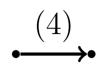
MATH-4140 EXAM

Problem 1 (4 points) Let k be a field. Up to isomorphism, find all 6-dimensional path k-algebras of connected graphs.

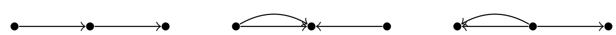
Solution: Since for a graph E the basis of the path algebra kE consists of all finite paths, we need two find connected graphs such that the number of all their paths, including the 0-paths (vertices), equals 6. Consider a graph with:

(a) 1 vertex. Then, if $E^1 \neq \emptyset$, there are no finite-dimensional path algebras for such graphs.

(b) 2 vertices. Then, to avoid creating a loop, the only possibility is to have 4 edges between these vertices arranged like this:

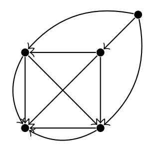


(c) 3 vertices. Then we have the following graphs:



(d) 4 or more vertices. If there are four or more vertices, then one needs more than two edges to make it connected. Hence, there no 6-dimensional path algebra for a connected graph with 4 or more vertices.

Problem 2 (3 points) Compute the number of all paths of length two for the following graph with 11 edges:



Is this the maximal number of paths of length two that one can obtain for a graph with 11 edges and no loops? If not, find a graph that maximizes this number.

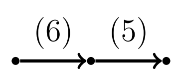
Solution: Call the above graph E. It has the following adjacency matrix:

$$A(E) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

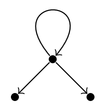
To find out the number of all paths of length 2, we need to compute

$$A(E)^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, there are $1 + 2 + 5 + 1 + 4 + 2 = 15$ paths of length two. The maximum number of 2-paths for a graph with no loops and 11 edges is $(5 + 1)^1 5^{(2-1)} = 30$. A graph maximizing this number is

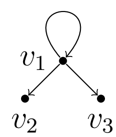


Problem 3 (6 points) Let E be the following graph:



Find all admissible subgraphs of E (i.e. all subgraphs of E whose inclusion in E is admissible) with proof.

Solution: We label vertices as follows:



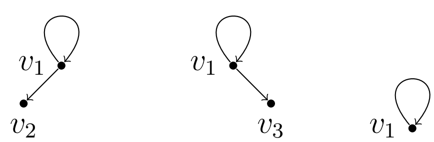
Of course, the empty subgraph and the whole graph are admissible. It remains to consider all non-empty proper subsets of E^0:

$$\{v_i\},\ i = 1, 2, 3,\ \{v_1, v_2\},\ \{v_1, v_3\},\ \{v_2, v_3\}.$$

Out of these 6 subsets only the following 3 subsets are hereditary and saturated:

$$\{v_2\},\ \{v_3\},\ \{v_2, v_3\}.$$

Now, every admissible subgraph yields a hereditary and saturated subset of missing veritices, and given a hereditary saturated subset $H \subseteq E^0$, there is only one way to obtain an admissible subgraph: $F^0 = E^0 \setminus H$ and $F^1 = t_E^{-1}(E^0 \setminus H)$. Hence we have only the following 3 admissible non-empty proper subgraphs of E:



Problem 4 (4 points) Let k be a field and let $E = (E^0, E^1, s, t)$ be a non-empty connected graph. Show that the path algebra kE is commutative if and only if

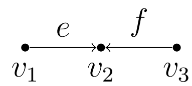
$$|E^0| = 1 \text{ and } |E^1| \leq 1.$$

Solution: Assume that $E^1 = \emptyset$ and $|E^0| = 1$. Then the graph E is connected and its path algebra $kE \cong k$ is commutative. Assume next that $|E^1| = 1$ and $|E^0| = 1$. Then the graph E consists of one vertex and one loop-edge attached to it, so it is connected and its path algebra $kE \cong k[\mathbb{N}]$ is commutative. Suppose now that $|E^0| > 1$ and the graph E is connected. Then there exists an edge e that is not a loop. It follows that kE is noncommutative because

$$\chi_{s(e)}\chi_e = \chi_e \neq 0 = \chi_e\chi_{s(e)} .$$

Suppose next that $|E^1| > 1$ and the graph E is connected. If there is an edge that is not a loop, then we already know that kE is noncommutative. If all edges are loops and E is connected, then there is only one vertex Hence there are at least two different loop-edges α and β starting from the same vertex. Consequently, kE is noncommutative because $\chi_\alpha\chi_\beta = \chi_{\alpha\beta} \neq \chi_{\beta\alpha} = \chi_\beta\chi_\alpha$.

Problem 5 (5 points) Let k be a field and E be the following graph:



Show that any element in the Leavitt path algebra $L_k(E)$ is a linear combination of

$$[\chi_{v_1}], \quad [\chi_{v_2}], \quad [\chi_{v_3}], \quad [\chi_e], \quad [\chi_f], \quad [\chi_{e^*}], \quad [\chi_{f^*}], \quad [\chi_{ef^*}], \quad [\chi_{fe^*}] \quad \in \quad L_k(E).$$

Solution: Since all elements of $L_k(E)$ corresponding to paths of length at most one in the extended graph $\bar{E}$ are already listed, it suffices to check that, if $p \in FP_n(\bar{E})$ and $n > 1$, then $[\chi_p]$ is a linear combination of the above elements. If p contains a subpath x^*y, where x and y are edges, then $[\chi_p] = 0$, if $x \neq y$, and $[\chi_p] = [\chi_q]$, if $x = y$ and q is a path obtained from p by removing x^*x. Consequently, the only elements of $L_k(E)$ corresponding to paths of length at least two are of the form $[\chi_{e_1 \cdots e_n f_m^* \cdots f_1^*}]$, where $e_1 \cdots e_n$ and $f_1 \cdots f_m$ are paths in E. As the only such paths are e and f, the only elements of $L_k(E)$ corresponding to paths of length at least two are

$$[\chi_{ef^*}], \quad [\chi_{fe^*}], \quad [\chi_{ee^*}] = [\chi_{v_1}], \quad [\chi_{ff^*}] = [\chi_{v_3}].$$