Non-Hamiltonian systems separable by Hamilton-Jacobi method

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Abstract

We show that with every separable calssical Stäckel system of Benenti type on a Riemannian space one can associate, by a proper deformation of the metric tensor, a multi-parameter family of non-Hamiltonian systems on the same space, sharing the same trajectories and related to the seed system by appropriate reciprocal transformations. These system are known as bi-cofactor systems and are integrable in quadratures as the seed Hamiltonian system is. We show that with each class of bi-cofactor systems a pair of separation curves can be related. We also investigate conditions under which a given flat bi-cofactor system can be deformed to a family of geodesically equivalent flat bi-cofactor systems.

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1 Introduction

A significant progress in the geometric separability theory for classical Hamiltonian systems separable by Hamilton-Jacobi method has been achieved in recent years (see for example [1] - [4]). Among other things a new class of non-Hamiltonian Newton systems was introduced [5], [6]. These systems were shown to have very interesting geometric properties when considered as systems on Riemann spaces [7],[8] (see also [9]). In [10] we showed that they can be separated by the Hamilton-Jacobi method after certain reparametrization of the evolution parameter (see also [11]). Originally these systems were called quasi-Lagrangian systems. In the present literature they are called bi-cofactor systems or cofactor-pair systems. In [12] it was further shown that each bi-cofactor system is *geodesically equivalent* (in the classical sense of Levi-Civita [13]) to some separable Lagrangian system which means that it has the same trajectories on the underlying configuration manifold as the Lagrangian system only traversed with a different speed and moreover that the metric tensors associated with both systems are equivalent i.e. have the same geodesics (considered as unparametrized curves). In the same paper one can also find a thorough geometric theory of bi-cofactor systems on an arbitrary pseudoriemannian space.

In the present paper we demonstrate on the level of differential equations the geodesic equivalence properties of cofactor and bi-cofactor systems expressed by an appropriate class of reciprocal transformations. We clarify and systematize their bi-quasihamiltonian formulation on the phase space. We show explicitly that a bi-cofactor system is geodesically equivalent to two different separable Hamiltonian systems of Benenti type and we show explicitly the transformation between all geometric structures associated with these two Benenti systems and the original bi-cofactor system. We further demonstrate that with each bi-cofactor system one

can relate two different separation curves and we find a map between these curves. From this point of view we therefore show that with each pair of separation curves that are related through the above mentioned map we can associate a whole class of geodesically equivalent bi-cofactor systems. Every such class contains at least two separable Hamiltonian systems and on the phase space all the members of a given class are related by a composition of an appropriate non-canonical transformation and a reciprocal transformation. Further, we investigate geodesically equivalent families of flat (in the sense of the underlying metric tensor) cofactor systems and find a sufficient condition for a so called J-tensor to generate from any given flat bi-cofactor system a multi-parameter family of flat bi-cofactor systems. Finally, we illustrate our considerations by presenting a thorough example of the class of separable bi-cofactor systems geodesically equivalent to the Henon-Heiles system and then specify this example to the flat case.

2   Cofactor systems

Let us consider the following Newton system

$$\frac{d^2 q^i}{dt^2} + \Gamma^i_{jk} \frac{dq^j}{dt} \frac{dq^k}{dt} = F^i, \qquad i = 1, \dots, n \tag{1}$$

where q^i are some coordinates on an n-dimensional pseudo-Riemannian manifold Q endowed with a metric tensor $g = (g_{ij})$ and where $F = (F^i)$ is a vector field on Q representing the force which we assume time- and velocity-independent. Here and in what follows we use the Einstein summation convention if not stated otherwise. The functions Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection associated with the metric tensor g and if all Γ^i_{jk} are zero we call the system (1) a *flat Newton system*. In case that $F = 0$ (1) is the equation of geodesic motion on Q and the variable t becomes an affine parameter of geodesic lines.

If the force F is conservative (potential) i.e. if

$$F = -\nabla V = -GdV, \tag{2}$$

where $G = g^{-1}$ is the contravariant form of the metric tensor g and where $V = V(q)$ is a potential function, then (1) is equivalent to the Lagrangian system

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0, \quad v^i = \frac{d}{dt} q^i, \quad i = 1, \dots, n \tag{3}$$

on the tangent bundle TQ endowed with coordinates $(q, v) = (q^1, \dots q^n, v^1, \dots, v^n)$, where $\mathcal{L} = \frac{1}{2} g_{ij}(q) v^i v^j - V(q)$ is a Lagrangian of the system. By the Legendre map $p_i = g_{ij} v^j$ the system (3) is transformed to the Hamiltonian dynamical system

$$\frac{d}{dt} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = \Pi_c \, dH \tag{4}$$

on the cotangent bundle T^*Q endowed with coordinates $(q, p) = (q^i, p_j)$ where $H = \frac{1}{2} G^{ij}(q) p_i p_j + V(q)$ is the Hamiltonian of the system, Π_c is the canonical Poisson tensor and dH is the differential of H.

We will now remind the notion of a J-tensor.

Definition 1 *A $(1,1)$-tensor $\mathbf{J} = (J^i_j)$ on Q is called a J-tensor associated with the metric g or $G = g^{-1}$ (we often write that $\mathbf{J}$ is a J_G-tensor when emphasizing the underlying metric) if its contravariant form $J^{ij} = J^i_k G^{kj}$ is a symmetric $(2,0)$-tensor and if $\mathbf{J}$ itself satisfies the following* characteristic equation

$$\nabla_h J^i_j = \left(\alpha_j \delta^i_h + \alpha^i g_{jh} \right) \tag{5}$$

where ∇_h is the covariant derivative associated with the metric g and where α_i is some 1-form.

From (5) it follows that the Nijenhuis torsion of $\mathbf{J}$ vanishes:

$$J^h_{[i} \nabla_{|h|} J^k_{j]} - J^k_l \nabla_{[i} J^l_{j]} = 0$$

(the square brackets denote skew-symmetric permutations of indices i, j; the index h is not permuted) and that $\mathbf{J}$ is a conformal Killing tensor of trace type which means that $J_{ij} = J^k_i g_{kj}$ satisfies the relation $\nabla_{(h} J_{ij)} = \alpha_{(h} g_{ij)}$ with $\alpha_i = \partial_i \operatorname{tr} \mathbf{J}$ (the brackets denote symmetric permutations of indices h, i, j).

Remark 2 *All J-tensors of a given metric tensor g constitute an **R**-linear vector space of dimension less or equal to $\frac{1}{2}(n+1)(n+2)$. This space attains its maximum dimension for metrics of constant curvature. In case the metric g is pseudoeuclidean so that $g = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$ in its Cartesian coordinates, the general form of **J** in these coordinates is [12]*

$$J^{ij} = mq^i q^j + \beta^i q^j + \beta^j q^i + \gamma^{ij} \tag{6}$$

*where m, β^i and $\gamma^{ij} = \gamma^{ji}$ are $\frac{1}{2}(n+1)(n+2)$ independent constants and where $J^{ij} = J^i_k G^{kj}$ is the contravariant form of **J**.*

If a J-tensor **J** has n real and simple eigenvalues then it is called L-tensor and its signed eigenvalues $(\lambda^1, \ldots, \lambda^n)$ given by $\det(\mathbf{J} + \lambda(q)I) = 0$ define a coordinate web on Q. Such webs will turn out to be separation webs for our systems (see below). See [12] for further details on J-tensors and L-tensors.

The system (1) is called *cofactor* if the force F has the following form

$$F = -\left(\mathrm{cof}\,\mathbf{J}\right)^{-1} \nabla V \tag{7}$$

for some J-tensor **J**, where $\mathrm{cof}\,\mathbf{J}$ is the cofactor matrix of **J** (i.e. the transposed matrix of signed minors of **J**) so that $\mathbf{J}\,\mathrm{cof}\,\mathbf{J} = (\mathrm{cof}\,\mathbf{J})\,\mathbf{J} = (\det \mathbf{J})\,I$ or in case that **J** is invertible $\mathrm{cof}\,\mathbf{J} = (\det \mathbf{J})\,\mathbf{J}^{-1}$. In the case $\mathbf{J} = I$ the system (7) becomes Lagrangian (potential).

In our further considerations the notion of equivalent metric tensors will play an important role. Two metric tensors G and $\overline{G}$ on manifold Q are said to be *equivalent* if their geodesics locally coincide as unparametrized curves. As it was shown in [12], a metric G admits an equivalent metric $\overline{G}$ if and only if it admits a nonsingular J-tensor **J**. In such a case $\overline{G}^{ij} = \sigma J^i_k G^{kj} = \sigma J^{ij}$ or in the matrix form

$$\overline{G} = \sigma \mathbf{J} G$$

with $\sigma = \det \mathbf{J} = \frac{dt}{d\bar{t}}$ where t and $\bar{t}$ are affine parameters associated with the (parametrized) geodesic of G and $\overline{G}$ respectively. Moreover, $\mathbf{J}^{-1}$ is a J-tensor for the new metric $\overline{G}$.

Two dynamical systems (g, F) and $(\overline{g}, \overline{F})$ of the form (1) on Q are said to be equivalent if their trajectories coincide up to a reparametrization of the evolution parameter. Moreover, they are called *geodesically equivalent* if also metrics g and $\overline{g}$ are equivalent. As it was proved in [12] two systems (g, F) and $(\overline{g}, \overline{F})$ are geodesically equivalent if and only if the metric g admits a nonsingular J-tensor **J** such that

$$\overline{G}^{ij} = \sigma J^i_k G^{kj}, \quad \overline{F} = \sigma^2 F, \quad \sigma = \det \mathbf{J}.$$

Also in this case the evolution parameters t and $\bar{t}$ of the systems (g, F) and $(\overline{g}, \overline{F})$ are related through the above mentioned reciprocal transformation

$$\frac{dt}{d\bar{t}} = \sigma.$$

We will now show that every cofactor system belongs to a whole class of geodesically equivalent cofactor systems.

