Simply-transitive CR real hypersurfaces in $\mathbb{C}^3$

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An equivalence problem

Up to local biholomorphism, classify all homogeneous real hypersurfaces $M^{2n+1} \subset \mathbb{C}^{n+1}$.

Symmetry: we have the real Lie algebra

$$\mathfrak{hol}(M) = \{X \text{ hol. v.f. on } \mathbb{C}^{n+1} : (X + \overline{X})\big|_M \text{ tangent to } M\}.$$

If $\forall p \in M$, the evaluation map $\mathfrak{hol}(M) \to T_p M$, $X \mapsto (X + \overline{X})|_p$ is surjective, then M is (holomorphically) homogeneous.

- multiply-transitive (MT): hom. & $\dim(M) < \dim \mathfrak{hol}(M)$.
- simply-transitive (ST): hom. & $\dim(M) = \dim \mathfrak{hol}(M)$.

Our focus: simply-transitive, 'Levi non-degenerate' $M^5 \subset \mathbb{C}^3$

Examples: Tubes

Given a real hypersurface $\mathcal{S}^n = \{x : \mathcal{F}(x) = 0\} \subset \mathbb{R}^{n+1}$ ('*base*'), its associated tubular CR hypersurface (or '*tube*') is:

$$M_{\mathcal{S}}^{2n+1} = \mathcal{S} + i\mathbb{R}^{n+1} := \{z : \mathcal{F}(\mathfrak{Re}\, z) = 0\} \subset \mathbb{C}^{n+1}, \quad \dim_{\mathbb{R}} M_{\mathcal{S}} = 2n+1.$$

- Always: $i\partial_{z_1}, \ldots, i\partial_{z_{n+1}} \in \mathfrak{hol}(M_{\mathcal{S}})$.
- If $\mathbf{S} = (A_{k\ell}x_\ell + b_k)\partial_{x_k} \in \mathfrak{aff}(\mathcal{S})$ is a *(real) affine symmetry* of $\mathcal{S}$, then $\mathbf{S}^{\mathrm{cr}} := (A_{k\ell}z_\ell + b_k)\partial_{z_k} \in \mathfrak{hol}(M_{\mathcal{S}})$.
- If $\mathcal{S}$ is affinely homogeneous, then $M_{\mathcal{S}}$ is homogeneous.

Example ($\mathcal{S}^2 \subset \mathbb{R}^3$ affinely hom., $M_{\mathcal{S}}^5 \subset \mathbb{C}^3$ simply-transitive)

$\mathcal{S}: \; u = x_1 \ln x_2, \;\; \mathfrak{aff}(\mathcal{S}) = \langle x_1\partial_{x_1} + u\partial_u, x_2\partial_{x_2} + x_1\partial_u \rangle,$
$\mathfrak{hol}(M_{\mathcal{S}}) = \langle i\partial_{z_1}, i\partial_{z_2}, i\partial_w, z_1\partial_{z_1} + w\partial_w, z_2\partial_{z_2} + z_1\partial_w \rangle.$

Example ($\mathcal{S}^2 \subset \mathbb{R}^3$ affinely inhom., $M_{\mathcal{S}}^5 \subset \mathbb{C}^3$ multiply-transitive)

$\mathcal{S}: \; u = x_1x_2 + x_1^3 \ln(x_1), \;\; \mathfrak{aff}(\mathcal{S}) = \langle \partial_{x_2} + x_1\partial_u \rangle, \; \mathfrak{hol}(M_{\mathcal{S}}) = \langle i\partial_{z_1}, i\partial_{z_2}, i\partial_w,$
$\partial_{z_2} + z_1\partial_w, \quad iz_1\partial_{z_2} + i\frac{z_1^2}{2}\partial_w, \;\; z_1\partial_{z_1} + (2z_2 - \frac{3}{2}z_1^2)\partial_{z_2} + (3w - \frac{1}{2}z_1^3)\partial_w \rangle.$

CR structure and its Levi form

$M^{2n+1} \subset \mathbb{C}^{n+1}$ inherits a (integrable) CR structure (M, C, J):

- $C := TM \cap J(TM)$, $J^2 = -1$; $n = \mathrm{rk}_{\mathbb{C}}\, C$ = CR-dim of M.
- $C^{\mathbb{C}} = C^{1,0} \oplus C^{0,1}$. These $\pm i$ eigenspaces for J are integrable.

