HOMEWORK 1

Problem 1 Let X and Y be non-empty sets. A map $f : X \to Y$ is defined as a subset of $X \times Y$ satisfying the condition

$$\forall\ x \in X\ \exists!\ y \in Y : (x, y) \in f.$$

Let f, g be maps from X to Y. Show that $f \subseteq g \Rightarrow f = g$.

Solution: Assume that $f \subseteq g$. Let $(x_1, y_1) \in g$. Since $f : X \to Y$, for $x_1 \in X$, $\exists y_2 \in Y$ such that $(x_1, y_2) \in f$. Now $f \subseteq g$, so $(x_1, y_2) \in g$ and $(x_1, y_1) \in g$. Hence $y_1 = y_2$ to ensure that g is well defined. Thus $(x_1, y_1) \in f$ implying $g \subseteq f$, and so $f = g$.

Problem 2 Let $(R, +, 0, \cdot, 1)$ be a ring such that $(R, +, 0)$ is a cyclic group $(\mathbb{Z}/N\mathbb{Z}, +, 0)$. Show that R is a commutative ring.

Solution: Consider ab for any $a, b \in R$. We can represent each element of R as a sum of 1's since $(R, +, 0) \cong (\frac{\mathbb{Z}}{N\mathbb{Z}}, +, 0)$. Thus $ab = a(1+1+...+1) = (a1+a1+...+a1) = (1a+1a+...+1a) = (1+1+...+1)a = ba$. Therefore $ab = ba$, so R is commutative.

Problem 3 Construct a noncommutative ring consisting of 8 elements.

Solution: Let R be the set of upper triangular 2x2-matrices with entries in $\frac{\mathbb{Z}}{2\mathbb{Z}}$. It is a ring as a subring of the ring of all 2x2-matrices with entries in $\frac{\mathbb{Z}}{2\mathbb{Z}}$. It is easy to see that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are elements of R which do not commute, so R is a noncommutative ring. The number of elements in R is $2^3 = 8$, and so R is a noncommutative ring of order 8.

Problem 4 Let R be a ring, and r, $s \in R$. Prove that $1 - rs$ is invertible if and only if $1 - sr$ is invertible.

Solution: Assume that $1-rs$ is invertible, then $\exists c \in R$ such that $(1-rs)c = c(1-rs) = 1$. Then

$$(1 - sr)(1 + scr) = (1 - sr) + (1 - sr)scr = (1 - sr) + (s - srs)cr$$
$$= (1 - sr) + s(1 - rs)cr = (1 - sr) + sr = 1 - sr + sr = 1.$$

Thus $(1 - sr)(1 + scr) = 1$, and similarly $(1 + scr)(1 - sr) = 1$. Hence $(1 - sr)$ is invertible. Switching the roles of rs and sr gives the other direction of the proof. Therefore $(1 - rs)$ is invertible if and only if $(1 - sr)$ is invertible.

HOMEWORK 2

Problem 1 Let K be a field and X be a non-empty set. Prove that the set $\mathrm{Map}(X, K)$ with pointwise addition and multiplication is a field if and only if X has one element.

Solution: If $X = \{x\}$, then the rings $\mathrm{Map}(X, K)$ and K are isomorphic via $f \mapsto f(x)$, with the inverse given by $\alpha \mapsto (x \mapsto \alpha)$, so $\mathrm{Map}(X, K)$ is a field. For the other direction, assume that X has at least 2 different elements x_1, x_2. Take $f \in \mathrm{Map}(X, K)$ such that $f(x_1) = 1$ and $f(x_2) = 0$. Then, for any $g \in \mathrm{Map}(X, K)$ we have $(gf)(x_2) = g(x_2)f(x_2) = g(x_2)0 = 0 \neq 1 = 1_{\mathrm{Map}(X,K)}(x_2)$. Hence $f \neq 0$ is not invertible, so $\mathrm{Map}(X, K)$ is not a field.

Problem 2 Show that the subset $M \subseteq M_2(\mathbb{R})$ of all two by two matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is a field with the ring operations of $M_2(\mathbb{R})$.

Solution: Note first that

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

is a linear basis of M. Sending the first matrix I to $1 \in \mathbb{C}$ and the second matrix J to $i \in \mathbb{C}$ defines a linear bijection $f\colon M \to \mathbb{C}$. To see that it preserves the multiplication, we compute:

$$f(aI + bJ)f(a'I + b'J) = (a + bi)(a' + b'i) = aa' - bb' + (ab' + a'b)i$$

$$f\big((aI + bJ)(a'I + b'J)\big) = f\big((aa' - bb')I + (ab' + a'b)J\big) = aa' - bb' + (ab' + a'b)i$$

Since $\mathbb{C}$ is a field and f is a linear bijection preserving the unit and multiplication, M is a field.

Problem 3 Let K be a field of characteristic $p > 0$. Show that $(x + y)^p = x^p + y^p$ for any $x, y \in K$.