Theorem 3 *Consider the cofactor system*

$$\frac{d^2 q^i}{dt^2} + \Gamma^i_{jk} \frac{dq^j}{dt} \frac{dq^k}{dt} = -\left(\left(\mathrm{cof}\,\mathbf{J}\right)^{-1} \nabla V\right)^i, \quad i = 1, \ldots, n. \tag{8}$$

Assume that $\mathbf{J}_1$ is another J-tensor for the metric G and denote by $G_1 = \sigma_1 \mathbf{J}_1 G$ (with $\sigma_1 = \det \mathbf{J}_1$) a new metric tensor equivalent to G. In a new independent variable t_1 defined through the reciprocal transformation

$$dt_1 = \frac{dt}{\sigma_1}$$

the cofactor system (8) attains the form

$$\frac{d^2 q^i}{dt_1^2} + (\Gamma^{(1)})^i_{jk} \frac{dq^j}{dt_1} \frac{dq^k}{dt_1} = -\left(\left[\mathrm{cof}\left(\mathbf{J}\mathbf{J}_1^{-1}\right)\right]^{-1} \nabla^{(1)} V\right)^i, \quad i = 1, \ldots, n \tag{9}$$

where $(\Gamma^{(1)})^i_{jk}$ are Christoffel symbols of the metric G_1 and $\nabla^{(1)} = G_1 d$ is the gradient operator associated with the metric G_1.

Proof. Since $dt_1 = dt/\sigma_1$ we have, by the chain rule,

$$\frac{dq^i}{dt} = \frac{1}{\sigma_1}\frac{dq^i}{dt_1}, \quad \frac{d^2q^i}{dt^2} = \frac{1}{\sigma_1^2}\frac{d^2q^i}{dt_1^2} - \frac{1}{\sigma_1^3}\frac{dq^i}{dt_1}\frac{\partial\sigma_1}{\partial q_l}\frac{dq_l}{dt_1}.$$

Moreover (see for example [14]) the Christoffel symbols of G and G_1 are related by

$$\Gamma^i_{jk} = (\Gamma^{(1)})^i_{jk} + \frac{1}{2\sigma_1}\left(\delta^i_j\frac{\partial\sigma_1}{\partial q_k} + \delta^i_k\frac{\partial\sigma_1}{\partial q_j}\right). \tag{10}$$

Further

$$\nabla^{(1)}V = G_1 dV = \sigma_1\mathbf{J}_1 GdV = \sigma_1\mathbf{J}_1\nabla V,$$

so that

$$\begin{aligned}
(\operatorname{cof}\mathbf{J})^{-1}\nabla V &= \frac{1}{\sigma_1}(\operatorname{cof}\mathbf{J})^{-1}\mathbf{J}_1^{-1}\nabla^{(1)}V = \frac{1}{\sigma_1^2}(\operatorname{cof}\mathbf{J})^{-1}\operatorname{cof}\mathbf{J}_1\nabla^{(1)}V = \\
&= \frac{1}{\sigma_1^2}\operatorname{cof}(\mathbf{J}^{-1})\operatorname{cof}\mathbf{J}_1\nabla^{(1)}V = \frac{1}{\sigma_1^2}\operatorname{cof}(\mathbf{J}_1\mathbf{J}^{-1})\nabla^{(1)}V = \\
&= \frac{1}{\sigma_1^2}\left(\operatorname{cof}\left(\mathbf{J}\mathbf{J}_1^{-1}\right)^{-1}\right)\nabla^{(1)}V.
\end{aligned}$$

Plugging all this into (8) we obtain (9). ∎

Remark 4 *The tensor $\mathbf{J}\mathbf{J}_1^{-1}$ is a J_{G_1}-tensor i.e. a J-tensor for the metric G_1. It means that the system (9) is a cofactor system geodesically equivalent to (8) with G_1 as the underlying metric.*

Note that in the particular case $\mathbf{J}_1 = \mathbf{J}$ the system (9) becomes potential

$$\frac{d^2q^i}{d\bar{t}^2} + \overline{\Gamma}^i_{jk}\frac{dq^j}{d\bar{t}}\frac{dq^k}{d\bar{t}} = -\left(\overline{\nabla}V\right)^i, \qquad i = 1,\dots,n. \tag{11}$$

with the affine parameter

$$dt_1 = d\bar{t} = \frac{dt}{\sigma}$$

and with $\overline{\Gamma}^i_{jk}$ and $\overline{\nabla} = \overline{G}d$ defined by the new metric $\overline{G} = \sigma\mathbf{J}G$ with $\sigma = \det(\mathbf{J})$. This shows that every cofactor system is geodesically equivalent (in the sense of the definition above) to a potential system. This fact yields us a possibility of determining a quasi-hamiltonian formulation for the cofactor system (8).

Proposition 5 *The cofactor system (8) has on T^*Q the following quasi-Hamiltonian representation:*

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{\sigma}\Pi_{nc}dH \tag{12}$$

with the noncanonical Poisson operator

$$\Pi_{nc} = \begin{pmatrix} 0 & \mathbf{J} \\ -\mathbf{J}^T & \Omega \end{pmatrix}, \quad \Omega^i_j = \left(\frac{\partial J^k_i}{\partial q^j} - \frac{\partial J^k_j}{\partial q^i}\right)p_k$$

and with the Hamiltonian

$$H(q,p) = \frac{1}{2}p^T(\operatorname{cof}\mathbf{J})Gp + V(q). \tag{13}$$

Proof. The systems (8) and (11) are related by the reciprocal transformation $d\bar{t} = dt/\sigma$ with $\sigma = \sigma(q)$ yielding that $dq^i/d\bar{t} = \sigma dq^i/dt$. Let us thus introduce new variables on TQ:

$$\bar{q} = q, \quad \bar{v} = \sigma v. \tag{14}$$

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The Lagrangian of (11) written in coordinates $(\bar{q}, \bar{v})$ is:

$$\overline{\mathcal{L}} = \frac{1}{2}\bar{g}_{ij}(\bar{q})\bar{v}^i\bar{v}^j - V(\bar{q})$$

This Lagrangian defines a new Legendre map from TQ to T^*Q that is just the fiberwise isomorphism between TQ and T^*Q induced by the new metric $\bar{g}$ i.e. $\bar{p} = \bar{g}\,\bar{v}$. From $\overline{G} = \sigma\mathbf{J}G$ we have

$$\bar{g} = \overline{G}^{-1} = \frac{1}{\sigma}g\mathbf{J}^{-1} = \frac{1}{\sigma}(\mathbf{J}^T)^{-1}g$$

so that

$$\bar{p} = \bar{g}\,\bar{v} = \frac{1}{\sigma}(\mathbf{J}^T)^{-1}g\sigma v = (\mathbf{J}^T)^{-1}gv = (\mathbf{J}^T)^{-1}p.$$

Thus, the map (14) on TQ induces the following non-canonical map on T^*Q:

$$\bar{q} = q, \ \ \bar{p} = \left(\mathbf{J}^T\right)^{-1}p. \tag{15}$$

In the coordinates $(\bar{q}, \bar{p})$ the system (11) has the following canonical Hamiltonian representation (cf (4)):

$$\frac{d}{d\bar{t}}\left(\begin{array}{c} \bar{q} \\ \bar{p} \end{array} \right) = \overline{\Pi}_c d\overline{H}, \tag{16}$$

with the usual Hamiltonian $\overline{H} = \frac{1}{2}\bar{p}^T\,\overline{G}\,\bar{p} + V(\bar{q})$. In order to obtain the quasi-hamiltonian form (12) of (8) it is enough to transform the system (16) back to the variables (q, p, t). The map between these variables is

$$\bar{q} = q, \ \ \bar{p} = \left(\mathbf{J}^T\right)^{-1}p, \ \ d\bar{t} = \frac{dt}{\sigma}. \tag{17}$$

or equivalently

$$q = \bar{q}, \ \ p = \mathbf{J}^T\bar{p}, \ \ dt = \sigma d\bar{t}. \tag{18}$$

Note that this map consists of a "space" part (11) that involves only (q, p) and $(\bar{q}, \bar{p})$ variables followed by the reciprocal transformation (reparametrization of evolution parameter) $d\bar{t} = dt/\sigma$. By using that $d/d\bar{t} = \sigma d/dt$ (which generates the factor $1/\sigma$ in (12)) and after some calculations that exploit the fact that $\mathbf{J}$ is torsionfree we obtain (12) with H denoting the function $\overline{H}$ written in (q, p)-coordinates. Since

$$\bar{p}^T\,\overline{G}\,\bar{p} = p^T\mathbf{J}^{-1}\sigma\mathbf{J}G(\mathbf{J}^T)^{-1}p = p^T\sigma\mathbf{J}^{-1}Gp = p^T\,\mathrm{cof}(\mathbf{J})Gp$$

we get that H is of the form (13). ∎

3 Bi-cofactor systems

The system of Newton equations of the form

$$\frac{d^2q^i}{dt^2} + \Gamma^i_{jk}\frac{dq^j}{dt}\frac{dq^k}{dt} = -\left(\left(\mathrm{cof}\,\mathbf{J}_1\right)^{-1}\nabla V\right)^i = -\left(\left(\mathrm{cof}\,\mathbf{J}_2\right)^{-1}\nabla W\right)^i \tag{19}$$

with two independent J_G-tensors $\mathbf{J}_1$ and $\mathbf{J}_2$ and with two different potentials V and W is called a *bi-cofactor system* on Q. It means that the force F has two different cofactor representations of the form (7). The following is a simple corollary of Theorem 3.

Proposition 6 *Assume that the metric G has a third J-tensor $\mathbf{J}_3$ and denote by $G_3 = \sigma_3\mathbf{J}_3G$ (with $\sigma_3 = \det\mathbf{J}_3$) a new metric tensor equivalent to G. In the new independent variable t_3 defined through*

$$dt_3 = \frac{dt}{\sigma_3} \tag{20}$$

the bi-cofactor system (19) attains the form

$$\frac{d^2 q^i}{dt_3^2} + (\Gamma^{(3)})^i_{jk}\frac{dq^j}{dt_3}\frac{dq^k}{dt_3} = -\left(\left[\mathrm{cof}\left(\mathbf{J}_1\,\mathbf{J}_3^{-1}\right)\right]^{-1}\nabla^{(3)}V\right)^i$$

$$= -\left(\left[\mathrm{cof}\left(\mathbf{J}_2\,\mathbf{J}_3^{-1}\right)\right]^{-1}\nabla^{(3)}W\right)^i \tag{21}$$

where $(\Gamma^{(3)})^i_{jk}$ are Christoffel symbols of the metric G_3 and $\nabla^{(3)} = G_3 d$.

As before, both the tensor $\mathbf{J}_1\,\mathbf{J}_3^{-1}$ and $\mathbf{J}_2\,\mathbf{J}_3^{-1}$ are J_{G_3}-tensors so that G_3 is the underlying metric of the system (21).