Levi form (on $C^{0,1}$): $\mathcal{L}(\xi, \eta) = [\xi, \bar{\eta}] \bmod C^{\mathbb{C}}$. (Want 'ndg'.)

	$M^3 \subset \mathbb{C}^2$	$M^5 \subset \mathbb{C}^3$
max sym	8	15
submax sym	3	$\begin{cases} 8, & \text{Levi indefinite} \\ 7, & \text{Levi definite} \end{cases}$

(See Kruglikov (2015) for symmetry gaps for higher dim CR.)

Example ($M^5 \subset \mathbb{C}^3$)

Hyperquadric $\mathfrak{Im}(w) = |z_1|^2 \pm |z_2|^2$ (15-dim sym);
Winkelmann hypersurface: $\mathfrak{Im}(w + \overline{z_1} z_2) = |z_1|^4$ (8-dim sym).
These are tubular, i.e. equivalent to tubes on $u = x_1^2 + x_2^2$, $u = x_1 x_2$, and $u = x_1 x_2 + x_1^4$ respectively.

Historical Summary

- Poincaré (1907): Not all $M^3 \subset \mathbb{C}^2$ are locally equivalent.
- Cartan (1932): Classified all homogeneous $M^3 \subset \mathbb{C}^2$.
- Cartan (1935): Bounded homogeneous domains $D \subset \mathbb{C}^k$:
  - $k = 2$: are Hermitian symmetric spaces. Equivalent to either $|z_1| < 1, |z_2| < 1$ or $|z_1|^2 + |z_2|^2 < 1$.
  - $k = 3$: announced to be Hermitian symmetric. Apparently, proof was long and he decided not to publish it. Investigating multiply-transitive $M^5 = \partial D \subset \mathbb{C}^3$ is an ingredient.
- Piatetski-Shapiro (1959): $\exists$ bounded homogeneous domain in $\mathbb{C}^k$ for $k \geq 4$ that are not Hermitian symmetric.
- Loboda (2000-2003): Most of the MT, Levi ndg case settled. Incomplete: 6-dim Levi indefinite case.
- Fels–Kaup (2008): Levi rank 1 & 2-ndg. All hom. models are tubular.
- Doubrov–Medvedev-T. (2017): All MT, Levi ndg.
- Kossovskiy–Loboda (2019): ST, Levi definite. All tubular.
- Loboda et al. (2019-2020): ST, Levi indefinite. Two non-tubular models.
- Doubrov–Merker–T. (2020): All ST, Levi ndg. (Independent approach.)

Multiply-transitive, Levi non-degenerate $M^5 \subset \mathbb{C}^3$

Theorem (Doubrov–Medvedev-T. 2017)

Any multiply-transitive Levi non-degenerate hypersurface $M^5 \subset \mathbb{C}^3$ is locally biholomorphically equivalent to:

1. *Hyperquadric $\mathfrak{Im}(w) = |z_1|^2 \pm |z_2|^2$ (15-dim sym);*
2. *A tube (extensive list[a]; 6, 7, or 8-dim sym);*
3. *Cartan hypersurfaces ($\mathfrak{so}(4), \mathfrak{so}(1,3), \mathfrak{so}(2,2)$ sym) or a related quaternionic model ($\mathfrak{so}^*(4)$ sym);*
4. *hypersurface of Winkelmann type with 6-dim sym.*

[a] Base may be affinely inhomogeneous!

Strategy:

- Study 'complexified' CR str. ('ILC' str. / PDE) via Cartan reduction.
- Classify CR real forms.
- Recognize most tubes via classification of affinely hom. surfaces $\mathcal{S}^2 \subset \mathbb{R}^3$, see Doubrov–Komrakov–Rabinovitch (1995) & Eastwood–Ezhov (1999).