Solution: Note first that

$$(x + y)^p = x^p + \sum_{k=1}^{p-1} \left(x^{p-k} y^k \frac{p!}{(p-k)!k!} \right) + y^p.$$

Every term in both $(p-k)!$ and is $k!$ is less than p. Furthermore, since p is a prime, it will remain in $\frac{p!}{(p-k)!k!}$ after dividing out all repeating primes and simplifying the fraction. Hence p divides $\frac{p!}{(p-k)!k!}$, so $\frac{p!}{(p-k)!k!} = 0$ in K, as K is of characteristic p. Thus $(x + y)^p = x^p + 0 + y^p = x^p + y^p$.

Problem 4 Let p be a prime number. Show that $q^p = q$ for any $q \in \mathbb{Z}/p\mathbb{Z}$.

Solution: Since p is prime, $\mathbb{Z}/p\mathbb{Z}\setminus\{0\}$ is a multiplicative group of order $p-1$. The order of every element must divide the order of the group. Hence $q^{p-1} = q^{n\frac{p-1}{n}} = 1^{\frac{p-1}{n}} = 1$, where n is the order of q. It follows that $q^p = q$ for any $q \neq 0$, and the case $q = 0$ is trivial.

HOMEWORK 3

Problem 1 Let k be a field, and $M_n(k)$ the matrix algebra over k. Define E_{ij}, for any $i, j \in \{1, \ldots, n\}$, to be the matrix whose only non-zero entry is one at the intersection of the i-th row with the j-the column. Prove that $E_{ij}E_{kl} = \delta_{jk}E_{il}$, where δ_{jk} is the Kronecker delta.

Solution: For any matrix M, $E_{ij}M$ is a matrix whose i-th row is the j-th row of M and all other rows are zero. Hence $E_{ij}E_{kl} = 0$ for $j \neq k$ and, if $j = k$, then the only non-zero row of $E_{ij}E_{jl}$ is the i-th row whose only non-zero entry is 1 in the l-th column. Therefore, $E_{ij}E_{kl} = \delta_{jk}E_{il}$.

Problem 2 Find a linear bijective map between the k-algebras $A := M_n(\mathrm{Map}(X, k))$ and $B := \mathrm{Map}(X, M_n(k))$ that intertwines their multiplications.

Solution: Since the set of all elmentary matrices is a basis of $M_n(k)$, any element of B can be uniquely written as $\sum_{i,j=1}^{n} f_{ij}E_{ij}$, where $f_{ij} \in \mathrm{Map}(X, k)$. Let us define a map $F : B \to A$ as follows: $F(f)_{ij}(x) := f_{ij}(x)$. The map is clearly bijective. Also, for any $i, j = 1, \ldots, n$ and any $x \in X$, we obtain:

$$\begin{aligned} F(af + bg)_{ij}(x) &:= (af + bg)_{ij}(x) = af_{ij}(x) + bg_{ij}(x) \\ &= aF(f)_{ij}(x) + bF(g)_{ij}(x) = \big(aF(f) + bF(g)\big)_{ij}(x), \end{aligned}$$

so F is linear. Again, for any $i, j = 1, \ldots, n$ and any $x \in X$, we compute:

$$\begin{aligned} F(fg)_{ij}(x) &:= (fg)_{ij}(x) = \big(f(x)g(x)\big)_{ij} \\ &= \sum_{l=1}^{n} f_{il}(x)g_{lj}(x) = \sum_{l=1}^{n} F(f)_{il}(x)F(g)_{lj}(x) \\ &= \big(\sum_{l=1}^{n} F(f)_{il}F(g)_{lj}\big)(x) = \big(F(f)F(g)\big)_{ij}(x), \end{aligned}$$

so F intertwines the multiplications of B and A.

Problem 3 Let A be a unital finite-dimensional algebra over a field k. Show that

$$a, b \in A, \qquad ab = 1 \quad \Rightarrow \quad ba = 1\,.$$

Solution: $L_a : A \ni x \mapsto ax \in A$ is a linear map. It is surjective because for any $y \in A$ we have $L_a(by) = aby = y$. Since $\dim A < \infty$, and the dimension of the image of a linear map from a finite-dimensional vector space is equal to the dimension of this space minus the dimension of the kernel of the map, we conclude that $\dim A - \dim\ker L_a = \dim L_a(A) = \dim A$. Hence $\dim\ker L_a = 0$, so $\ker L_a = 0$, whence L_a is also injective. Therefore, as $aba = a$ means $L_a(ba) = L_a(1)$, we conclude that $ba = 1$.

Problem 4 Take $V := \{f \in \mathrm{Map}(\mathbb{R}, \mathbb{R}) \mid f(-x) = -f(x) \quad \forall\, x \in \mathbb{R}\}$ as a vector space with respect to the pointwise structure. Prove or disprove that V is an algebra with respect to the pointwise multiplication.

Solution: The identity map is in V, but its pointwise square $x \mapsto x^2$ is not because $(-x)^2 = x^2 \neq -x^2$ for $x \neq 0$. Hence V is not an algebra with respect to the pointwise multiplication.

HOMEWORK 4

Problem 1 Let $I := (2,5)R$ be an ideal of the ring $R = \mathbb{Z}/16\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$. Count the number of elements in the quotient ring R/I.

Solution: The ring R has $16{\cdot}15$ elements. There are 8 elements in $2\mathbb{Z}/16\mathbb{Z}$ and 3 elements in $5\mathbb{Z}/15\mathbb{Z}$, so the ideal I has $8 \cdot 3$ many elements. Consequently, the number of elements in any equivalence class defined by I will also have $8 \cdot 3$ many elements. Hence the number of elements in R/I is $(16 \cdot 15)/(8 \cdot 3) = 10$.