In case that $\mathbf{J}_3 = \mathbf{J}_1$ the system (21) attains the potential-cofactor form

$$\frac{d^2 q^i}{d\bar{t}^2} + \bar{\Gamma}^i_{jk}\frac{dq^j}{d\bar{t}}\frac{dq^k}{d\bar{t}} = -\left(\overline{\nabla}V\right)^i = -\left(\left(\mathrm{cof}\,\overline{\mathbf{J}}\right)^{-1}\overline{\nabla}W\right)^i \tag{22}$$

with the affine parameter $d\bar{t} = dt/\bar{\sigma}$ and with $\overline{\mathbf{J}} = \mathbf{J}_2\,\mathbf{J}_1^{-1}$ being a $J_{\overline{G}}$-tensor for the new metric $\overline{G} = \bar{\sigma}\mathbf{J}_1 G$ with $\bar{\sigma} = \det(\mathbf{J}_1) = \sigma_1$.

If $\mathbf{J}_3 = \mathbf{J}_2$ then the system (21) attains the cofactor-potential form

$$\frac{d^2 q^i}{d\widetilde{t}^2} + \widetilde{\Gamma}^i_{jk}\frac{dq^j}{d\widetilde{t}}\frac{dq^k}{d\widetilde{t}} = -\left(\left(\mathrm{cof}\,\widetilde{\mathbf{J}}\right)^{-1}\widetilde{\nabla}V\right)^i = -\left(\widetilde{\nabla}W\right)^i \tag{23}$$

with the affine parameter $d\widetilde{t} = dt/\widetilde{\sigma}$ and with $\widetilde{\mathbf{J}} = \mathbf{J}_1\,\mathbf{J}_2^{-1} = \overline{\mathbf{J}}^{-1}$ being a $J_{\widetilde{G}}$-tensor for the new metric $\widetilde{G} = \widetilde{\sigma}\mathbf{J}_2 G$ with $\widetilde{\sigma} = \det(\mathbf{J}_2) = \sigma_2$.

Proposition 7 *The bi-cofactor system (19) has on T^*Q the following bi-quasihamiltonian representation:*

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{\sigma_1}\Pi_{nc}(\mathbf{J}_1)dH = \frac{1}{\sigma_2}\Pi_{nc}(\mathbf{J}_2)dF, \tag{24}$$

with two compatible noncanonical Poisson operators $\Pi_{nc}(\mathbf{J}_1)$ and $\Pi_{nc}(\mathbf{J}_2)$ given by

$$\Pi_{nc}(\mathbf{J}) = \begin{pmatrix} 0 & \mathbf{J} \\ -\mathbf{J}^T & \Omega \end{pmatrix}, \quad \Omega^i_j = \left(\frac{\partial J^k_i}{\partial q^j} - \frac{\partial J^k_j}{\partial q^i}\right)p_k$$

and with the Hamiltonians

$$H = \frac{1}{2}p^T(\mathrm{cof}\ \mathbf{J}_1)Gp + V(q), \quad F = \frac{1}{2}p^T(\mathrm{cof}\ \mathbf{J}_2)Gp + W(q).$$

The representation (24) follows directly from Proposition 5 applied independently to both cofactor representations of (19). The fact that the operators $\Pi_{nc}(\mathbf{J}_1)$ and $\Pi_{nc}(\mathbf{J}_2)$ are compatible (i.e. that any linear combination $\eta_1\Pi_{nc}(\mathbf{J}_1) + \eta_2\Pi_{nc}(\mathbf{J}_2)$ is Poisson) is shown below. In the particular case of potential-cofactor systems (22) and (23) this proposition yields their well-known quasi-bi-Hamiltonian representation [15], [16].

Theorem 8   *1. The system (24) has n constants of motion*

$$H_r = E_r + V_r(q) = \frac{1}{2}p^T K_r Gp + V_r(q), \qquad r = 1, ..., n, \tag{25}$$

(with $H = H_1$ and $F = H_n$) where K_r are $(1,1)$-Killing tensors (for the metric G) defined by

$$\mathrm{cof}(\mathbf{J}_2 + \xi\mathbf{J}_1) = \sum_{i=0}^{n-1} K_{n-i}\xi^i \tag{26}$$

(so that $K_1 = \mathrm{cof}\ \mathbf{J}_1$, $K_n = \mathrm{cof}\ \mathbf{J}_2$) and where the potentials V_r can be obtained from two equivalent formulas

$$\nabla V_r = \frac{1}{\sigma_1}K_r\mathbf{J}_1\nabla V_1 \quad or \ \ \nabla V_r = \frac{1}{\sigma_2}K_r\mathbf{J}_2\nabla V_n, \quad V = V_1, W = V_n. \tag{27}$$

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2. The constants H_r are in involution with respect to both operators $\Pi_{nc}(\mathbf{J}_1)$ and $\Pi_{nc}(\mathbf{J}_2)$:

$$\{H_r, H_s\}_{\Pi_{nc}(\mathbf{J}_1)} = \{H_r, H_s\}_{\Pi_{nc}(\mathbf{J}_2)} = 0 \quad \text{for all } \ r, s = 1, \dots, n.$$

To prove this theorem, we will first need

Proposition 9 *In the variables $(\bar{q}, \bar{p}, \bar{t})$ related with (q, p, t) through the map*

$$\bar{q} = q, \ \ \bar{p} = \left(\mathbf{J}_1^T\right)^{-1} p, \ \ d\bar{t} = \frac{dt}{\sigma_1} \tag{28}$$

the system (24) attains the quasi-bihamiltonian form

$$\frac{d}{d\bar{t}} \left(\begin{array}{c} \bar{q} \\ \bar{p} \end{array} \right) = \overline{\Pi}_c d\overline{H} = \frac{1}{\det(\overline{\mathbf{J}})} \overline{\Pi}_{nc}(\overline{\mathbf{J}}) d\overline{F}, \tag{29}$$

with $\overline{H} = H$ and $\overline{F} = F$ (as functions on T^*Q) and with

$$\overline{\Pi}_c = \Pi_{nc}(\mathbf{J}_1), \quad \overline{\Pi}_{nc}(\overline{\mathbf{J}}) = \Pi_{nc}(\mathbf{J}_2) \tag{30}$$

(as tensors on T^*Q). Moreover, the tensor $\overline{\Pi}_c$ is canonical in $(\bar{q}, \bar{p})$-variables. Similarly, in the variables $(\tilde{q}, \tilde{p}, \tilde{t})$ defined by

$$\tilde{q} = q, \ \ \tilde{p} = \left(\mathbf{J}_2^T\right)^{-1} p, \ \ d\tilde{t} = \frac{dt}{\sigma_2}. \tag{31}$$

(24) attains the form

$$\frac{d}{d\tilde{t}} \left(\begin{array}{c} \tilde{q} \\ \tilde{p} \end{array} \right) = \frac{1}{\det(\tilde{\mathbf{J}})} \widetilde{\Pi}_{nc}(\tilde{\mathbf{J}}) d\widetilde{H} = \widetilde{\Pi}_c d\widetilde{F}, \tag{32}$$

with $\widetilde{H} = H$ and $\widetilde{F} = F$ (as functions on T^*Q) and with

$$\widetilde{\Pi}_c = \Pi_{nc}(\mathbf{J}_2), \quad \widetilde{\Pi}_{nc}(\tilde{\mathbf{J}}) = \Pi_{nc}(\mathbf{J}_1) \tag{33}$$

(again considered as tensors on T^*Q) so that

$$\widetilde{\Pi}_c = \overline{\Pi}_{nc}(\overline{\mathbf{J}}), \ \overline{\Pi}_c = \widetilde{\Pi}_{nc}(\tilde{\mathbf{J}}).$$

Again, the tensor $\widetilde{\Pi}_c$ is canonical in $(\tilde{q}, \tilde{p})$-variables.

This proposition can be proved either by direct calculation or by observing that the underlying bi-cofactor system (19) has in the variables $(\bar{q}, \bar{t})$ the potential-cofactor form (22) and in the variables $(\tilde{q}, \tilde{t})$ the cofactor-potential form (23) and using arguments similar to those used in the proof of Proposition 5.

Proof. (of Theorem 8). By Proposition 9, the system (24) has in variables $(\bar{q}, \bar{p}, \bar{t})$ the form (29) so that it is a so called Benenti system and therefore (see [3]) $\overline{\Pi}_c = \Pi_{nc}(\mathbf{J}_1)$, $\overline{\Pi}_{nc}(\overline{\mathbf{J}}) = \Pi_{nc}(\mathbf{J}_2)$ are compatible and the system has n constants of motion of the form

$$\overline{H}_r = \overline{E}_r + \overline{V}_r(\bar{q}) = \frac{1}{2} \bar{p}^T \overline{K}_r \overline{G} \bar{p} + \overline{V}_r(\bar{q}), \qquad r = 1, ..., n, \tag{34}$$

with $\overline{G} = \sigma_1 \mathbf{J}_1 G$ and where the Killing tensors $\overline{K}_r$ of the metric $\overline{G}$ are determined by the expansion

$$\text{cof} \left(\overline{\mathbf{J}} + \xi I \right) = \sum_{i=0}^{n-1} \overline{K}_{n-i} \xi^i \tag{35}$$

(so that $\overline{K}_1 = I$, $K_n = \text{cof } \overline{\mathbf{J}}$) while $\overline{V}_r$ are separable potentials satisfying

$$\overline{K}_r \overline{\nabla} \overline{V}_1 = \overline{\nabla} \overline{V}_r. \tag{36}$$

On the other hand,

$$\mathrm{cof}\left(\overline{\mathbf{J}}+\xi I\right) = \mathrm{cof}\left(\mathbf{J}_2\mathbf{J}_1^{-1}+\xi I\right) = \mathrm{cof}(\mathbf{J}_1)^{-1}\,\mathrm{cof}\left(\mathbf{J}_2+\xi\mathbf{J}_1\right) = K_1^{-1}\sum_{i=0}^{n-1}K_{n-i}\xi^i,$$

so that by comparing with (35) we obtain

$$\overline{K}_i = K_1^{-1}K_i, \quad i = 1,\ldots,n \tag{37}$$

and

$$\overline{E}_r = \frac{1}{2}\overline{p}^T\overline{K}_r\overline{G}\overline{p} = \frac{1}{2}p^T\mathbf{J}_1^{-1}K_1^{-1}K_r\sigma_1\mathbf{J}_1G\left(\mathbf{J}^{-1}\right)^T p = E_r.$$