The simply-transitive classification

Theorem (Loboda et al. 2019-2020 & Doubrov–Merker–T. 2020)

Any simply-transitive Levi non-degenerate hypersurface $M^5 \subset \mathbb{C}^3$ is locally biholomorphically equivalent to precisely one of:

1. *Either one hypersurface among the 6 families of tubes with affinely simply-transitive base (for $\alpha, \beta \in \mathbb{R}$ and $\epsilon = \pm 1$):*

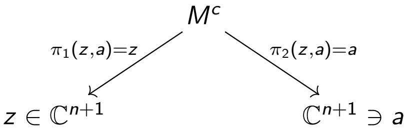
T1	$u = x_1^\alpha x_2^\beta$	$\alpha\beta(1-\alpha-\beta) \neq 0$, $(\alpha,\beta) \neq (1,1),(1,-1),(-1,1)$ *Redundancy:* $(\alpha,\beta) \sim (\beta,\alpha) \sim (\frac{1}{\alpha}, -\frac{\beta}{\alpha})$
T2	$u = (x_1^2 + x_2^2)^\alpha \exp(\beta \arctan(\frac{x_2}{x_1}))$	$\alpha \neq \frac{1}{2}; (\alpha,\beta) \neq (0,0),(1,0)$ *Redundancy:* $(\alpha,\beta) \sim (\alpha,-\beta)$
T3	$u = x_1(\alpha \ln(x_1) + \ln(x_2))$	$\alpha \neq -1$
T4	$(u - x_1 x_2 + \frac{x_1^3}{3})^2 = \alpha(x_2 - \frac{x_1^2}{2})^3$	$\alpha \neq 0, -\frac{8}{9}$
T5	$x_1 u = x_2^2 + \epsilon x_1^\alpha$	$\alpha \neq 0, 1, 2$
T6	$x_1 u = x_2^2 + \epsilon x_1^2 \ln(x_1)$	–

2. $\mathfrak{Im}(w) = |\mathfrak{Im}(z_2) - w\,\mathfrak{Im}(z_1)|^2$. *(Levi indefinite, non-tubular, symmetry $\mathfrak{saff}(2,\mathbb{R}) := \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2$.)*

Loboda (2020): Another ST (non-tube) model. (False: it's intransitive.)

Segré varieties

If $M^{2n+1} = \{z : \Phi(z,\bar{z}) = 0\} \subset \mathbb{C}^{n+1}$, define its 'complexification' $M^c := \{(z,a) : \Phi(z,a) = 0\} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ (or 'Segré variety'). We have M = fixed-point set of $\tau(z,a) = (\bar{a},\bar{z})$ restricted to M^c.



This induces rank n distributions $E = \ker(d\pi_2)$ and $F = \ker(d\pi_1)$. Let $\mathrm{sym}(M^c) = \{X = \xi^k(z)\partial_{z_k} + \sigma^k(a)\partial_{a_k} : X \text{ tangent to } M^c\}$.

Example ($\mathcal{S}:\ u = x_1 \ln x_2$)

$\mathfrak{hol}(M_{\mathcal{S}}) = \langle i\partial_{z_1},\, i\partial_{z_2},\, i\partial_w,\, z_1\partial_{z_1} + w\partial_w, z_2\partial_{z_2} + z_1\partial_w \rangle$.

$$\begin{aligned}\mathrm{sym}(M^c_{\mathcal{S}}) = \langle & \partial_{z_1} - \partial_{a_1},\, \partial_{z_2} - \partial_{a_2},\, \partial_w - \partial_c, \\ & z_1\partial_{z_1} + w\partial_w + a_1\partial_{a_1} + c\partial_c,\, z_2\partial_{z_2} + z_1\partial_w + a_2\partial_{a_2} + a_1\partial_c \rangle.\end{aligned}$$

We always have $\boxed{\dim_{\mathbb{R}} \mathfrak{hol}(M) = \dim_{\mathbb{C}} \mathrm{sym}(M^c)}$.

ILC structures & PDE

Definition

A Legendrian contact (LC) structure is a complex contact manifold (N, C) with $C = E \oplus F$ where E, F are maximally isotropic. If one or both of E, F are integrable, it is SILC or ILC respectively.

Example

For $M \subset \mathbb{C}^{n+1}$ as before, $(M^c; E, F)$ is an ILC structure.

Locally, $C = \{\sigma = 0\}$, $\eta = d\sigma|_C$ ndg. $\exists (z^j, w, w_j)$ with $\sigma = dw - w_j dz^j$. If F is integ., we may assume $F = \langle \partial_{w_j} \rangle$, so

$$E = \langle \mathcal{D}_j := \partial_{z^j} + w_j \partial_w + f_{jk} \partial_{w_k} \rangle, \quad \exists f_{jk} = f_{kj}.$$

Equivalently, we have a complete 2nd order PDE system $\frac{\partial^2 w}{\partial z_j \partial z_k} = f_{jk}(z^\ell, w, w_\ell)$, considered up to point transformations.