Problem 2 Prove that, in the algebra $M_2(k)$ over a field k, all non-zero proper left ideals are of the form

$$\begin{pmatrix} ax & ay \\ bx & by \end{pmatrix}, \text{ where } (x,y) \neq 0 \text{ and } a,b \in k \text{ are arbitrary.} \tag{0.1}$$

Likewise, show that all non-zero proper right ideals are of the form

$$\begin{pmatrix} ax & bx \\ ay & by \end{pmatrix}, \text{ where } (x,y) \neq 0 \text{ and } a,b \in k \text{ are arbitrary.}$$

Solution: Let I be a proper non-zero left ideal of $M_2(k)$. Since it is proper, all its elements must be non-invertible, which means that their rows are linearly dependent. Hence these are matrices that can be written as in (0.1), which is

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix}. \tag{0.2}$$

As $I \neq 0$, it contains an element M given by $(a,b) \neq 0$, say, with $a \neq 0$. Then an arbitrary matrix of the form (0.1) can obtained by the left multiplication on M:

$$\begin{pmatrix} p \\ q \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix}. \tag{0.3}$$

If $b \neq 0$, we take the row $(0\ b^{-1})$. Hence I must be the set of all matrices of the form (0.1). It is a left ideal because it is a vector subspace of $M_2(k)$ that is closed under the left multiplication. The right-hand-side case is completely analogous, and can be obtained by applying the matrix transposition to the above reasoning.

Problem 3 Let I be an ideal of a k-algebra A and let $X \neq \emptyset$. Show that

$$\widetilde{I} := \{f \in \mathrm{Map}(X,A) \mid f(X) \subseteq I\}$$

is an ideal of the pointwise algebra $\mathrm{Map}(X,A)$.

Solution: For any $f, g \in \tilde{I}$, $h \in \mathrm{Map}(X,A)$, $m \in k$, and $x \in X$, we have

$$(f+g)(x) = f(x) + g(x) \in I, \quad (mf)(x) = mf(x) \in I,$$
$$(fh)(x) = f(x)h(x) \in I, \quad (hf)(x) = h(x)f(x) \in I.$$

Hence mf, $f+g$, hf, $fh \in \tilde{I}$. Finally, as $0 \in I$, the constant zero function is in $\tilde{I}$.

Problem 4 Prove that, if R is a commutative ring with a unique non-zero proper ideal I, then R/I is a field.

Solution: Since we know that R/I is a commutative ring, it suffices to show that every non-zero element of R/I is invertible. Let $[x] \in R/I$ be non-zero, so $x \notin I$. We show that the additive subgroup $K := I + Rx$ is an ideal of R. Indeed, for any $j \in I$ and $r, r' \in R$, we have that

$$r'(j + rx) = r'j + (r'r)x \in K, \quad (j + rx)r' = jr' + (rr')x \in K.$$

Here we used that I is an ideal and R is commutative. Furthermore, since $I \subseteq K$ and I is the unique non-zero proper ideal, we conclude that $K = R$. Hence $K \ni 1 = j + sx$ for some $j \in I$ and $s \in R$, so, again by commutativity of R,

$$(x + I)(s + I) = (s + I)(x + I) = sx + I = 1 + I.$$

Therefore $[s]$ is the multiplicative inverse of $[x]$ in R/I.

HOMEWORK 5

Problem 1 Let I and J be ideals of a k-algebra A. Prove that $I \cap J$, $I + J$, and

$$IJ := \left\{ \sum_{i \in F} \alpha_i x_i y_i \mid x_i \in I, y_i \in J, \alpha_i \in k, F \text{ is a finite set} \right\}$$

are ideals of A.

Solution: Let I, J be ideals of a k-algebra A. All sets $I \cap J$, $I + J$ and IJ are, clearly, vector subspaces of A. Take $j \in I \cap J$ and $r \in A$. Since $rj, jr \in I$ and $rj, jr \in J$, we infer that $rj, jr \in I \cap J$, so $I \cap J$ is an ideal of A. Next, $A(I + J) \subseteq I + J$ and $(I + J)A \subseteq I + J$, so $I + J$ is an ideal of A. Finally, take $\phi := \sum_{i \in F} \alpha_i x_i y_i \in IJ$. Then, for any $r \in A$, we obtain

$$r\phi = r \sum_{i \in F} \alpha_i x_i y_i = \sum_{i \in F} \alpha_i (r x_i) y_i \in IJ,$$

$$\phi r = \left(\sum_{i \in F} \alpha_i x_i y_i \right) r = \sum_{i \in F} \alpha_i x_i (y_i r) \in IJ.$$

Hence IJ is an ideal of A.

Problem 2 Show an example of a homomorphism of monoids that is <u>not</u> injective, but whose kernel is trivial (i.e., it contains only of the neutral element).