The last equality follows from $\mathbf{J}_1^{-1}K_1^{-1} = \sigma_1 I$ and $\mathbf{J}_1 G\left(\mathbf{J}^{-1}\right)^T = \mathbf{J}_1\mathbf{J}^{-1}G = G$, so that indeed $\overline{E}_r = E_r$ if we define K_i as in (26). Thus, if we put $V_r = \overline{V}_r$ we obtain that $\overline{H}_r = H_r$ (as functions on T^*Q). Now, substituting (37) into (36) we get

$$K_1^{-1}K_r\sigma_1 J_1\nabla V_1 = \sigma_1 J_1\nabla V_r \quad \text{or} \quad K_r J_1\nabla V_1 = K_1 J_1\nabla V_r$$

which yields the first formula in (27). Naturally, the functions H_r Poisson-commute with respect to both Poisson tensors $\Pi_{nc}(\mathbf{J}_i)$ in (24) since $\overline{H}_r = H_r$ Poisson-commute with respect to both Poisson tensors $\overline{\overline{\Pi}}_c, \overline{\overline{\Pi}}_{nc}(\overline{\mathbf{J}})$ in (29) and since these tensors are just $\Pi_{nc}(\mathbf{J}_i)$ written in the variables $(\overline{q},\overline{p})$, according to (30). That proves all the statements in Theorem 8 except the second formula in (27). Consider now the system (32). It is also a Benenti system so it also has n constants of motion of the form

$$\widetilde{H}_r = \widetilde{E}_r + \widetilde{V}_r(\widetilde{q}) = \frac{1}{2}\widetilde{p}^T\widetilde{K}_r\widetilde{G}\widetilde{p} + \widetilde{V}_r(\widetilde{q}), \qquad r = 1,\ldots,n, \tag{38}$$

(with $\widetilde{G} = \sigma_2\mathbf{J}_2G$ and with $\widetilde{H} = \widetilde{H}_n$, $\widetilde{F} = \widetilde{H}_1$) where the Killing tensors $\widetilde{K}_r$ of the metric $\widetilde{G}$ are determined by

$$\mathrm{cof}\left(\widetilde{\mathbf{J}}+\xi I\right) = \sum_{i=0}^{n-1}\widetilde{K}_{n-i}\xi^i$$

(so that $\widetilde{K}_1 = I$, $\widetilde{K}_n = \mathrm{cof}\,\widetilde{\mathbf{J}}$) while $\widetilde{V}_r$ are separable potentials satisfying

$$\widetilde{K}_r\widetilde{\nabla}V_1 = \widetilde{\nabla}\widetilde{V}_r. \tag{39}$$

Repeating the procedure above we obtain an equivalent proof of Theorem 8. However, this time we get

$$\begin{aligned}
\sum_{i=0}^{n-1}\widetilde{K}_{n-i}\xi^i &= \mathrm{cof}\left(\widetilde{\mathbf{J}}+\xi I\right) = \mathrm{cof}(\mathbf{J}_1\mathbf{J}_2^{-1}+\xi I) = K_n^{-1}\,\mathrm{cof}(\mathbf{J}_1+\xi\mathbf{J}_2) = \\
&= K_n^{-1}\xi^{n-1}\sum_{i=0}^{n-1}K_{n-i}\xi^{-i} = K_n^{-1}\sum_{i=0}^{n-1}K_{n-i}\xi^{n-i-1},
\end{aligned}$$

which gives

$$\widetilde{K}_i = K_n^{-1}K_{n-i+1}, \quad i = 1,\ldots,n. \tag{40}$$

This also yields $\widetilde{E}_i = E_{n-i+1}$ and thus $\widetilde{V}_i = V_{n-i+1}$, $\widetilde{H}_i = H_{n-i+1} = \overline{H}_{n-i+1}$ for all $i = 1,\ldots,n$. By transforming the formula (39) to (q,p)-coordinates (similarly as we did for bar-coordinates) we obtain the second formula in (27). Finally, the functions K_i defined by (26) must be $(1,1)$-Killing tensors for G since $\mathrm{cof}(\mathbf{J}_2+\xi\mathbf{J}_1)$ is a $(1,1)$-Killing tensor for any value of the parameter ξ and since Killing tensors of G constitute a vector space. ∎

Note also that the direct map between variables $(\overline{q},\overline{p},\overline{t})$ and $(\widetilde{q},\widetilde{p},\widetilde{t})$ is obtained by composing the map (28) with the map (31). It attains the form

$$\widetilde{q} = \overline{q}, \quad \widetilde{p} = \left(\overline{\mathbf{J}}^T\right)^{-1}\overline{p}, \quad d\widetilde{t} = \frac{d\overline{t}}{\det(\overline{\mathbf{J}})} \quad \text{or} \quad \overline{q} = \widetilde{q}, \quad \overline{p} = \left(\widetilde{\mathbf{J}}^T\right)^{-1}\widetilde{p}, \quad d\overline{t} = \frac{d\widetilde{t}}{\det(\widetilde{\mathbf{J}})}. \tag{41}$$

Further, by comparing (37) and (40) we obtain

$$\widetilde{K}_i = \overline{K}_n^{-1} \overline{K}_{n-i+1}.$$

In the remaining part of this chapter will shortly discuss how the two equivalent systems (29) and (32) can be embedded in quasi-bihamiltonian chains and discuss the relation between these chains.

Since the system (29) has n commuting with respect to both operators $\overline{\Pi}_c$ and $\overline{\Pi}_{nc}$ integrals of motion $\overline{H}_r$ it belongs to the set of n commuting Hamiltonian vector fields

$$\frac{d}{d\bar{t}_r} \left(\begin{array}{c} \bar{q} \\ \bar{p} \end{array} \right) = \overline{\Pi}_c \, d\overline{H}_r \equiv \overline{X}_r, \quad r = 1, ..., n, \tag{42}$$

(where $d\bar{t}_1 = d\bar{t} = dt/\det(\mathbf{J}_1)$) and the system (29) itself defines the first vector field $\overline{X}_1$. Similarly, since the system (32) has n commuting with respect to both $\widetilde{\Pi}_c$ and $\widetilde{\Pi}_{nc}$ integrals of motion $\widetilde{H}_r$ it belongs to the set of n commuting Hamiltonian vector fields

$$\frac{d}{d\tilde{t}_r} \left(\begin{array}{c} \tilde{q} \\ \tilde{p} \end{array} \right) = \widetilde{\Pi}_c \, d\widetilde{H}_r \equiv \widetilde{X}_r, \quad r = 1, ..., n, \tag{43}$$

(where $d\tilde{t}_1 = d\tilde{t} = dt/\det(\mathbf{J}_2)$ so that $d\tilde{t}_1 = d\bar{t}_1/\det(\overline{\mathbf{J}})$) and it also is the first vector field $\widetilde{X}_1$. By the above construction, the vector fields $\overline{X}_1$ and $\widetilde{X}_1$ are parallel

$$\widetilde{X}_1 = \det(\overline{\mathbf{J}}) \, \overline{X}_1 \quad \text{or} \quad \overline{X}_1 = \det(\widetilde{\mathbf{J}}) \, \widetilde{X}_1$$

which once again reflects the geodesic equivalence of the systems (22) and (23) on Q.

Moreover, vector fields (42) belong to the following quasi-bihamiltonian chain:

$$\begin{aligned} \overline{X}_1 &= \overline{\Pi}_c \, d\overline{H}_1 = \frac{1}{\bar{\rho}_n} \overline{\Pi}_{nc}(\overline{\mathbf{J}}) d\overline{H}_n \\ \overline{X}_r &= \overline{\Pi}_c \, d\overline{H}_r = \frac{\bar{\rho}_{r-1}}{\bar{\rho}_n} \overline{\Pi}_{nc}(\overline{\mathbf{J}}) d\overline{H}_n - \overline{\Pi}_{nc}(\overline{\mathbf{J}}) d\overline{H}_{r-1}, \quad r = 2, ..., n, \end{aligned}$$

where the functions $\bar{\rho}_r$ are defined through the polynomial expansion of $\det(\overline{\mathbf{J}} + \xi I)$:

$$\det(\overline{\mathbf{J}} + \xi I) = \sum_{i=0}^{n} \bar{\rho}_i \xi^{n-i}$$

(so that $\bar{\rho}_n = \det \overline{\mathbf{J}}$). Similarly, vector fields (43) belong to a similar quasi-bihamiltonian chain:

$$\begin{aligned} \widetilde{X}_1 &= \widetilde{\Pi}_c \, d\widetilde{H}_1 = \frac{1}{\tilde{\rho}_n} \widetilde{\Pi}_{nc}(\widetilde{\mathbf{J}}) d\overline{H}_n \\ \widetilde{X}_r &= \widetilde{\Pi}_c \, d\widetilde{H}_r = \frac{\tilde{\rho}_{r-1}}{\tilde{\rho}_n} \widetilde{\Pi}_{nc}(\widetilde{\mathbf{J}}) d\widetilde{H}_n - \widetilde{\Pi}_{nc}(\widetilde{\mathbf{J}}) d\widetilde{H}_{r-1}, \quad r = 2, ..., n, \end{aligned}$$

where $\tilde{\rho}_r$ are defined through

$$\det(\widetilde{\mathbf{J}} + \xi I) = \sum_{i=0}^{n} \tilde{\rho}_i \xi^{n-i}$$

(so that $\tilde{\rho}_n = \det \widetilde{\mathbf{J}}$). Since $\widetilde{\mathbf{J}} = \overline{\mathbf{J}}^{-1}$ we have that $\bar{\rho}_r$ and $\tilde{\rho}_r$ are related via

$$\tilde{\rho}_r = \frac{\bar{\rho}_{n-r}}{\bar{\rho}_n} \quad \text{or} \quad \bar{\rho}_r = \frac{\tilde{\rho}_{n-r}}{\tilde{\rho}_n}.$$

Comparing both chains we obtain that the vector fields $\overline{X}_r$ and $\widetilde{X}_r$ are related through

$$\widetilde{X}_1 = \bar{\rho}_n \overline{X}_1, \qquad \widetilde{X}_i = \bar{\rho}_{n-i+1} \overline{X}_1 - \overline{X}_{n-i+2}, \quad i = 2, \ldots, n.$$

4   Flat bi-cofactor systems.

Let us recall that a pseudoriemannian space is called space of constant curvature if the curvature tensor R_{ijkl} has the form

$$R_{ijkl} = K \left(g_{jl} g_{ik} - g_{jk} g_{il} \right) \tag{44}$$

for some scalar function K. By Bianchi identity it follows then that K is a constant, related to scalar (Ricci) curvature $\varkappa = R_{ik} g^{ik}$ through $\varkappa = K n (n - 1)$. Thus, for such spaces the condition $\varkappa = 0$ or $K = 0$ implies that the Riemann tensor R_{ijkl} is zero i.e. that the metric g is flat.