M^c and the PDE solution spaces

PDE compatibility $\Leftrightarrow$ integrability of E.

In fact, $M^c = \{(z,a) : \Phi(z,a) = 0\}$ is the solution space of a 2nd order PDE system. How to find it?

1. Regard $w := z_{n+1}$ as a function of $(z_1, \ldots, z_n)$, treat $a \in \mathbb{C}^{n+1}$ as parameters.
2. Find $w_j := \frac{\partial w}{\partial z_j}$. Solve for a in terms of $(w, w_1, \ldots, w_n)$.
3. Find $w_{jk} := \frac{\partial^2 w}{\partial z_j \partial z_k}$ and sub. in a.

Example ($\mathcal{S}: \ u = x_1 \ln x_2$)

$$M_{\mathcal{S}} : \ \mathfrak{Re}(w) = \mathfrak{Re}(z_1) \ln \mathfrak{Re}(z_2), \quad M^c_{\mathcal{S}} : \ \frac{w+c}{2} = \left(\frac{z_1+a_1}{2}\right) \ln\left(\frac{z_2+a_2}{2}\right)$$

$$\Rightarrow (w_1, w_2, w_{11}, w_{12}, w_{22}) = \left(\ln\left(\frac{z_2+a_2}{2}\right), \frac{z_1+a_1}{z_2+a_2}, 0, \frac{1}{z_2+a_2}, -\frac{z_1+a_1}{(z_2+a_2)^2} \right).$$

$$\Rightarrow w_{11} = 0, \quad w_{12} = \frac{1}{2} e^{-w_1}, \quad w_{22} = -\frac{1}{2} w_2 e^{-w_1}.$$

Simply-transitive classification strategy - 1

Hom. ILC $(G/K; E, F) \leftrightarrow$ ILC quadruple $(\mathfrak{g}, \mathfrak{k}; \mathfrak{e}, \mathfrak{f})$. When $\mathfrak{k} = 0$:

Definition (ILC triple & ASD-ILC triple)

Let $\dim \mathfrak{g} = 2n + 1$. *An ILC triple* $(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})$ *consists of n-dim subalgs* $\mathfrak{e}, \mathfrak{f} \subset \mathfrak{g}$ *s.t.* $C = \mathfrak{e} \oplus \mathfrak{f}$ *is ndg, i.e.* $\eta(x, y) = [x, y] \bmod C$ *is ndg on* C. *An ILC triple* $(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})$ *is ASD if* $\exists$ *anti-involution* τ *of* $\mathfrak{g}$ *that swaps* $\mathfrak{e}$ *and* $\mathfrak{f}$.

Want: ASD-ILC triples $(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})$ with $5 = \dim(\mathfrak{g}) = \dim \mathrm{sym}_{\mathrm{ILC}}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})$.

How to tell if symmetry jumps up? i.e. $\dim \mathrm{sym}_{\mathrm{ILC}}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f}) > \dim(\mathfrak{g}) = 5$.

1. Find embedding of $(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})$ into $(\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{k}}; \widetilde{\mathfrak{e}}, \widetilde{\mathfrak{f}})$, where $\widetilde{\mathfrak{k}} \neq 0$.
2. New coord-indep. formula for fundamental quartic $\mathcal{Q}_4$. Moreover,

$\mathcal{Q}_4$ root type	O	N	D	III	II	I
max sym	15	8	7	6	5	5

3. Direct computation via two methods: PDE syms or power series.

Simply-transitive classification strategy - 2

Kossovskiy–Loboda (2019): In the Levi-definite case, if 5-dim $\mathfrak{hol}(M)$ contains a 3-dim abelian ideal, then M is a tube over an affinely hom. base $\mathcal{S}$. (Proof does not extend to indefinite case.)

