde{\alpha}, -\tilde{\gamma}, \delta). \tag{4}$$

$$w = b_0(z) + C\,\exp(\varphi(z)) + \sum_{k=2}^{\infty} b_k(z) C^k \exp(k\varphi(z)).$$

$$b_{0,l}(z) = (-1)^l \sqrt{\frac{\beta}{\delta}}\frac{1}{z} + \left(-\frac{2\beta}{\delta} + (-1)^l \frac{\gamma}{2\delta}\sqrt{\frac{\beta}{\delta}}\right)\frac{1}{z^2} + \sum_{s=3}^{\infty}\frac{c_{-s,l}}{z^s},$$

$$\exp(\varphi_{l,m}(z)) = Cz^{\rho_{l,m}}\exp\{(-1)^m\sqrt{-2\delta}z + \sum_{s=1}^{\infty}\frac{d_{s,l,m}}{z^s}\},$$

$$\rho_{l,m} = -1 + (-1)^{m+l}2\sqrt{-2\beta} + \frac{(-1)^{m+1}\gamma}{\sqrt{-2\delta}};$$

Gevrey asymptotic expansions

Let $\Omega_R(\varphi_1, \varphi_2)$ — be an open sector with a vertex in the infinity on extended complex plane or on Riemann surface of logarithm, i. e.

$$\Omega_R(\varphi_1, \varphi_2) = \{z : |z| > R, \operatorname{Arg} z \in (\varphi_1, \varphi_2)\}.$$

w — holomorphic on $\Omega_R(\varphi_1, \varphi_2)$ function and $\hat{w} \in \mathbb{C}[[1/z]]$.
A function w is said to be *asymptotically approximated by a series $\hat{w}$ on* $\Omega_R(\varphi_1, \varphi_2)$, if for the points z of any closed subsector Y of $\Omega_R(\varphi_1, \varphi_2)$ and for any $n \in \mathbb{N}$ there exist the constants $M_{Y,n} > 0$:

$$|z^n||w(z) - \sum_{p=0}^{n-1} a_p z^{-p}| < M_{Y,n}.$$

If there exist the constants A_Y, C i. e. $M_{Y,n} = C(n!)^{1/k} A_Y^n$ a series $\hat{w}$ is *an asymptotical Gevrey series of order $1/k$ for a function w on* $\Omega_R(\varphi_1, \varphi_2)$.

For $G(z, Y, Y_1, \ldots, Y_n)$ being an analytic function of $n+2$ variables we consider an equation

$$G(z, w, Dw, \ldots, D^n w) = 0. \quad (5)$$

Let $\hat{w} \in \mathbb{C}[[1/z]]$ be a formal series being a solution to the differential equation (5) and D be an operator $z\frac{d}{dz}$.

Theorem

(O. Perron, J.-P. Ramis, B. Malgrange, Y. Sibuya in different cases) Let $\hat{w} \in \mathbb{C}[[z]]_{1/k}$ be a solution to the equation (5). Then there exist $k' > 0$ i. e. for every open sector V with the vertex in the infinity, having an angle less than $\min(\pi/k, \pi/k')$ and for a sufficiently big R there exist a function w, being a solution to the equation (5) which is asymptotically approximated of Gevrey order $1/k$ by a series $\hat{w}$.

The next theorem contains conditions on the Newton polygon. We will describe the process of its construction.

Let us be given a linear differential operator

$$L = \sum_{k=0}^{n} a_k(z) D^k, \text{ where } a_k(z) \in \mathbb{C}[[z]][1/z],\ a_k(z) = \sum_{j_k = j_{k,\,0}}^{\infty} a_{j_k} z^{-j},$$

$$a_{j_k} = \mathrm{const} \in \mathbb{C}.$$

We will put in correspondence a set of points on the plane: $(k,\, j_{k,\,0})$, $k = 0,\, \ldots,\, n$ — *a support of the operator* L. We will define a set

$$N = \bigcup_{k=0}^{n} \{(q_1, q_2) : q_1 \leq k,\ q_2 \geq j_{k,\,0}\}$$

and then we construct a convex hull of this set in half-plane $q_1 \geq 0$. A boundary this set is called *a Newton polygon of the linear differential operator* L.

Theorem

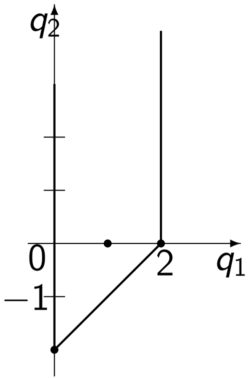
(J.-P. Ramis) Let $\hat{w} \in \mathbb{C}[[z]]$ *be a formal solution to the equation* (5), *then the series* $\hat{w}$ *converges or has a Gevrey order equal to* s, *where* $s \in \{0, \frac{1}{k_1}, \ldots, \frac{1}{k_N}\}$ *and* $0 < k_1 < \ldots < k_N < +\infty$ *are all positive slopes of the edges of the* $N(G, \hat{w})$ *to the X-axis.*

The previous theorem is formulated for a nonlinear differential equation and its formal solution $\hat{w}$.

In a particular case $\frac{\partial G}{\partial Y_n}(z, \hat{w}, \dots, \hat{w}^{(n)}) \neq 0$ a Newton polygon of this equation on a formal solution can be constructed as a polygon of an operator

$$L_0 = \sum_{k=0}^{n} \left(\frac{\partial G}{\partial Y_k}(z, \hat{w}, D\hat{w}, \dots, D^n \hat{w}) \right) D^k. \tag{6}$$

We can show that this operator coincides with the operator $\mathcal{M}$ used to construct exponential expansions of solutions using the methods of power geometry.


q_2
0
2
q_1
-1

Gevrey asymptotic expansions

Gevrey orders of the series obtained as formal power series solutions as $\alpha\delta \neq 0$ are equal to one. The Puiseaux series obtained obtained as formal power series solutions as $\alpha \neq 0$, $\delta = 0$, considered as the series of $\sqrt{z}$ also have Gevrey order equal to one.

$\alpha\delta \neq 0$
There exist $k' \geq 1$ and $R_0 \in \mathbb{R}_+$ i. e. for every open sector $\Omega_R(\varphi_1, \varphi_2)$, $R \geq R_0$, $\varphi_2 - \varphi_1 < \pi/k' \leq \pi$ there exist a solution to the fifth Painlevé equation approximated by this Gevrey-1 series by each of the power series obtained.

$\alpha \neq 0$, $\delta = 0$
There exist $k' \geq 1/2$ and $R_0 \in \mathbb{R}_+$ i. e. for every open sector $\Omega_R(\varphi_1, \varphi_2)$, $R \geq R_0$, $\varphi_2 - \varphi_1 < \pi/k' \leq 2\pi$ there exist a solution to the fifth Painlevé equation Gevrey asymptotically approximated of order one by each of the power series obtained.

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