Solution: Consider a homomorphism of additive monoids

$$f : \mathbb{N} \oplus \mathbb{N} \ni (m, n) \longmapsto m + n \in \mathbb{N},$$

where the addition in $\mathbb{N} \oplus \mathbb{N}$ is defined componentwise. It is not injective because $f((3,1)) = f((2,2))$. However, its kernel is still trivial because

$$f((n_1, n_2)) = n_1 + n_2 = 0$$

implies that $n_1 = n_2 = 0$.

Problem 3 Let $A \xrightarrow{f} B$ be a homomorphism of algebras, and let C be a subalgebra of B. Prove that $f^{-1}(C)$ is a subalgebra of A.

Solution: For any $u, v \in f^{-1}(C)$, $m \in k$, we have that $f(u + v) = f(u) + f(v) \in C$, $f(mu) = mf(u) \in C$, and $f(uv) = f(u)f(v) \in C$, because f is an algebra homomorphism and C is a subalgebra of B. Hence $f^{-1}(C)$ is a subalgebra of A.

Problem 4 Let $\mathbb{Z} \xrightarrow{f} \mathbb{Z}$ be a homomorphism of groups given by $f(m) := 3m$. Compute the preimage subgroup $f^{-1}(6\mathbb{Z})$.

Solution: Since $a \in f^{-1}(6\mathbb{Z}) \iff 3a \in 6\mathbb{Z} \iff a \in 2\mathbb{Z}$, we conclude that

$$f^{-1}(6\mathbb{Z}) = 2\mathbb{Z}.$$

HOMEWORK 6

Problem 1 Show that the short exact sequence

$$0 \longrightarrow n\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

does <u>not</u> split for any $n \in \mathbb{N} \setminus \{0, 1\}$.

Solution: Suppose that there exists a spliting homomorphism $s : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$. Then

$$ns([1]) = s([n]) = s(0) = 0 \;\Rightarrow\; s([1]) = 0 \;\Rightarrow\; s = 0 \;\Rightarrow\; 0 = f \circ s = \mathrm{id},$$

which is a contradiction. Here, we used $n \neq 0$ in the first implication, and we used $n \neq 1$ to claim $\mathrm{id} \neq 0$.

Problem 2 Show that a short exact sequence $0 \to K \to M \xrightarrow{f} N \to 0$ of vector spaces and linear maps always splits.

Solution: Let $\{e_i\}_{i\in I}$ be a basis of N. Since f is surjective, $\forall\, i \in I\; \exists\, m_i \in f^{-1}(e_i)$. Now we can define a linear map $s : N \to M$ by assigning m_i to each e_i. The map s is a splitting because, for any basis element e_i, we obtain

$$(f \circ s)(e_i) = f(s(e_i)) = f(m_i) = e_i\,,$$

which is equivalent to $f \circ s = \mathrm{id}$.

Problem 3 Let $0 \to K \to M \to N \to 0$ be a short exact sequence of vector spaces and linear maps. Prove that, if $\dim M < \infty$, then $\dim M = \dim K + \dim N$.

Solution: From Problem 2, we know that the above sequence splits. Spliting of the sequence implies that $M \cong K \oplus N$, so the claim follows.

Problem 4 Show an example of a non-zero polynomial $\alpha \in \mathbb{Z}/2\mathbb{Z}\,[\mathbb{N}]$ whose polynomial function $f_\alpha \in \mathrm{Map}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is zero.

Solution: Consider $0 \neq \alpha = x + x^2 \in \mathbb{Z}/2\mathbb{Z}\,[\mathbb{N}]$. Its polynomial function f_α is defined by $f_\alpha(r) = \mathrm{ev}_r(\alpha)$. Since

$$f_\alpha(0) = \mathrm{ev}_0(x + x^2) = 0 + 0 = 0, \quad f_\alpha(1) = \mathrm{ev}_1(x + x^2) = 1 + 1 = 0,$$

we infer that $f_\alpha = 0$.

HOMEWORK 7

Problem 1 Show that the only bijection $f : \mathbb{N} \to \mathbb{N}$ that preserves the order ($m \geq n \Rightarrow f(m) \geq f(n)$) is the identity map.

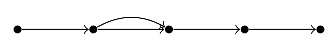
Solution: Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection different from the identity map. Then there exists the smallest $i \in \mathbb{N}$ such that $f(i) \neq i$. In fact, $f(i) > i$. Indeed, if $i = 0$, it is obvious. Otherwise, if $f(i) = j$ for $j < i$, then $f(j) = j = f(i)$, which contradicts the injectivity of f. Next, since f is order preserving, for any $k \geq i$, we have $f(k) \geq f(i) > i$. This shows that i is not in the image of f, as for $j < i$ we have $f(j) = j \neq i$ and $f(i) \neq i$. Thus we obtain a contradiction to the surjectivity of f.

Problem 2 Let $P(X)$ be the set of all subsets of a set X, and let $f : P(X) \to P(X)$ be an inclusion-preserving map: $A \subseteq B \implies f(A) \subseteq f(B)$. Prove that there exists $C \in P(X)$ such that $f(C) = C$.

Solution: It follows from $\emptyset \subseteq f(\emptyset)$ that $Y := \{A \in P(X) \,|\, A \subseteq f(A)\} \neq \emptyset$. Define $C := \cup_{A \in Y} A$. Then $C \subseteq \cup_{A \in Y} f(A)$. Furthermore, as $A \subseteq C$ for any $A \in Y$, we infer that $\cup_{A \in Y} f(A) \subseteq f(C)$. Consequently, $C \subseteq f(C)$. Hence $f(C) \subseteq f(f(C))$, so $f(C) \in Y$. Therefore $f(C) \subseteq C$, and we conclude the desired equality: $f(C) = C$.