Let $\mathfrak{g}$ be a complex 5-dim Lie alg. Want: ASD-ILC triples $(\mathfrak{g}; \mathfrak{e}, \mathfrak{f})$ with admissible τ and $\dim \mathsf{sym}_{\mathsf{ILC}}(\mathfrak{g}; \mathfrak{e}, \mathfrak{f}) = 5$. Overview:

1. $\mathfrak{g}$ has no 3-dim abelian ideal: $\exists!$ model.
2. $\mathfrak{g}$ has a 3-dim abelian ideal $\mathfrak{a}$:
   (a) $\mathfrak{a} \neq \tau(\mathfrak{a})$: $\nexists$ models.
   (b) $\mathfrak{a} = \tau(\mathfrak{a})$:
      - $\mathfrak{e} \cap \mathfrak{a} \neq 0$ (or $\mathfrak{f} \cap \mathfrak{a} \neq 0$): $\nexists$ models.
      - $\mathfrak{e} \cap \mathfrak{a} = \mathfrak{f} \cap \mathfrak{a} = 0$: All must be tubes on affinely hom. $\mathcal{S}^2 \subset \mathbb{R}^3$.

Tube strategy: From DKR list, remove $\mathcal{S}$ with MT $M_{\mathcal{S}}$ & restrict to affinely ST $\mathcal{S}$ with ndg Hessians. Get tube list in Main Thm as a *candidate list*. Need to test for symmetry jumps.

The fundamental quartic $\mathcal{Q}_4$

All 5-dim LC structures admit a fundamental curvature invariant that is a binary quartic field. (Typically computed via Chern–Moser normal form or via PDE realization.) Geometric interpretation?

Key idea: Lift $(N^5; E, F)$ to a $\mathbb{P}^1$-bundle $\widetilde{N}^6 \to N^5$.

$$\widetilde{N}_x := \{(\ell_E, \ell_F) \in \mathbb{P}(E_x) \times \mathbb{P}(F_x) : \eta(\ell_E, \ell_F) = 0\}$$

(In fact, $\ell_F = F \cap (\ell_E)^{\perp_\eta}$, so $\widetilde{N} \to N$ is a $\mathbb{P}^1$-bundle.) On $\widetilde{N}$:

(i) rank 1: $V = \ker(\pi_*)$;
(ii) rank 3: $D|_{\widetilde{x}} := (\pi_*)^{-1}(\ell_E \oplus \ell_F)$ for $\widetilde{x} = (\ell_E, \ell_F)$;
(iii) rank 5: $\widetilde{C} := (\pi_*)^{-1} C$ for $C = E \oplus F$.

Indeed, we get an instance of a Borel geometry (R^6, D):

- weak derived flag $D \subset D^2 \subset D^3 = TR$ has growth $(3, 5, 6)$.
- symbol algebra modelled on $\begin{pmatrix} 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix} \subset \mathfrak{sl}(4)$.

Geometric interpretation of $\mathcal{Q}_4$

Proposition

Given any Borel geometry (R^6, D), we canonically have:

(a) *rk 2:* $\sqrt{D} \subset D$ *satisfying* $[\sqrt{D}, \sqrt{D}] \equiv 0 \mod D$.

(b) *rk 1:* $V = \{X \in D : [X, D^2] \subset D^2\}$. *Have* $D = V \oplus \sqrt{D}$.

(c) $\sqrt{D} = L_1 \oplus L_2$ *(unique up to ordering) into null lines* L_1, L_2 *for a ndg conformal symmetric bilinear form on* $\sqrt{D}$.

Corollary

The map $\Gamma(L_1) \times \Gamma(L_2) \to \Gamma(V)$, $(X, Y) \mapsto \mathrm{proj}_V([X, Y])$ *is tensorial, so determines a vector bundle map* $\Phi : L_1 \otimes L_2 \to V$. *Geometrically,* Φ *obstructs Frobenius-integrability of* $\sqrt{D}$.

For LC str. on N, Φ on $\widetilde{N}$ is a quartic tensor field $\mathcal{Q}_4(t)$ wrt affine coord t on $\mathbb{P}^1$. Homog. cases are easily computed in terms of algebraic data.

Abstract realization of the exceptional model

Proposition

Any 5-dimensional complex Lie algebra $\mathfrak{g}$ without 3-dimensional abelian ideals is isomorphic to one of:

- $\mathfrak{sl}(2,\mathbb{C}) \times \mathbb{C}^2$;
- $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{r}_2$;
- $\mathfrak{saff}(2,\mathbb{C}) := \mathfrak{sl}(2,\mathbb{C}) \ltimes \mathbb{C}^2$;
- *upper-triangular matrices in* $\mathfrak{sl}(3,\mathbb{C})$.

