MATH-5140 EXAM

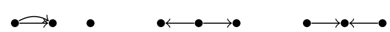
Problem 1 (6 points) Let k be a field. Up to isomorphism, find all 5-dimensional path algebras over k.

Solution: Since for a graph E the basis of the path algebra kE consists of all finite paths, we need to find all graphs such that the number of all their paths, including the 0-paths (vertices), equals 5. Consider a graph with:

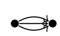
(a) 0 edges. The only possibility is to have 5 vertices.

(b) 1 edge. We need to have at least two vertices, because otherwise the edge would would be a loop. Then, the only possibility is to have two more disconnected vertices.

(c) 2 edges. Again, we need to have at least two vertices so that both edges are not loops. If both edges start at the same vertex, they both can end at some other vertex or end at two different vertices. If they both end at the same vertex, we need to add a disconnected vertex. If they start at two different vertices, they need to end at the same vertex.

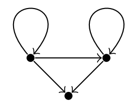


(d) 3 edges. The only possibility is the following:



(e) 4 or more edges. This is impossible because then there would be only one vertex, whence edges would be loops.

Problem 2 (3 points) Compute the number of all paths of a fixed length $k > 1$ for the following graph:



Solution: Call the above graph E. It has the following adjacency matrix:

$$A(E) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

To obtain the number of k-paths we need to raise $A(E)$ to the k-th power. We claim that

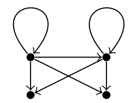
$$A(E)^k = \begin{bmatrix} 1 & k & k \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and prove it by induction. For $k = 1$ the equality is satisfied and

$$\begin{bmatrix} 1 & k & k \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & k+1 & k+1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

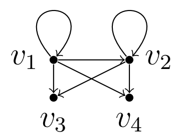
proves the inductive step. Hence, there are $2k + 3$ many k-paths.

Problem 3 (4 points) Let E be the following graph:



Find all admissible subgraphs of E (i.e. all subgraphs of E whose inclusion in E is admissible) with proof.

Solution: We label vertices as follows:



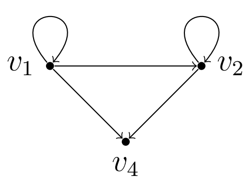
Of course, the empty subgraph and the whole graph are admissible. It remains to consider all non-empty proper subsets of E^0:

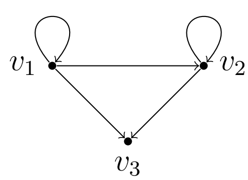
$$\{v_i\},\ i=1,2,3,4,\ \{v_1,v_2\},\ \{v_1,v_3\},\ \{v_1,v_4\},\ \ \{v_2,v_3\},\ \{v_2,v_4\},\ \{v_3,v_4\},$$
$$\{v_1,v_2,v_3\},\ \{v_1,v_2,v_4\},\ \{v_1,v_3,v_4\},\ \{v_2,v_3,v_4\}.$$

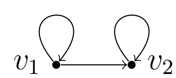
Out of these 14 subsets only the following 4 subsets are hereditary and saturated:

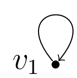
$$\{v_3\},\ \{v_4\},\ \{v_3,v_4\},\ \{v_2,v_3,v_4\}.$$

Now, every admissible subgraph yields a hereditary and saturated subset of missing veritices, and given a hereditary saturated subset $H \subseteq E^0$, there is only one way to obtain an admissible subgraph: $F^0 = E^0 \setminus H$ and $F^1 = t_E^{-1}(E^0 \setminus H)$. Hence we have only the following 4 admissible non-empty proper subgraphs of E:









Problem 4 (5 points) Let k be a field and let E be the following graph:

Compute all idempotents ($x^2 = x$) in the path algebra kE.