Problem 3 Construct a graph with 5 edges and no loops such that there are exactly 5 different paths of length 2.

Solution:



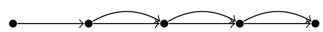
Problem 4 Let E be a finite graph whose all vertices emit at least one edge. Prove that there is a loop in E.

Solution: If every vertex emits at least one edge, then there exists an infinite path. This in turn means that the set of all finite paths $FP(E)$ is infinite. However, we proved that $FP(E)$ is finite if and only if there are no loops in E. Hence the claim follows.

HOMEWORK 8

Problem 1 Construct a graph with 7 edges, no loops, and whose longest path is longer than 3, but for which the number of paths of length 3 is still maximal.

Solution:

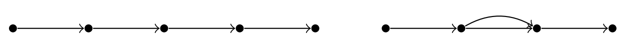


Problem 2 Let $x, y \in \mathbb{R}$ and $x + y = 1$. Find x and y for which the product $p := xy$ is maximal.

Solution: The product p is given by the parabola $p(x) = x(1-x) = x - x^2$, which is non-negative only on the interval $[0, 1]$. Hence it reaches its maximum at a point in $[0, 1]$. As $p'(x) = 1 - 2x = 0$ has its unique solution $x = \frac{1}{2}$, we conclude that p is maximized by $x = \frac{1}{2} = y$.

Problem 3 Construct all graphs with 4 edges, no loops, and all vertices in the image of the source map or the target map, such that the number of *all* positive-length finite paths is maximal.

Solution:



Problem 4 Let x and y be elements in a noncommutative $\mathbb{R}$-algebra satisfying $qxy = yx$ for some $q \in \mathbb{R} \setminus \{0, \pm 1\}$. Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}, \quad n \geq 0,$$

where

$$\binom{n}{k}_q := \frac{\prod_{m=0}^{k-1}(1-q^{n-m})}{\prod_{m=1}^{k}(1-q^m)} \quad \text{for } k \leq n \text{ and } \binom{l}{0}_q := 1 \quad \text{for } l \in \mathbb{N}.$$

Solution: We proceed by induction. For $n = 0$, both sides are equal to 1, and for $n = 1$, both sides are equal to $x + y$. For the induction step, we need to prove an analog of the Pascal identity. For $2 \le k \le n$, we compute:

$$\begin{aligned}
q^k \binom{n}{k}_q + \binom{n}{k-1}_q &= q^k \frac{\prod_{m=0}^{k-1}(1-q^{n-m})}{\prod_{m=1}^{k}(1-q^m)} + \frac{\prod_{m=0}^{k-2}(1-q^{n-m})(1-q^k)}{\prod_{m=1}^{k-1}(1-q^m)(1-q^k)} \\
&= \frac{\prod_{m=0}^{k-2}(1-q^{n-m})}{\prod_{m=1}^{k}(1-q^m)} \left(q^k(1-q^{n-k+1}) + 1 - q^k\right) \\
&= \frac{\prod_{m=1}^{k-1}(1-q^{n+1-m})}{\prod_{m=1}^{k}(1-q^m)} \left(1-q^{n+1}\right) \\
&= \frac{\prod_{m=0}^{k-1}(1-q^{n+1-m})}{\prod_{m=1}^{k}(1-q^m)} \\
&= \binom{n+1}{k}_q .
\end{aligned}$$

For $k = 1$, we have:

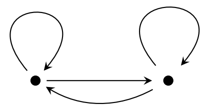
$$q\binom{n}{1}_q + \binom{n}{0}_q = q\frac{1-q^n}{1-q} + 1 = \frac{q(1-q^n)+1-q}{1-q} = \frac{1-q^{n+1}}{1-q} = \binom{n+1}{1}_q .$$

Now, we can prove the induction step:

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n = (x+y)\sum_{k=0}^{n}\binom{n}{k}_q x^k y^{n-k} \\
&= \sum_{k=0}^{n}\left(\binom{n}{k}_q x^{k+1}y^{n-k} + \binom{n}{k}_q q^k x^k y^{n+1-k}\right) \\
&= \sum_{k=1}^{n+1}\binom{n}{k-1}_q x^k y^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k}_q q^k x^k y^{n+1-k} \\
&= \binom{n}{0}_q y^{n+1} + \sum_{k=1}^{n}\left(\binom{n}{k-1}_q + q^k\binom{n}{k}_q\right) x^k y^{n+1-k} + \binom{n}{n}_q x^{n+1} \\
&= \binom{n+1}{0}_q y^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}_q x^k y^{n+1-k} + \binom{n+1}{n+1}_q x^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}_q x^k y^{n+1-k} .
\end{aligned}$$

HOMEWORK 9

Problem 1 For the graph given below, find the number of all paths of length $k \geq 2$.