Proof is indep. of Mubarakzyanov classification of 5-dim (real) Lie alg.

Theorem

For list above, only $\mathfrak{g} = \mathfrak{saff}(2,\mathbb{C})$ supports an ASD-ILC triple $(\mathfrak{g};\mathfrak{e},\mathfrak{f})$ with $\dim \text{sym}_{\text{ILC}}(\mathfrak{g};\mathfrak{e},\mathfrak{f}) = 5$. Up to Aut$(\mathfrak{g})$-equivalence, $(\mathfrak{g};\mathfrak{e},\mathfrak{f})$ is unique and admits a unique admissible anti-involution τ.

Wrt 'usual' basis H, X, Y, v_1, v_2: $\mathfrak{e} = \langle H + v_1, X\rangle$, $\mathfrak{f} = \langle H - v_2, Y\rangle$ ($\mathcal{Q}_4$ has root type I), and $(H, X, Y, v_1, v_2) \stackrel{\tau}{\longmapsto} (-H, Y, X, v_2, v_1)$.

Derivation of the exceptional model

Std. action of $\mathfrak{g} = \mathfrak{saff}(2,\mathbb{C})$ on $\mathbb{C}^2 = J^0(\mathbb{C},\mathbb{C})$:

$$H = z_1\partial_{z_1} - z_2\partial_{z_2}, \quad X = z_1\partial_{z_2}, \quad Y = z_2\partial_{z_1}, \quad v_1 = \partial_{z_1}, \quad v_2 = \partial_{z_2}.$$

Prolong these to $J^1(\mathbb{C},\mathbb{C})$, i.e. $(z_1, z_2, w := z_2')$-space. Induce the joint action on two copies of $J^1(\mathbb{C},\mathbb{C})$, i.e. $(z_1, z_2, w, a_1, a_2, c)$-space.

$$\begin{aligned} H &= z_1\partial_{z_1} - z_2\partial_{z_2} - 2w\partial_w + a_1\partial_{a_1} - a_2\partial_{a_2} - 2c\partial_c, \\ X &= z_1\partial_{z_2} + \partial_w + a_1\partial_{a_2} + \partial_c, \qquad v_1 = \partial_{z_1} + \partial_{a_1}, \\ Y &= z_2\partial_{z_1} - w^2\partial_w + a_2\partial_{a_1} - c^2\partial_c, \quad v_2 = \partial_{z_2} + \partial_{a_2}. \end{aligned}$$

This prolonged $\mathfrak{g}$-action admits the joint differential invariant:

$$\mathcal{A} := \frac{(z_2 - a_2 - w(z_1 - a_1))(z_2 - a_2 - c(z_1 - a_1))}{2(w - c)}.$$

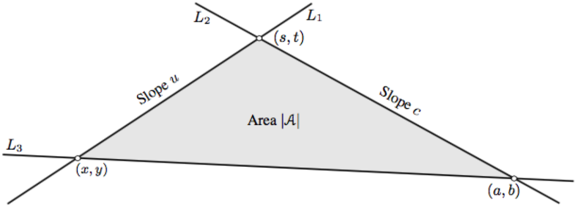
Consider $\boxed{\mathcal{A} = \lambda \in \mathbb{C}^\times}$. Rescale variables to normalize $\boxed{\lambda \text{ to } i}$. Intersect with the fixed-point set of $(z_1, z_2, w, a_1, a_2, c) \overset{\tau}{\mapsto} (\bar{a}_1, \bar{a}_2, \bar{c}, \bar{z}_1, \bar{z}_2, \bar{w})$ to get the exceptional model $\mathfrak{Im}(w) = |\mathfrak{Im}(z_2) - w\,\mathfrak{Im}(z_1)|^2$.