Suppose now that g is a metric of constant curvature and that $\overline{g}$ is another metric tensor obtained by deforming g through

$$\overline{G} = \sigma \mathbf{J} G \tag{45}$$

(with $\mathbf{J}$ being a J_G-tensor $\mathbf{J}$ and with $\sigma = \det \mathbf{J}$). Then, by the classical result of Beltrami [17] we know that $\overline{g}$ is also of constant curvature. Moreover, for two metrics g and $\overline{g}$ that are geodesically equivalent and of constant curvature their scalar curvatures $\varkappa$ and $\overline{\varkappa}$ are related by the formula

$$\overline{\varkappa} \, \overline{g}_{ij} = \varkappa g_{ij} - \nabla_i f_j + f_i f_j \tag{46}$$

(see [14] p. 293) where the covector f_i is defined as

$$f_i = \frac{1}{2(n+1)} \frac{\partial}{\partial q_i} \left(\ln \frac{\det \overline{g}}{\det g} \right). \tag{47}$$

A simple calculation shows that for our choice of $g, \overline{g}$ we have

$$f_i = -\frac{1}{n+1} \sigma_i \text{ where } \sigma_i = \frac{1}{\sigma} \frac{\partial \sigma}{\partial q_i}.$$

Substituting this into (46) and performing contraction with $\overline{G}$ we obtain

$$\overline{\varkappa} = \frac{\sigma}{n} \left[\varkappa \operatorname{tr} \mathbf{J} + \tfrac{1}{n+1} J^{ij} \left(\sigma_i \sigma_j + \tfrac{1}{n+1} \nabla_i \sigma_j \right) \right].$$

(the summation convention applies as usual). Thus, we see that if $\varkappa = 0$ then a sufficient condition for $\overline{\varkappa}$ to be zero is

$$J^{ij} \left(\sigma_i \sigma_j + \tfrac{1}{n+1} \nabla_i \sigma_j \right) = 0. \tag{48}$$

Let us now assume that the metric G of the system (19) is flat (i.e. $\varkappa = 0$) so that in some coordinate system (q^i) it assumes the form

$$G = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n) \text{ with } \varepsilon_i = \pm 1. \tag{49}$$

(note that then $g = G^{-1} = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ in this particular coordinate system too while $\Gamma^i_{jk} = 0$). Suppose now that we want to "deform" this system as in Proposition 6 by introducing the new independent variable $dt_3 = dt/\sigma$ where $\sigma = \det \mathbf{J}$ for some new J_G-tensor $\mathbf{J}$ but in such a way that the resulting equivalent metric $\overline{G} = \sigma \mathbf{J} G$ is also flat so that the geodesically equivalent system (21) is a flat Newton system (in this section we will use $\mathbf{J}$, σ and $\overline{G}$ instead of $\mathbf{J}_3$, $\det \mathbf{J}_3$ and G_3 to shorten the notation). A sufficient condition for doing this is to take $\mathbf{J}$ that satisfies (48). In the Cartesian (with respect to g) coordinates (q^i) the contravariant form of the tensor $\mathbf{J}$ is given by (6). However, by Theorem B.4.3 in [12] we know that $m = 0$ or else $\overline{\varkappa} \neq 0$. Thus, our aim is to find a more explicit form of the condition (48) for $\mathbf{J}$ given by (6) with $m = 0$.

Let us for the moment denote the $(2,0)$-form of $\mathbf{J}$ as given in (6) by $\mathbf{J}_c$ ($\mathbf{J}$-contravariant) so that $\mathbf{J}_c = \mathbf{J} G$ or $\mathbf{J} = \mathbf{J}_c g$. We have then

Theorem 10 *Assume that G is of the form (49) and that $\mathbf{J}$ is a J_G-tensor such that its contravariant form $\mathbf{J}_c$ is given by (6). Then for the metric $\overline{G} = \sigma \mathbf{J} G$ to be flat it is sufficient that $m = 0$ and*

$$\beta^T g \left(\operatorname{cof} \mathbf{J} \right) \beta = 0 \text{ or } \beta^T \left(\operatorname{cof} \mathbf{J}_c \right) \beta = 0. \tag{50}$$

Proof. Both conditions in (50) are equivalent since $\operatorname{cof}\mathbf{J}_c = \operatorname{cof}(\mathbf{J}G) = \operatorname{cof}G\,\operatorname{cof}\mathbf{J} = \det(G)\,g\,\operatorname{cof}\mathbf{J}$. We have to show that the condition (48) in our setting attains the form (50). Since $\mathbf{J}$ is torsionless it satisfies the identity

$$\sigma\frac{\partial\left(\operatorname{tr}\mathbf{J}\right)}{\partial q^i} = \mathbf{J}_i^h\frac{\partial\sigma}{\partial q^h},$$

or in the matrix form

$$\sigma\,d(\operatorname{tr}\mathbf{J}) = \mathbf{J}^T d\sigma. \tag{51}$$

Since $J_j^i = J^{ik}g_{kj} = \beta^i\varepsilon_j q^j + \beta^j\varepsilon_j q^i + \gamma^{ij}\varepsilon_j$ (no summation) we have $\operatorname{tr}(\mathbf{J}) = J_i^i = 2\beta^i\varepsilon_i q^i + \gamma^{ii}\varepsilon_i$ so that $d(\operatorname{tr}\mathbf{J}) = (\varepsilon_1\beta^1,\dots,\varepsilon_n\beta^n)^T$. Thus, (51) reads

$$d\sigma = 2\left(\operatorname{cof}\mathbf{J}\right)^T g\beta = 2g\left(\operatorname{cof}\mathbf{J}\right)\beta. \tag{52}$$

Therefore

$$\sigma_i J^{ij}\sigma_j = \frac{1}{\sigma^2}\frac{\partial\sigma}{\partial q^i}J^{ij}\frac{\partial\sigma}{\partial q^j} = \frac{1}{\sigma^2}(d\sigma)^T\mathbf{J}\,d\sigma = \frac{4}{\sigma^2}\beta^T g\left(\operatorname{cof}\mathbf{J}\right)\beta. \tag{53}$$

Further

$$J^{ij}\nabla_i\sigma_j = J^{ij}\frac{\partial}{\partial q^i}\left(\frac{1}{\sigma}\frac{\partial\sigma}{\partial q^j}\right) = -\sigma_i J^{ij}\sigma_j + \frac{1}{\sigma}J^{ij}\frac{\partial^2\sigma}{\partial q^i\partial q^j}.$$

But, using (52) twice and (49) we obtain

$$\begin{aligned}
J^{ij}\frac{\partial^2\sigma}{\partial q^i\partial q^j} &= J^{ij}\frac{\partial}{\partial q^i}\left(\frac{\partial\sigma}{\partial q^j}\right) = 2J^{ij}g_{jk}\frac{\partial}{\partial q^i}\left(\operatorname{cof}\mathbf{J}\right)_s^k\beta^s = 2J_k^i\frac{\partial}{\partial q^i}\left(\operatorname{cof}\mathbf{J}\right)_s^k\beta^s = \\
&= 2\left[\frac{\partial}{\partial q^i}\left(J_k^i\left(\operatorname{cof}\mathbf{J}\right)_s^k\right) - \left(\operatorname{cof}\mathbf{J}\right)_s^k\frac{\partial}{\partial q^i}J_k^i\right]\beta^s = 2\frac{\partial\sigma}{\partial q^i}\beta^i - 2(n+1)\beta^k\varepsilon_k\left(\operatorname{cof}\mathbf{J}\right)_s^k\beta^s = \\
&= 4\beta^T g\left(\operatorname{cof}\mathbf{J}\right)\beta - 2(n+1)\beta^T g\left(\operatorname{cof}\mathbf{J}\right)\beta.
\end{aligned}$$

so that

$$J^{ij}\frac{\partial^2\sigma}{\partial q^i\partial q^j} = 2(1-n)\beta^T g(\operatorname{cof}\mathbf{J})\beta.$$

Thus,

$$J^{ij}\nabla_i\sigma_j = -\frac{2}{\sigma}(1+n)\beta^T g(\operatorname{cof}\mathbf{J})\beta. \tag{54}$$

Plugging (53) and (54) into (48) we immediately obtain (50). ∎

Therefore, we have showed that for any flat bi-cofactor system (19) there exists a multi-parameter family (with $\frac{1}{2}n(n+3)-1$ parameters) of geodesically equivalent (but algebraically very different) flat bi-cofactor systems.

Remark 11 *The condition (50) can be written as*

$$\beta^T g\left(\operatorname{cof}\gamma g\right)\beta = 0\ \text{ or }\ \beta^T\left(\operatorname{cof}\gamma\right)\beta = 0. \tag{55}$$

5 Separation curves for bi-cofactor systems

A system of n algebraic equations of the form

$$\varphi_i(\lambda^i,\mu_i;a_1,\dots,a_n) = 0,\qquad i = 1,\dots,n,\quad \det\left[\frac{\partial\varphi_i}{\partial a_j}\right] \neq 0, \tag{56}$$

each containing only one pair (λ^i,μ_i) of coordinates (λ,μ) on T^*Q (and with real coefficients a_i) is called *separation relations*. The condition in (56) means that we can solve the equations (56) with respect to a_i obtaining n independent functions on T^*Q of the form $a_i = H_i(\lambda,\mu)$, $i = 1,\dots,n$. If the functions $W_i(\lambda^i,a)$ are solutions of a system of n decoupled ODE's

$$\varphi_i\left(\lambda^i,\mu_i = \frac{dW_i(\lambda^i,a)}{d\lambda^i},a_1,\dots,a_n\right) = 0,\ \ i = 1,\dots,n, \tag{57}$$

then the function $W(\lambda, a) = \sum_{i=1}^{n} W_i(\lambda^i, a)$ is a solution of all the equations (57) and simultaneously it is an additively separable solution of all Hamilton-Jacobi equations

$$H_i\left(\lambda^1, ..., \lambda^n, \frac{\partial W}{\partial \lambda^1}, ..., \frac{\partial W}{\partial \lambda^n}\right) = a_i, \qquad i = 1, ..., n. \tag{58}$$

simply because solving (56) to the form $a_i = H_i(\lambda, \mu)$ is a purely algebraic operation. The Hamiltonians H_i Poisson-commute by the classical theorem of Jacobi. The function $W(\lambda, a)$ is a generating function for the canonical transformation $(\lambda, \mu) \rightarrow (b, a)$ to the new set of coordinates that simultaneously linearize all the Hamiltonian equations

$$u_{t_i} = \Pi_c \, dH_i = X_{H_i}, \qquad i = 1, ..., n. \tag{59}$$

The coordinates (λ, μ) are thus called the *separation coordinates* for the Hamiltonian systems (59).