Solution: The adjacency matrix for the above graph is

$$A(E) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

We prove by induction that, for any $k \geq 1$, we have

$$A(E)^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{bmatrix}.$$

Indeed, the result holds for $k = 1$, and the computation

$$\begin{bmatrix} 2^{k-1} & 2^{k-1} \\ 2^{k-1} & 2^{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2^{k} & 2^{k} \\ 2^{k} & 2^{k} \end{bmatrix}$$

proves the inductive step. Hence, the number of all k-paths in E equals

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \left(A(E)^k\right)_{ij} = 4 \cdot 2^{k-1} = 2^{k+1}.$$

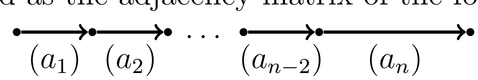
Problem 2 Interpreting the matrix

$$A(E) = \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & & & \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & & a_n \\ 0 & & & \dots & & 0 \end{pmatrix}$$

as the adjacency matrix of a certain graph E, prove that

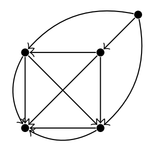
$$A(E)^n = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \prod_{i=1}^{n} a_i \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 & & & \dots & & 0 \end{pmatrix}.$$

Solution: $A(E)$ can be viewed as the adjacency matrix of the following graph E:



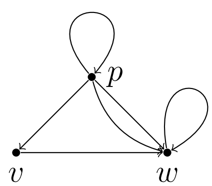
There are $\prod_{i=1}^{n} a_i$ many paths of length n, so the sum of all entires of $A(E)^n$ equals $\prod_{i=1}^{n} a_i$. Furthermore, as all paths of length n start at the first vertex and end at the n-th vertex, the only non-zero entry of $A(E)^n$ is the last entry in the first row. Hence $A(E)^n$ is of the claimed form.

Problem 3 Prove that the adjacency matrix of the graph given below raised to the 5-th power is zero.



Solution: Call the above graph E. Since there are no loops in E and E is finite, there exists a longest path. Any longest path must end in a sink. There is only one sink in E, and one can easily check that its longest path is of length 4. Hence, there are no paths of length 5, so the adjacency matrix of E raised to the 5-th power is zero.

Problem 4 Find all hereditary subsets and all saturated subsets for the graph given below.



Solution: Call the above graph E, and consider all possible subsets of E^0.

(a) $\{p\} \subseteq E^0$ is not hereditary because p emits arrows that end not at p. It is saturated since there are no arrows ending at p which start not at p.

(b) $\{v\} \subseteq E^0$ is not hereditary because v emits an arrow that ends not at v. It is saturated since the only arrow ending in v starts at p which emits arrows ending not at v.

(c) $\{w\} \subseteq E^0$ is hereditary because w emits only an edge ending at w. It is not saturated since v emits only one edge, and the edge ends at w.

(d) $\{p, v\} \subseteq E^0$ is not hereditary because both p and v emit an arrow that ends in w. It is saturated since w emits only one edge ending at w.

(e) $\{p, w\} \subseteq E^0$ is not hereditary because p emits an arrow that ends at v. It is also not saturated since v emits only one edge, and the edge ends at w.

(f) $\{v, w\} \subseteq E^0$ is hereditary because v emits only an arrow that ends at wm, and w emits only one edge, and the edge ends at w. It is also saturated since there is an edge emitted by p that ends at p.

(g) Both $\emptyset$ and E^0 are hereditary and saturated.

Conclusion: There are four herditary subsets $\emptyset, \{w\}, \{v, w\}, E^0$, and six saturated subsets $\emptyset, \{p\}, \{v\}, \{p, v\}, \{v, w\}, E^0$.

HOMEWORK 10

Problem 1 Let $(f_0, f_1) : E \to F$ a graph homomorphism with both f_0 and f_1 bijective. Prove that $(f_0^{-1}, f_1^{-1}) : F \to E$ is a graph homomorphism.

Solution: Since $(f_0, f_1) : E \to F$ is a graph homomorphism, we have

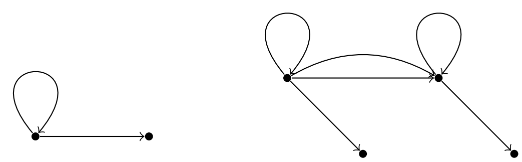
$$s_F \circ f_1 = f_0 \circ s_E, \qquad t_F \circ f_1 = f_0 \circ t_E \,.$$

Composing both of the above equlities with f_0^{-1} on the left and f_1^{-1} on the right yields

$$f_0^{-1} \circ s_F = s_E \circ f_1^{-1}, \qquad f_0^{-1} \circ t_F = t_E \circ f_1^{-1} \,.$$

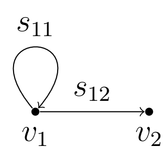
This means that $(f_0^{-1}, f_1^{-1}) : F \to E$ is a graph homomorphism.

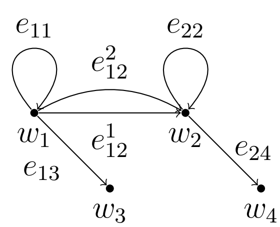
Problem 2 Consider two graphs E and F:



Define two injective graph homomorphisms from E to F such that one of them is an admissible inclusion and the other one is not.