Related real equi-affine geometry

Fix $(x, y, u, a, b, c) \in \mathbb{R}^6 \simeq_{loc} J^1(\mathbb{R}, \mathbb{R}) \times J^1(\mathbb{R}, \mathbb{R})$, define

$$\mathcal{A} := \frac{(y - b - u(x - a))(y - b - c(x - a))}{2(u - c)}.$$

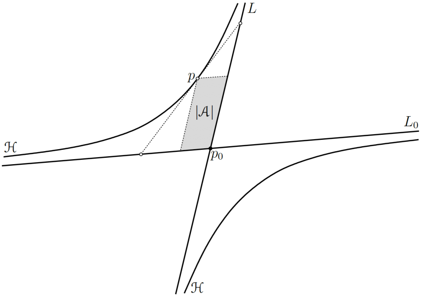
For $\mathcal{A} \in \mathbb{R}^\times$, this data determines a triangle in $\mathbb{R}^2$ with area $|\mathcal{A}|$:

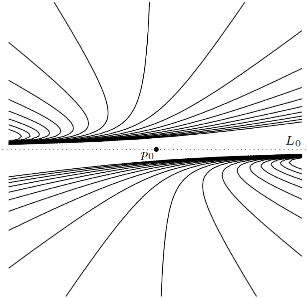


Setting $\mathcal{A} = \lambda \in \mathbb{R}^\times$ defines a family of planar triangles that is invariant under the planar equi-affine group $\mathrm{SAff}(2, \mathbb{R})$.

Related real equi-affine geometry - 2

Recall: For any planar hyperbola $\mathcal{H}$, its 'asymptotes-parallelogram' has area $Area(\mathcal{H})$ independent of $p \in \mathcal{H}$.





Fix $\mathcal{A}$. Any $(a, b, c) \in \mathbb{R}^3 \simeq_{loc} J^1(\mathbb{R}, \mathbb{R})$ yields $p_0 = (a, b) \in \mathbb{R}^2$ and a line L_0 through it with slope c. Get a local foliation $\{\mathcal{H} : Area(\mathcal{H}) = |\mathcal{A}|,\ L_0 \text{ an asymptote for } \mathcal{H}\}$. The collection of all such foliations is $\mathrm{SAff}(2, \mathbb{R})$-invariant.

Tubes

Real affine hypersurface
$\mathcal{S} = \{x : \mathcal{F}(x) = 0\} \subset \mathbb{R}^{n+1}, \quad d\mathcal{F} \neq 0$ on $\mathcal{S}$;
Real affine symmetry $\mathbf{S} = (A_{k\ell} x_\ell + b_k)\partial_{x_k} \in \mathfrak{aff}(\mathcal{S})$

Tubular CR hypersurface
$M_{\mathcal{S}} = \{z : \mathcal{F}(\mathfrak{Re}\, z) = 0\} \subset \mathbb{C}^{n+1}$;
$i\partial_{z_1}, \ldots, i\partial_{z_{n+1}} \in \mathfrak{hol}(M_{\mathcal{S}})$,
$\mathbf{S}^{\rm cr} := (A_{k\ell} z_\ell + b_k)\partial_{z_k} \in \mathfrak{aff}(\mathcal{S})^{\rm cr}$

Tubular ILC hypersurface
$M^c_{\mathcal{S}} = \{(z,a) : \mathcal{F}(\frac{z+a}{2}) = 0\} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$;
$\partial_{z_1} - \partial_{a_1}, \ldots, \partial_{z_{n+1}} - \partial_{a_{n+1}} \in \mathrm{sym}(M^c_{\mathcal{S}})$,
$\mathbf{S}^{\rm lc} := (A_{k\ell} z_\ell + b_k)\partial_{z_k} + (A_{k\ell} a_\ell + b_k)\partial_{a_k} \in \mathfrak{aff}(\mathcal{S})^{\rm lc}$

Note that $\mathfrak{a} = \langle \partial_{z_1} - \partial_{a_1}, \ldots, \partial_{z_{n+1}} - \partial_{a_{n+1}} \rangle$ is transverse to the projections $\pi_1(z,a) = z$ and $\pi_2(z,a) = a$.

Abstract description of tubes

Given any affine hypersurface $\mathcal{S} \subset \mathbb{R}^{n+1}$ with homogeneous CR / ILC tubes $M_{\mathcal{S}}$ and $M_{\mathcal{S}}^c$, we get the following abstract structure:

Definition

A tubular CR realization for an ILC quadruple $(\mathfrak{g}, \mathfrak{k}; \mathfrak{e}, \mathfrak{f})$ *in dimension* $\dim(\mathfrak{g}/\mathfrak{k}) = 2n+1$ *is a pair* $(\mathfrak{a}, \tau)$*, where*

(T.1) $\mathfrak{a} \subset \mathfrak{g}$ *is an* $(n+1)$*-dim abelian subalgebra;*

(T.2) $\mathfrak{e} \cap \mathfrak{a} = \mathfrak{f} \cap \mathfrak{a} = 0$.