In the case that the relations (56) are affine in a_i the obtained systems belong to the well-known class of (generalized) Stäckel separable systems.

Let us now consider a special subclass of Stäckel systems given by the following separation relations:

$$H_1(\lambda^i)^{n-1} - H_2(\lambda^i)^{n-2} + ... + (-1)^{n-1} H_n = \frac{1}{2} f_i(\lambda^i)\mu_i^2 + \gamma_i(\lambda^i), \quad i = 1, ..., n, \tag{60}$$

where f_i and γ_i are smooth functions. Such systems are known as Benenti systems. In the particular case that $f_i(\lambda^i) = f(\lambda^i)$ and $\gamma_i(\lambda^i) = \gamma(\lambda^i)$, separation relations (60) are given by n copies of the so called separation curve

$$H_1\lambda^{n-1} - H_2\lambda^{n-2} + ... + (-1)^{n-1} H_n = \frac{1}{2} f(\lambda)\mu^2 + \gamma(\lambda). \tag{61}$$

so that now λ and $\mu \in \mathbf{R}$. By solving the system of n copies of this relation (with i-th copy containing variables labelled (λ_i, μ_i)) with respect to H_i we find that the Hamiltonians H_i attain the form

$$H_r = E_r + V_r(\lambda) = \frac{1}{2}\mu^T K_r G\mu + V_r(\lambda), \qquad r = 1, ..., n,$$

(cf (25)) with the metric tensor

$$G = \text{diag}\left(\frac{f(\lambda^1)}{\Delta_1}, ..., \frac{f(\lambda^n)}{\Delta_n}\right)$$

where $\Delta_i = \prod_{j \neq i}(\lambda^i - \lambda^j)$ while the $(1,1)$-tensors K_r are generated by the expansion

$$\text{cof}(\mathbf{J} + \xi I) = \sum_{i=0}^{n-1} K_{n-i}\xi^i$$

with the J_G-tensor $\mathbf{J} = \text{diag}(\lambda_1, ..., \lambda_n)$. Thus [19]:

$$K_r = \text{diag}\left(\frac{\partial \rho_r}{\partial \lambda^1}, ..., \frac{\partial \rho_r}{\partial \lambda^n}\right)$$

where the functions ρ_r can be obtained from

$$\det(\mathbf{J} + \xi I) = \sum_{i=0}^{n} \rho_i \xi^{n-i}.$$

Coordinate-free expression for K_r is as follows [18]

$$K_{r+1} = \rho_r I - \mathbf{J} K_r, \ r = 0, 1, ..., n-1, \ \rho_0 = 1, \ K_0 = 0,$$

or alternatively

$$K_r = \sum_{k=0}^{r-1} \rho_k \, (-\mathbf{J})^{r-1-k}, \qquad r = 1, ..., n.$$

It is important to stress that in case that eigenvalues of J are not simple the obtained tensors K_r will not be independent and thus will not generate an integrable system (see also below).

For a particular choice $\gamma(\lambda^i) = (\lambda^i)^k$, $k \in \mathbf{Z}$ in the separation curve (61) we obtain a family of separable potentials that can be constructed recursively by [19]

$$V_r^{(k+1)} = \rho_r V_1^{(k)} - V_{r+1}^{(k)} \text{ with } V_r^{(0)} = (-1)^{n-1} \delta_{rn}. \tag{62}$$

This recursion can be reversed

$$V_r^{(k-1)} = \frac{\rho_{r-1}}{\rho_n} V_n^{(k)} - V_{r-1}^{(k)}. \tag{63}$$

In both cases we put $V_r^{(k)} = 0$ for $r < 0$ or $r > n$. The above recursion can be written in a matrix form as

$$V^{(k)}(\lambda) = R^k(\lambda) V^{(0)}, \quad k \in Z, \tag{64}$$

where $V^{(k)}(\lambda) = (V_1^{(k)}(\lambda), ..., V_n^{(k)}(\lambda))^T$, $V^{(0)} = (0, ..., 0, (-1)^{n-1})^T$ and

$$R = \begin{pmatrix} \rho_1(\lambda) & -1 & 0 & \cdots & 0 \\ \rho_2(\lambda) & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \rho_{n-1}(\lambda) & 0 & 0 & \cdots & -1 \\ \rho_n(\lambda) & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{65}$$

This recursion is equivalent to (62) and (63) and is invariant with respect to any point change of variables on Q as R in (65) is expressed by coefficients of the characteristic polynomial of $\mathbf{J}$. The first nontrivial potentials in the positive hierarchy are $V_r^{(n)}(\lambda) = \rho_r(\lambda)$, while for the negative hierarchy $V_r^{(-1)} = \rho_{r-1}(\lambda)/\rho_n(\lambda)$.

Let us now once again consider our systems (22) and (23) and their Hamiltonian formulations (42) and (43) respectively. From now on we will additionally assume that the tensor $\overline{\mathbf{J}}$ (and hence $\widetilde{\mathbf{J}}$) has all its eigenvalues real and *simple* (i.e. both are so called L-tensors [12]). Then the tensors $\overline{K}_r$ (and $\widetilde{K}_r$ likewise) are independent and thus (42) and (43) are integrable. Moreover, both systems belong to the class of separable (in the sense of Hamilton-Jacobi theory) systems called Benenti systems. It is known that all Hamiltonian flows in (42) are separable in variables $(\overline{\lambda}, \overline{\mu})$ where the new coordinates $\overline{\lambda}^i$ are obtained from the characteristic equation of $\overline{\mathbf{J}}$:

$$\det(\overline{\mathbf{J}} + \overline{\lambda} I) = 0 \tag{66}$$

(i.e. are (signed) eigenvalues of $\overline{\mathbf{J}}$) while the corresponding momenta $\overline{\mu}^i$ are obtained from $\overline{\mu} = (\Phi'^{-1})^T \overline{p}$ where Φ' is the Jacobi matrix of the map $\Phi : q \to \overline{\lambda}$ given by (66). Similarly, all the flows in (43) are separable in variables $(\widetilde{\lambda}, \widetilde{\mu})$ where $\widetilde{\lambda}^i$ are obtained from

$$\det(\widetilde{\mathbf{J}} + \widetilde{\lambda} I) = 0 \tag{67}$$

with the corresponding momenta $\widetilde{\mu}^i$ obtained by $\widetilde{\mu} = (\Psi'^{-1})^T \widetilde{p}$ where Ψ' is the Jacobi matrix of the map $\Psi : q \to \widetilde{\lambda}$ given by (67).

Theorem 12 *The separation variables $(\overline{\lambda}, \overline{\mu})$ of (42) and the separation variables $(\widetilde{\lambda}, \widetilde{\mu})$ of (43) are related by the transformation*

$$\widetilde{\lambda}^i = \frac{1}{\overline{\lambda}^i}, \quad \widetilde{\mu}_i = -\overline{\lambda}^i \overline{\mu}_i \quad \text{(no summation)}, \quad i = 1, \ldots, n. \tag{68}$$

Proof. By comparing (66) with (67) we obtain that $\widetilde{\lambda}^i = \frac{1}{\overline{\lambda}^i}$. The map between momenta $\widetilde{\mu}$ and $\overline{\mu}$ can be found in the following way. We know (cf. (28) and (31)) that $\overline{p} = (\mathbf{J}_1^T)^{-1} p$ and $\widetilde{p} = (\mathbf{J}_2^T)^{-1} p$ which yields $\widetilde{p} = (\overline{\mathbf{J}}^T)^{-1} \overline{p}$. Thus

$$\widetilde{\mu} = (\Psi'^{-1})^T \widetilde{p} = (\Psi'^{-1})^T (\overline{\mathbf{J}}^T)^{-1} \Phi'^T \overline{\mu} = \Phi' \overline{\mathbf{J}}^{-1} (\Psi'^{-1})^T \overline{\mu}.$$

Since $\overline{\lambda}^i = 1/\widetilde{\lambda}^i$ we see that $\Psi' = \Theta\Phi'$ where

$$\Theta = -\operatorname{diag}\left(\frac{1}{(\overline{\lambda}^1)^2}, \ldots, \frac{1}{(\overline{\lambda}^n)^2}\right)$$

so that $\Psi'^{-1} = -\Phi'^{-1}\operatorname{diag}\left((\overline{\lambda}^1)^2, \ldots, (\overline{\lambda}^n)^2\right)$. Inserting it in the above formula yields

$$\widetilde{\mu} = -\left(\Phi'\overline{\mathbf{J}}^{-1}\Phi'^{-1}\operatorname{diag}((\overline{\lambda}^1)^2, \ldots, (\overline{\lambda}^n)^2)\right)^T \overline{\mu}.$$

But $\Phi'\overline{\mathbf{J}}^{-1}\Phi'^{-1} = \operatorname{diag}(1/\overline{\lambda}^1, \ldots, 1/\overline{\lambda}^n)$ since it is the inverse of the L-tensor $\overline{\mathbf{J}}$ written in its separation coordinates $\overline{\lambda}$. Inserting it into the above formula we get the map between momenta as in (68). ∎

According to the remarks above, the Benenti system (29) in variables $(\overline{\lambda}, \overline{\mu})$ has the separation curve

$$\overline{H}_1\overline{\lambda}^{n-1} - \overline{H}_2\overline{\lambda}^{n-2} + \ldots + (-1)^{n-1}\overline{H}_n = \frac{1}{2}\overline{f}(\overline{\lambda})\overline{\mu}^2 + \overline{\gamma}(\overline{\lambda}). \tag{69}$$