Solution: Let us label the vertices and edges in both graphs as follows:





First, we check that the inclusion

$$\iota_1 : E \to F, \quad v_1 \mapsto w_1, \quad v_2 \mapsto w_3, \quad s_{11} \mapsto e_{11}, \quad s_{12} \mapsto e_{13} \,,$$

is admissible: $F^0 \setminus \iota_1(E^0) = F^0 \setminus \{w_1, w_3\} = \{w_2, w_4\}$ is both hereditary (w_4 is a sink and all paths starting at w_2 end at w_2 or w_4) and saturated (w_3 is a sink and w_1 emits a loop), and

$$F^1 \setminus t_F^{-1}(F^0 \setminus \iota_1(E^0)) = t_F^{-1}(\iota_1(E^0)) = \{e_{11}, e_{13}\} = \iota_1(E^1) \,.$$

Next, we consider the inclusion

$$\iota_2 : E \to F, \quad v_1 \mapsto w_2, \quad v_2 \mapsto w_4, \quad s_{11} \mapsto e_{22}, \quad s_{12} \mapsto e_{24} \,,$$

which is not admissible because $F^0 \setminus \iota_2(E^0) = \{w_1, w_3\}$ is not hereditary (w_1 emits e_{12}^1 which ends at w_2).

Problem 3 Prove that the path algebra over a field k of the graph

is isomorphic to the polynomial algebra $k[\mathbb{N}]$.

Solution: Call the above graph E, denote the vertex in E by v and the loop in E by α . By definition, $\{\chi_v, \chi_\alpha, \chi_{\alpha^2}, \ldots\}$ is a basis of the path algebra kE. Note first that $\chi_v = 1$ because $\chi_v\chi_{\alpha^i} = \chi_{\alpha^i} = \chi_{\alpha^i}\chi_v$ for all $i \in \mathbb{N}$. Next, the multiplication is given by $\chi_{\alpha^i}\chi_{\alpha^j} = \chi_{\alpha^{i+j}}$. Much in the same way, $\{1, x, x^2, \ldots\}$ is a basis of $k[\mathbb{N}]$. Here

$$x^m(n) := \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

and the convolution product of two basis elements reads $x^i * x^j = x^{i+j}$. Hence, the linear bijection determined by

$$\varphi : k[\mathbb{N}] \longrightarrow kE, \quad 1 \longmapsto \chi_v, \quad x^i \longmapsto \chi_{\alpha^i},\ i \in \mathbb{N} \setminus \{0\},$$

is an algebra isomorphism.

Problem 4 Prove that the path algebra over a field k of the graph

is isomorphic to the algebra of upper triangular 2×2 matrices over k.

Solution: The algebra of upper triangular 2×2 matrices over k admits the following basis:

$$E_{11} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{22} := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The multiplication is given by

$\cdot$	E_{11}	E_{12}	E_{22}
E_{11}	E_{11}	E_{12}	0
E_{12}	0	0	E_{12}
E_{22}	0	0	E_{22}

Next, call the above graph E and denote by v its left vertex, by e its edge, and by w its right vertex. Then, by definition, $\{\chi_v, \chi_e, \chi_w\}$ is a basis of kE. The multiplication is given by

$\cdot$	χ_v	χ_e	χ_w
χ_v	χ_v	χ_e	0
χ_e	0	0	χ_e
χ_w	0	0	χ_w

Hence the linear map determined by

$$E_{11} \mapsto \chi_v, \quad E_{12} \mapsto \chi_e, \quad E_{22} \mapsto \chi_w,$$

is an algebra isomorphism.

HOMEWORK 11

Problem 1 Let $E = (E^0, E^1, s, t)$ be a graph and k be a field. Prove that the path algebra kE is unital if and only if E^0 is finite.

Solution: Assume that E^0 is finite. Then

$$1_{kE} = \sum_{v \in E^0} \chi_v.$$

Indeed, for any $p \in FP(E)$, we have

$$\chi_p \left(\sum_{v \in E^0} \chi_v \right) = \sum_{v \in E^0} \chi_p \chi_v = \sum_{v \in E^0} \chi_{pv} = \chi_{pt(p)} = \chi_p \,,$$

$$\left(\sum_{v \in E^0} \chi_v \right) \chi_p = \sum_{v \in E^0} \chi_v \chi_p = \sum_{v \in E^0} \chi_{vp} = \chi_{s(p)p} = \chi_p \,.$$

Assume now that kE is unital. Then 1_{kE} can be expressed as a finite linear combination of some basis elements:

$$1_{kE} = \sum_{i=1}^{n} \lambda_i \chi_{p_i} \,,$$

where each p_i is a path in E. Since $1_{kE}\chi_v = \chi_v \neq 0$ for all $v \in E^0$, we infer that that $E^0 \subseteq \{t(p_i)\}_{i=1}^n$. Consequently, E^0 is finite because any subset of a finite set is finite.

Problem 2 Let $E = (E^0, E^1, s, t)$ be a graph and let k be a field. Prove that the path algebra kE is commutative if and only if $E^1 = \emptyset$ or each edge is a loop starting/ending at a different vertex.