Solution: Let v be the left vertex of E, let w be the right vertex of E, and let e and f be the two edges in E. Every element of kE is of the form

$$x = \lambda_1\chi_v + \lambda_2\chi_w + \alpha_1\chi_e + \alpha_2\chi_f, \qquad \lambda_1, \lambda_2, \alpha_1, \alpha_2 \in k.$$

Therefore,

$$\begin{aligned} x^2 &= (\lambda_1\chi_v + \lambda_2\chi_w + \alpha_1\chi_e + \alpha_2\chi_f)(\lambda_1\chi_v + \lambda_2\chi_w + \alpha_1\chi_e + \alpha_2\chi_f) \\ &= \lambda_1^2\chi_v + \lambda_1\alpha_1\chi_e + \lambda_1\alpha_2\chi_f + \lambda_2^2\chi_w + \lambda_2\alpha_1\chi_e + \lambda_2\alpha_2\chi_f\,. \end{aligned}$$

Hence, remembering that for any finite path p the element χ_p is a basis element, the idempotent equation $x^2 = x$ yields

$$\lambda_1^2 = \lambda_1, \quad \lambda_2^2 = \lambda_2 \quad \Longleftrightarrow \quad \boxed{\lambda_1 = 0 \text{ or } 1, \quad \lambda_2 = 0 \text{ or } 1};$$

$$(\lambda_1 + \lambda_2)\alpha_1 = \alpha_1, \quad (\lambda_1 + \lambda_2)\alpha_2 = \alpha_2$$

$$\Longleftrightarrow \quad \boxed{\lambda_1 + \lambda_2 = 1 \quad \text{or} \quad \alpha_1 = 0} \quad \text{and} \quad \boxed{\lambda_1 + \lambda_2 = 1 \quad \text{or} \quad \alpha_2 = 0}.$$

We consider all possibilities:

(a) $\lambda_1 = \lambda_2 = 0$. Then $\alpha_1 = \alpha_2 = 0$, and consequently $x = 0$.

(b) $\lambda_1 = \lambda_2 = 1$. Then $\alpha_1 = \alpha_2 = 0$, and consequently $x = \chi_v + \chi_w = 1$.

(c) $\lambda_1 = 1$ and $\lambda_2 = 0$. Then α_1 and α_2 are arbitrary, and $x = \chi_v + \alpha_1\chi_e + \alpha_2\chi_f$.

(d) $\lambda_1 = 0$ and $\lambda_2 = 1$. Then α_1 and α_2 are arbitrary, and $x = \chi_w + \alpha_1\chi_e + \alpha_2\chi_f$.

Problem 5 (4 points) Using the pullback theorem (Theorem 7.9 in the lecture notes), prove that, for any two row-finite graphs E and F such that $E \cap F = \emptyset$, we have an isomorphism of algebras

$$L_k(E \cup F) \cong L_k(E) \oplus L_k(F).$$

If in addition both E and F are non-empty, show also that $E \cup F$ is *not* a connected graph.

Solution: Since the Leavitt path algebra $L_k(\emptyset)$ of the empty graph is zero, the canonical quotient maps $L_k(E) \overset{\pi_1}{\to} L_k(E \cap F) \overset{\pi_1}{\leftarrow} L_k(F)$ are zero. Furthermore, as both graphs are row finite, and the empty graph is always an admissible subgraph, Theorem 7.9 applies, so $L_k(E \cup F) \cong P(\pi_1, \pi_2) = L_k(E) \oplus L_k(F)$. Finally, as $E^0 \neq \emptyset \neq F^0$ and $E^0 \cap F^0 = \emptyset$, there exist $v \in E^0$ and $w \in F^0$ such that $v \neq w$. Suppose that $E \cup F$ is connected. Then there exists an unoriented path between v and w. It must contain an edge joining a vertex in E^0 with a vertex in F^0, but such an edge does not exist because, as $E^0 \cap F^0 = \emptyset$, it neither can belong to E^1 nor to F^1, and $(E \cup F)^1 = E^1 \cup F^1$.