Similarly, the separation curve for the Benenti system (32) is

$$\widetilde{H}_1\widetilde{\lambda}^{n-1} - \widetilde{H}_2\widetilde{\lambda}^{n-2} + \ldots + (-1)^{n-1}\widetilde{H}_n = \frac{1}{2}\widetilde{f}(\widetilde{\lambda})\widetilde{\mu}^2 + \widetilde{\gamma}(\widetilde{\lambda}). \tag{70}$$

Applying the map (68) to the separation curve (70), using that $\overline{H}_r = \widetilde{H}_{n-r+1}$ and comparing the result with (69) we obtain

$$\overline{H}_1\overline{\lambda}^{n-1} - \overline{H}_2\overline{\lambda}^{n-2} + \ldots + (-1)^{n-1}\overline{H}_n = \frac{(-1)^{n-1}}{2}\widetilde{f}\left(\overline{\lambda}^{-1}\right)\overline{\lambda}^{n+1}\overline{\mu}_i^2$$

$$+(-1)^{n-1}\widetilde{\gamma}(\overline{\lambda}^{-1})\overline{\lambda}^{n-1}.$$

Corollary 13 *If the functions $\overline{f}, \widetilde{f}$ and $\overline{\gamma}, \widetilde{\gamma}$ satisfy the conditions:*

$$\overline{f}(\xi) = (-1)^{n-1}\widetilde{f}\left(\xi^{-1}\right)\xi^{n+1}, \quad \overline{\gamma}(\xi) = (-1)^{n-1}\widetilde{\gamma}(\xi^{-1})\xi^{n-1}, \quad \xi \in \mathbf{R} \tag{71}$$

then the separation curves (69) and (70) generate two geodesically equivalent systems of Benenti type para-metrized by two different evolution parameters $\overline{t}$ and $\widetilde{t}$ such that $d\widetilde{t} = d\overline{t}/\det(\overline{\sigma})$, where $\overline{\sigma} = \rho_n = \prod_{i=1}^{n}\overline{\lambda}^i$. The corresponding families of separable potentials (64) for both systems are related by

$$\overline{V}^{(k)} = (-1)^{n-1}\widetilde{V}^{(n-k-1)} \quad or \quad \widetilde{V}^{(k)} = (-1)^{n-1}\overline{V}^{(n-k-1)} \quad \text{for all } r \in \mathbf{Z}.$$

It is known that the metric

$$\overline{G} = \operatorname{diag}\left(\frac{\overline{f}(\overline{\lambda}^1)}{\overline{\Delta}_1}, \ldots, \frac{\overline{f}(\overline{\lambda}^n)}{\overline{\Delta}_n}\right)$$

is of constant curvature if and only if $\overline{f}(\overline{\lambda}) = \sum_{k=0}^{n+1} c_k \overline{\lambda}^k$ for some constants c_k. From (71) it follows immediately that the equivalent metric

$$\widetilde{G} = \operatorname{diag}\left(\frac{\widetilde{f}(\widetilde{\lambda}^1)}{\widetilde{\Delta}_1}, \ldots, \frac{\widetilde{f}(\widetilde{\lambda}^n)}{\widetilde{\Delta}_n}\right)$$

is also of constant curvature, as in this case

$$\widetilde{f}(\widetilde{\lambda}) = (-1)^{n-1}\overline{f}(\widetilde{\lambda}^{-1})\widetilde{\lambda}^{n+1} = (-1)^{n-1}\sum_{k=0}^{n+1} c_k \widetilde{\lambda}^{n-k+1} = (-1)^{n-1}\sum_{k=0}^{n+1} c_{n-k+1}\widetilde{\lambda}^k.$$

6 Example: flat bi-cofactor systems geodesically equivalent to Henon-Heiles system

Let us illustrate the ideas of this paper on the example of the integrable case of the Henon-Heiles system. It has the potential-cofactor form (22):

$$\frac{d^2\overline{q}}{d\overline{t}^2} = -\overline{\nabla}V = -\left(\operatorname{cof}\overline{\mathbf{J}}\right)^{-1}\overline{\nabla}W = -\left(\begin{array}{c} 3(\overline{q}^1)^2 + \frac{1}{2}\left(\overline{q}^2\right)^2 \\ \overline{q}^1\overline{q}^2 \end{array}\right) \tag{72}$$

with $\overline{G} = I$ (so that $\Gamma^i_{jk} = 0$ and coordinates $\overline{q}$ are Euclidean) and with the $J_{\overline{G}}$-tensor $\overline{\mathbf{J}}$ of the form (6)

$$\overline{\mathbf{J}} = \left(\begin{array}{cc} -\overline{q}^1 & -\frac{1}{2}\overline{q}^2 \\ -\frac{1}{2}\overline{q}^2 & 0 \end{array}\right).$$

The potentials V and W are

$$V(\overline{q}) = \left(\overline{q}^1\right)^3 + \frac{1}{2}\overline{q}^1\left(\overline{q}^2\right)^2, \quad W(\overline{q}) = \frac{1}{4}\left(\overline{q}^1\overline{q}^2\right)^2 + \frac{1}{16}\left(\overline{q}^2\right)^4.$$

The system (72) has the quasi-bihamiltonian representation (29)

$$\frac{d}{d\overline{t}}\left(\begin{array}{c} \overline{q} \\ \overline{p} \end{array}\right) = \overline{\Pi}_c d\overline{H} = \frac{1}{\det(\overline{\mathbf{J}})}\overline{\Pi}_{nc}(\overline{\mathbf{J}})d\overline{F}, \tag{73}$$

with Hamiltonians $\overline{H}_r$ of the form (34). Explicitly:

$$\overline{H}_1 = \overline{H} = \frac{1}{2}\left(\overline{p}_1\right)^2 + \frac{1}{2}\left(\overline{p}_2\right)^2 + V(\overline{q}), \quad \overline{H}_2 = \overline{F} = \frac{1}{2}\overline{q}^2\overline{p}_1\overline{p}_2 - \frac{1}{2}\overline{q}^1\left(\overline{p}_2\right)^2 + W(\overline{q}).$$

The non-canonical Poisson operator $\overline{\Pi}_{nc}$ reads explicitly as

$$\overline{\Pi}_{nc}(\overline{\mathbf{J}}) = \left(\begin{array}{cc} 0 & \overline{\mathbf{J}} \\ -\overline{\mathbf{J}}^T & \overline{\Omega} \end{array}\right) \quad \text{with } \overline{\Omega} = \left(\begin{array}{cc} 0 & -\frac{1}{2}\overline{p}_2 \\ \frac{1}{2}\overline{p}_2 & 0 \end{array}\right).$$

The system (73) separates in variables $(\overline{\lambda}, \overline{\mu})$ that can be found from the characteristic equation (66) and are given by

$$\overline{q}^1 = -(\overline{\lambda}^1 + \overline{\lambda}^2), \quad \overline{q}^2 = 2\sqrt{-\overline{\lambda}^1\overline{\lambda}^2}$$

$$\overline{p}_1 = -\left(\frac{\overline{\lambda}^1\overline{\mu}_1 - \overline{\lambda}^2\overline{\mu}_2}{\overline{\lambda}^1 - \overline{\lambda}^2}\right), \quad \overline{p}_2 = \sqrt{-\overline{\lambda}^1\overline{\lambda}^2}\left(\frac{\overline{\mu}_1 - \overline{\mu}_2}{\overline{\lambda}^1 - \overline{\lambda}^2}\right)$$

while the separation curve (69) generating Hamiltonians $\overline{H}_r$ is

$$\overline{H}_1\overline{\lambda} - \overline{H}_2 = \frac{1}{2}\overline{\lambda}\overline{\mu}^2 - \overline{\lambda}^4. \tag{74}$$

Let us now take another, arbitrary $J_{\overline{G}}$-tensor $\mathbf{J}_3$. Since the metric $\overline{G} = I$ in variables $(\overline{q}^1, \overline{q}^2)$ the most general form of $\mathbf{J}_3$ is (6) which reads now as

$$\mathbf{J}_3 = \left(\begin{array}{cc} m\left(\overline{q}^1\right)^2 + 2\beta_1\overline{q}^1 + \gamma^{11} & m\overline{q}^1\overline{q}^2 + \beta_1\overline{q}^2 + \beta_2\overline{q}^1 + \gamma^{12} \\ m\overline{q}^1\overline{q}^2 + \beta_1\overline{q}^2 + \beta_2\overline{q}^1 + \gamma^{12} & m\left(\overline{q}^2\right)^2 + 2\beta_2\overline{q}^2 + \gamma^{22} \end{array}\right) \tag{75}$$

with 6 arbitrary constants $m, \beta_i, \gamma^{ij} = \gamma^{ji}$. Using Proposition 6 (with $\mathbf{J}_1 = I$ and $\mathbf{J}_2 = \overline{\mathbf{J}}$) we see that in a new independent variable defined by

$$dt_3 = \frac{dt}{\det(\mathbf{J}_3)}$$

our potential-cofactor system (72) attains the bi-cofactor form (21) with $(\Gamma^{(3)})^i_{jk}$ being Christoffel symbols of the metric $G_3 = (\det \mathbf{J}_3)\, \mathbf{J}_3 G$. They can be obtained from (10) which reads now (since $\Gamma^i_{jk} = 0$)

$$(\Gamma^{(3)})^i_{jk} = -\frac{1}{2\det \mathbf{J}_3}\left(\delta^i_j \frac{\partial(\det \mathbf{J}_3)}{\partial \bar{q}_k} + \delta^i_k \frac{\partial(\det \mathbf{J}_3)}{\partial \bar{q}_j}\right).$$

It is important to stress that for all the choices of $\mathbf{J}_3$ the obtained system has on Q exactly the same trajectories as Henon-Heiles system, only traversed with different speed. Moreover, the metric G_3 is of constant curvature since it is geodesically equivalent to the flat metric $\overline{G} = I$ [17].