Solution: Assume that $E^1 = \emptyset$. Then, for all $v, w \in E^0$, $v \neq w$, $\chi_v \chi_w = 0 = \chi_w \chi_v$, so kE is commutative. Now let E^1 consist only of loops starting at different vertices. Then, for any $p, q \in FP(E)$, with $t(p) \neq t(q)$, we have $\chi_p \chi_q = 0 = \chi_q \chi_p$. If $t(p) = t(q)$, then $\chi_p \chi_q = \chi_{pq} = \chi_{qp} = \chi_q \chi_p$. Hence kE is commutative. To prove the opposite implication, we need to negate the following statement:

$$E^1 = \emptyset \text{ or } \big(E^1 \neq \emptyset \text{ and } \forall\, e \in E^1 : s(e) = t(e) \text{ and } (e \neq f \Rightarrow s(e) \neq s(f))\big).$$

The negation reads

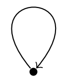
$$\exists\, e \in E^1 : s(e) \neq t(e) \text{ or } (e \neq f \text{ and } s(e) = s(f)).$$

First assume that there is an edge e which is not a loop. Then

$$\chi_{s(e)} \chi_e = \chi_e \neq 0 = \chi_e \chi_{s(e)} \,,$$

so kE is not commutative. Next, if there are two different edges e and f such that they are loops and $s(e) = s(f)$, then $\chi_e \chi_f \neq \chi_f \chi_e$ because they are two different paths. Hence, again, kE is noncommutative.

Problem 3 Prove that the Leavitt path algebra over a field k of the graph



is isomorphic to the Laurent polynomial algebra $k[\mathbb{Z}]$.

Solution: Call the above graph E and denote by v the vertex of E and by α the edge of E. Then $[\chi_v] = 1$ in the Leavitt path algebra $L_k(E)$, and $L_k(E)$ is spanned by $B := \{1, [\chi_\alpha], [\chi_{\alpha^2}], [\chi_{\alpha^*}], [\chi_{(\alpha^*)^2}], \ldots\}$ because $[\chi_{\alpha^*\alpha}] = 1 = [\chi_{\alpha\alpha^*}]$. The set B is also linearly independent by Corollary 1.5.12 in *Leavitt Path Algebras*, so it is a basis of $L_k(E)$. The multiplication of elements of B is given by

$$[\chi_{\alpha^i}][\chi_{\alpha^j}] = [\chi_{\alpha^{i+j}}], \quad [\chi_{(\alpha^*)^i}][\chi_{(\alpha^*)^j}] = [\chi_{(\alpha^*)^{i+j}}],$$

$$[\chi_{\alpha^i}][\chi_{(\alpha^*)^j}] = [\chi_{(\alpha^*)^j}][\chi_{\alpha^i}] = \begin{cases} [\chi_{\alpha^{i-j}}] & \text{for } i > j \\ 1 & \text{for } i = j \\ [\chi_{(\alpha^*)^{j-i}}] & \text{for } j > i \end{cases}$$

for all $i, j \in \mathbb{N}$, with the convention that $[\chi_{\alpha^0}] = 1 = [\chi_{(\alpha^*)^0}]$. Next, recall that $\{x^i\}_{i\in\mathbb{Z}}$ is a basis of $k[\mathbb{Z}]$, where

$$x^j(n) := \begin{cases} 1 & n = j \\ 0 & n \neq j \end{cases}.$$

The convolution product for the basis elements reads $x^i * x^j = x^{i+j}$ for all $i, j \in \mathbb{N}$. Hence, the linear map determined by

$$x^i \longmapsto \chi_{\alpha^i} \text{ for } i \geq 0, \quad x^i \longmapsto \chi_{(\alpha^*)^{-i}} \text{ for } i \leq 0,$$

is an algebra isomorphism.

Problem 4 Prove that the Leavitt path algebra over a field k of the graph

$$\bullet \longrightarrow \bullet$$

is isomorphic to the algebra of 2×2 matrices over k.

Solution: The algebra of 2×2 matrices over k admits the following basis:

$$E_{11} := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The multiplication is given by

$\cdot$	E_{11}	E_{12}	E_{21}	E_{22}
E_{11}	E_{11}	E_{12}	0	0
E_{12}	0	0	E_{11}	E_{12}
E_{21}	E_{21}	E_{22}	0	0
E_{22}	0	0	E_{21}	E_{22}

Next, call the above graph E and denote by v its left vertex, by e its edge, and by w its right vertex. Then $B := \{[\chi_v], [\chi_e], [\chi_{e^*}], [\chi_w]\}$ is a basis of the Leavitt path algebra $L_k(E)$ by Corollary 1.5.12 in *Leavitt Path Algebras*. The multiplication of elements of B is given by

$\cdot$	$[\chi_v]$	$[\chi_e]$	$[\chi_{e^*}]$	$[\chi_w]$
$[\chi_v]$	$[\chi_v]$	$[\chi_e]$	0	0
$[\chi_e]$	0	0	$[\chi_v]$	$[\chi_e]$
$[\chi_{e^*}]$	$[\chi_{e^*}]$	$[\chi_w]$	0	0
$[\chi_w]$	0	0	$[\chi_{e^*}]$	$[\chi_w]$

Hence the linear map determined by

$$E_{11} \mapsto [\chi_v], \quad E_{12} \mapsto [\chi_e], \quad E_{21} \mapsto [\chi_{e^*}], \quad E_{22} \mapsto [\chi_w],$$

is an algebra isomorphism.