(T.3) τ *is an admissible anti-involution of* $(\mathfrak{g}, \mathfrak{k}; \mathfrak{e}, \mathfrak{f})$ *that preserves* $\mathfrak{a}$.

If $\mathfrak{a}$ has normalizer $\mathfrak{n}(\mathfrak{a})$ in $\mathfrak{g}$, then $\mathfrak{n}(\mathfrak{a})/\mathfrak{a} \cong \mathfrak{aff}(\mathcal{S}) \otimes_{\mathbb{R}} \mathbb{C}$.

Theorem

If $M^5 \subset \mathbb{C}^3$ *is ST, Levi ndg with* $\mathfrak{hol}(M)$ *containing a 3-dim abelian ideal, then* $M \cong M_{\mathcal{S}}$ *for some affinely ST base* $\mathcal{S} \subset \mathbb{R}^3$.

Affinely homogeneous hypersurfaces to ILC structures

Proposition

Let $\mathcal{S} \subset \mathbb{R}^{n+1}$ be an affinely hom. hypersurface with ndg 2nd fundamental form. Then $M^c_{\mathcal{S}} \subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ is homogeneous and encoded by an ILC quadruple $(\mathfrak{g}, \mathfrak{k}; \mathfrak{e}, \mathfrak{f})$ given for any $p \in \mathcal{S}$ by

$$\mathfrak{e} := \mathfrak{aff}(\mathcal{S}) \otimes_{\mathbb{R}} \mathbb{C}, \quad \mathfrak{g} := \mathfrak{e} \ltimes \mathbb{C}^{n+1},$$
$$\mathfrak{f} := \{Y \in \mathfrak{g} : Y|_p = 0\}, \quad \mathfrak{k} := \mathfrak{e} \cap \mathfrak{f}.$$

Thus, $\mathcal{Q}_4$ can be efficiently computed for our tubes of study.

Example ($\mathcal{S}$: $u = x_1(\alpha \ln x_1 + \ln x_2)$; ndg: $\alpha \neq -1$; $p = (1,1,0) \in \mathcal{S}$)

$\mathfrak{e} = \langle \mathbf{S} := x_1\partial_{x_1} - \alpha x_2 \partial_{x_2} + u\partial_u, \mathbf{T} := x_2\partial_{x_2} + x_1\partial_u \rangle$, $[\mathbf{S}, \mathbf{T}] = 0$

$\mathfrak{f} = \langle \widetilde{\mathbf{S}} := \mathbf{S} - \partial_{x_1} + \alpha\partial_{x_2}, \widetilde{\mathbf{T}} := \mathbf{T} - \partial_{x_2} - \partial_u \rangle$. We use this to calculate:

$$\mathcal{Q}_4 = -t^4 - 4t^3 - \frac{2(\alpha+3)}{\alpha+1} t^2 - \frac{4}{\alpha+1} t - \frac{1}{(\alpha+1)^2} \quad \Rightarrow \quad \begin{cases} \text{I}: & \alpha \neq -1, 0, 8; \\ \text{II}: & \alpha = 8; \\ \text{N}: & \alpha = 0 \end{cases}$$

Conclusion: Type I and II have 5-dim sym. For type N, use `Maple` on the PDE system $(w_{11}, w_{12}, w_{22}) = (0, \frac{1}{2}e^{-w_1}, -\frac{1}{2}w_2 e^{-w_1})$.

Summary

The classification of homogeneous $M^5 \subset \mathbb{C}^3$ branches as:

1. $\mathbb{R} \times \mathbb{C} \times \mathbb{C}$ or $M^5 = \mathcal{M}^3 \times \mathbb{C}$, where $\mathcal{M}^3 \subset \mathbb{C}^2$ is Levi ndg.
2. Levi rank 1 & 2-nondegenerate
3. Levi non-degenerate (MT & ST)

This classification is now complete.

- We used a Lie algebraic approach that circumvents normal forms, is independent of the Mubarakzyanov classification, and takes advantage of the close relationship to ILC structures.
- A key new tool is a coordinate-independent formula / geometric interpretation for the fundamental quartic $\mathcal{Q}_4$. (See `Maple` supplement in `arXiv` submission.)