Among all possible choices of the deforming tensor $\mathbf{J}_3$ there is only one that leads to a new potential-cofactor system, namely $\mathbf{J}_3 = \overline{\mathbf{J}}$. This choice leads to cofactor-potential system (23) with $\widetilde{\mathbf{J}} = \overline{\mathbf{J}}^{-1}$ and with the metric $\widetilde{G} = (\det \overline{\mathbf{J}})\, \overline{\mathbf{J}}\, \overline{G}$. The metric $\widetilde{G}$ is flat since the deforming tensor $\mathbf{J}_3 = \overline{\mathbf{J}}$ satisfies the conditions (50) and (55). Explicitly, we have

$$\widetilde{\mathbf{J}} = \overline{\mathbf{J}}^{-1} = \frac{4}{(\widetilde{q}^2)^2}\begin{pmatrix} 0 & -\frac{1}{2}\widetilde{q}^2 \\ -\frac{1}{2}\widetilde{q}^2 & \widetilde{q}^1 \end{pmatrix}, \quad \widetilde{G} = \frac{1}{4}(\widetilde{q}^2)^2\begin{pmatrix} \widetilde{q}^1 & \frac{1}{2}\widetilde{q}^2 \\ \frac{1}{2}\widetilde{q}^2 & 0 \end{pmatrix}$$

(where of course $\bar{q}^i = \widetilde{q}^i$). This system has the quasi-bihamiltonian form (32) with the Hamiltonians as in (38) ($\widetilde{H}_1 = \overline{H}_2$ and $\widetilde{H}_2 = \overline{H}_1$) where the new momenta $\widetilde{p}$ are related with the old momenta through the map (41) and read explicitly as

$$\overline{p}_1 = -\widetilde{q}^1 \widetilde{p}_1 - \frac{1}{2}\widetilde{q}^2 \widetilde{p}_2, \quad \overline{p}_2 = -\frac{1}{2}\widetilde{q}^2 \widetilde{p}_1.$$

Our new system separates in variables $(\widetilde{\lambda}, \widetilde{\mu})$ that can be found from the characteristic equation (67) and are given by

$$\widetilde{q}^1 = -\left(\frac{1}{\widetilde{\lambda}^1} + \frac{1}{\widetilde{\lambda}^2}\right), \quad \widetilde{q}^2 = \frac{2}{\sqrt{-\widetilde{\lambda}^1 \widetilde{\lambda}^2}}$$

$$\widetilde{p}_1 = -\frac{\widetilde{\lambda}^1 \widetilde{\lambda}^2}{\widetilde{\lambda}^1 - \widetilde{\lambda}^2}\left(\widetilde{\lambda}^1 \widetilde{\mu}_1 - \widetilde{\lambda}^2 \widetilde{\mu}_2\right), \quad \widetilde{p}_2 = -\frac{\sqrt{-\widetilde{\lambda}^1 \widetilde{\lambda}^2}}{\widetilde{\lambda}^1 - \widetilde{\lambda}^2}\left(\left(\widetilde{\lambda}^1\right)^2 \widetilde{\mu}_1 - \left(\widetilde{\lambda}^2\right)^2 \widetilde{\mu}_2\right).$$

Our system can be obtained from the separation curve of the form (70) that explicitly reads as

$$\widetilde{H}_1 \widetilde{\lambda} - \widetilde{H}_2 = -\frac{1}{2}\widetilde{\lambda}^2 \widetilde{\mu}^2 + \widetilde{\lambda}^{-3}$$

and can also be obtained from the separation curve (74) by the transformation (71). Let us now introduce a new coordinates (r^1, r^2) on Q defined through

$$\widetilde{q}^1 = -2\frac{r^1}{r^2}, \quad \widetilde{q}^2 = \frac{4}{r^2}, \tag{76}$$

(see [20]). In (r^1, r^2) the metric $\widetilde{G}$ attains the antidiagonal form

$$\widetilde{G}(r) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{77}$$

(so that (r^1, r^2) are flat coordinates for $\widetilde{G}$ and $(\widetilde{\Gamma}(r))^i_{jk} = 0$) while the $J_{\widetilde{G}}$-tensor $\widetilde{\mathbf{J}}$ becomes

$$\widetilde{\mathbf{J}}(r) = \frac{1}{4}\begin{pmatrix} r^1 r^2 & (r^1)^2 + 4 \\ (r^2)^2 & r^1 r^2 \end{pmatrix}.$$

Our cofactor-potential system (geodesically equivalent to (72)) attains in variables (r^1, r^2) the flat Newton form

$$\frac{d^2}{d\widetilde{t}^2} \begin{pmatrix} \widetilde{r}^1 \\ \widetilde{r}^2 \end{pmatrix} = -\left(\operatorname{cof} \widetilde{\mathbf{J}}(r) \right)^{-1} \widetilde{\nabla} \widetilde{V}_2 = -\widetilde{\nabla} \widetilde{V}_1 = \left(\frac{2}{r^2} \right)^5 \begin{pmatrix} 2\left((r^1)^2 + 1 \right) \\ -r^1 r^2 \end{pmatrix} \tag{78}$$

with potentials

$$\widetilde{V}_1(r) = V(r) = 16 \frac{\left((r^1)^2 + 1 \right)}{(r^2)^4}, \quad \widetilde{V}_2(r) = W(r) = -8 \frac{\left((r^1)^2 + 2 \right) r^1}{(r^2)^3},$$

while the corresponding Hamiltonians are

$$
\begin{aligned}
\widetilde{H}_1(r,s) &= \overline{H}_2(r,s) = s_1 s_2 + \widetilde{V}_1(r), \\
\widetilde{H}_2(r,s) &= \overline{H}_1(r,s) = \left(\frac{1}{8} (r^1)^2 + \frac{1}{2} \right) s_1^2 - \frac{1}{4} r^1 r^2 s_1 s_2 + \frac{1}{8} (r^2)^2 s_2^2 + \widetilde{V}_2(r),
\end{aligned}
$$

where the momenta (s_1, s_2) are obtained from the point transformation (76) and are

$$\widetilde{p}_1 = -\frac{2}{r^2} s_1, \quad \widetilde{p}_2 = 2 \frac{r^1}{(r^2)^2} s_1 - \frac{4}{(r^2)^2} s_2.$$

Let us finally make another choice of the deforming tensor $\mathbf{J}_3$, namely $m = 0$, $\gamma^{22} = a$, $\beta^2 = b \neq 0$ and $\beta^1 = \gamma^{11} = \gamma^{12} = 0$ so that

$$\mathbf{J}_3 = \begin{pmatrix} 0 & b\overline{q}^1 \\ b\overline{q}^1 & 2b\overline{q}^2 + a \end{pmatrix}, \quad G_3 = (\det \mathbf{J}_3) \mathbf{J}_3 \overline{G} = \begin{pmatrix} 0 & -b^3 (\overline{q}^1)^3 \\ -b^3 (\overline{q}^1)^3 & -b^2 (\overline{q}^1)^2 (2b\overline{q}^2 + a) \end{pmatrix}.$$

The metric G_3 is again flat since the deforming tensor $\mathbf{J}_3$ satisfies the conditions (50) and (55). The respective J_{G_3}-tensors for the related bi-cofactor system (19) are

$$\mathbf{J}_1 = \mathbf{J}_3^{-1} = \frac{1}{b\overline{q}^1} \begin{pmatrix} -\frac{2b\overline{q}^2 + a}{b\overline{q}^1} & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{J}_2 = \overline{\mathbf{J}} \mathbf{J}_3^{-1} = \frac{1}{b\overline{q}^1} \begin{pmatrix} \frac{2b\overline{q}^2 + a}{b} - \frac{1}{2}\overline{q}^2 & -\overline{q}^1 \\ \frac{1}{2} \overline{q}^2 \frac{(2b\overline{q}^2 + a)}{b\overline{q}^1} & -\frac{1}{2}\overline{q}^2 \end{pmatrix}.$$

Let us now perform a parameter-dependent change of variables to the coordinates (x^1, x^2) on Q defined through

$$x^1 = \frac{1}{2} \frac{2b\overline{q}^2 + a}{b^2 \overline{q}^1}, \quad x^2 = \frac{1}{b^2 \overline{q}^1}.$$

In (x^1, x^2) the metric G_3 attains the antidiagonal form (77), so that (x^1, x^2) are flat coordinates for G_3 and the J_{G_3}-tensors $\mathbf{J}$ become

$$\mathbf{J}_1 = -b^2 \begin{pmatrix} x^1 x^2 & (x^1)^2 + \frac{1}{b^2} \\ (x^2)^2 & x^1 x^2 \end{pmatrix}, \quad \mathbf{J}_2 = \begin{pmatrix} \frac{1}{2} x^1 + \frac{1}{4} a x^2 & \frac{1}{2} a x^1 \\ x^2 & \frac{1}{2} x^1 + \frac{1}{4} a x^2 \end{pmatrix}.$$

Hence, our two-parameter family of flat bi-cofactor systems attains in variables (x^1, x^2) the flat Newton form

$$
\begin{aligned}
\frac{d^2}{dt_3^2} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} &= -\left((\operatorname{cof} \mathbf{J}_1)^{-1} \nabla^{(3)} V \right)^i = -\left((\operatorname{cof} \mathbf{J}_2)^{-1} \nabla^{(3)} W \right)^i \\
&= \begin{pmatrix} \frac{1}{8} \frac{x^1 [4(x^1)^2 - 4ax^1 x^2 + a^2 (x^2)^2]}{b^4 (x^2)^5} + \frac{1}{2} \frac{4x^1 + ax^2}{b^6 (x^2)^5} \\ \frac{1}{8} \frac{4(x^1)^2 - 4ax^1 x^2 + a^2 (x^2)^2}{b^4 (x^2)^4} + \frac{3}{b^6 (x^2)^4} \end{pmatrix}
\end{aligned} \tag{79}
$$

with the potentials

$$
\begin{aligned}
V &= \frac{1}{8} \frac{4(x^1)^2 - 4ax^1 x^2 + a^2 (x^2)^2}{b^4 (x^2)^3} + \frac{1}{b^6 (x^2)^3}, \\
W &= \frac{1}{16^2} \frac{(ax^2 - 2x^1)[4(x^1)^2 - 4ax^1 x^2 + a^2 (x^2)^2]}{b^4 (x^2)^4} + \frac{1}{16} \frac{ax^2 - 2x^1}{b^6 (x^2)^4}.
\end{aligned}
$$

All the flat bi-cofactor systems in the 2-parameter family (79) are geodesically equivalent to both the Henon-Heiles system (72) and the Hamiltonian system (78). They also belong to the whole family of such systems generated by (75).

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