# Graph algebras

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# Introduction

GOAL: Gaining understanding & developing the ability of precise and refined logical thinking.

MOTIVATION: (after JFK's 1962 speech)

We choose to do mathematics not because it is easy but because it is hard, because it is a challenge we are willing to accept, one we are unwilling to postpone, because it serves to organize and measure the best of our energies and skills.

PLAN:

1. Recap basic knowledge of fields and linear algebra
2. Define and explore the concept of an algebra
3. Graphs (quivers) and path algebras
4. Elements of representation theory
5. Leavitt path algebras

# 1 Rings

## 1.1 Definition, examples, remarks

**Definition 1.1.** *Let $R$ be a non-empty set equipped with two binary operations $R \times R \xrightarrow{+} R$, $R \times R \xrightarrow{\cdot} R$ and two special elements $0, 1 \in R$. The quintuple $(R, +, 0, \cdot, 1)$ is called a ring iff:*

$(R_1)$ $(R, +, 0)$ *is an abelian group, i.e.*

$$\forall\, r, s, t \in R : (r+s)+t = r+(s+t), \quad r+s = s+r, \quad 0+r = r, \quad \forall\, r \in R\ \exists\, r' \in R : r+r' = 0.$$

$(R_2)$ $(R, \cdot, 1)$ *is a monoid, i.e.*

$$\forall\, r, s, t \in R : (r \cdot s) \cdot t = r \cdot (s \cdot t), \quad 1 \cdot r = r = r \cdot 1.$$

$(R_3)$ *The multiplication is distributive over addition, i.e.*

$$\forall\, r, s, t \in R : (r + s) \cdot t = r \cdot t + s \cdot t, \quad r \cdot (s + t) = r \cdot s + r \cdot t.$$

*We say that the ring $R$ is commutative iff the monoid $(R, \cdot, 1)$ is commutative ($r \cdot s = s \cdot r$).*

Elementary observations:

1. If there are at least two elements in a ring $R$, then $0 \neq 1$. Indeed,

$$0 \cdot x = 0 \cdot x + x - x = 0 \cdot x + 1 \cdot x - x = (0+1) \cdot x - x = 1 \cdot x - x = x - x = 0,$$

   so, if $0 = 1$, then $x = 1 \cdot x = 0 \cdot x = 0$ and $0$ is the only element of $R$.

2. $(-1) \cdot x = -x$ because

$$x + (-1) \cdot x = 1 \cdot x + (-1) \cdot x = (1-1) \cdot x = 0 \cdot x = 0$$

   and the claim follows by the uniqueness of the inverse element.

Examples of rings:

1. The ring of integers $(\mathbb{Z}, +, 0, \cdot, 1)$. It is a commutative ring. So is every quotient ring

$$\mathbb{Z}/N\mathbb{Z} := \mathbb{Z}/R_N, \quad R_N := \{(m,n) \in \mathbb{Z}^2 \mid m - n \in N\mathbb{Z}\}.$$

   For $N = 0$, we get $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$, and for $N = 1$ we obtain $\mathbb{Z}/\mathbb{Z} = \{0\}$. For $N = 24$ (or 12), we get the "clock" ring.

2. If $R$ is a ring and $X \neq \emptyset$, then $Map(X, R)$ is a ring with respect to the pointwise operations given by

$$\forall\, x \in X : (f+g)(x) := f(x) + g(x), \quad (f \cdot g)(x) := f(x) \cdot g(x).$$

   The neutral elements are the constant functions $x \mapsto 0$ and $x \mapsto 1$ for the addition and multiplication respectively.

## 1.2 Convolution multiplication

Let $R$ be a ring and $G$ be a monoid. Define

$$R[G] := \{\alpha \in Map(G, R) \mid \alpha(g) \neq 0 \text{ only for finitely many } g \in G\}.$$

The binary operation $R[G] \times R[G] \xrightarrow{*} R[G]$ given by

$$\boxed{(\alpha * \beta)(g) := \sum_{\substack{h,h' \in G \\ h \cdot h' = g}} \alpha(h)\beta(h')}$$

is called the convolution multiplication.

1. It is well defined, i.e. the sum is finite, and again a map with finite support.

2. The convolution is associative:

$$((\alpha * \beta) * \gamma)(g) = \sum_{\substack{h,z \in G \\ hz=g}} (\alpha * \beta)(h)\gamma(z) = \sum_{\substack{h,z \in G \\ hz=g}} \sum_{\substack{x,y \in G \\ xy=h}} \alpha(x)\beta(y)\gamma(z) = \sum_{\substack{x,y,h,z \in G \\ hz=g,\ xy=h}} \alpha(x)\beta(y)\gamma(z)$$
$$= \sum_{\substack{x,y,z \in G \\ xyz=g}} \alpha(x)\beta(y)\gamma(z) = \sum_{\substack{x,y,z,k \in G \\ xk=g,\ yz=k}} \alpha(x)\beta(y)\gamma(z) = \sum_{\substack{x,k \in G \\ xk=g}} \alpha(x) \sum_{\substack{y,z \in G \\ yz=k}} \beta(y)\gamma(z)$$
$$= \sum_{\substack{x,k \in G \\ xk=g}} \alpha(x)(\beta * \gamma)(k) = (\alpha * (\beta * \gamma))(g)\,.$$

The neutral element for $*$ is given by

$$\delta_e(g) := \begin{cases} 0 & \text{for } g \neq e \\ 1 & \text{for } g = e \end{cases},$$

where $e$ is the neutral element of $G$.

Hence $(R[G], *, \delta_e)$ is a monoid.

3. The convolution is distributive with respect to the pointwise multiplication, so $(R[G], +, 0, *, \delta_e)$ is a ring. We call it a monoidal ring. If $G$ is a group, we call it a group ring.

Examples of monoidal rings:

1. For the additive monoid of natural numbers $(\mathbb{N}, +, 0)$, $R[\mathbb{N}]$ is the polynomial ring with coeffitiens in $R$. Elements of $R[\mathbb{N}]$ are called polynomials. A key polynomial is given by

$$x(m) := \begin{cases} 1 & \text{for } m = 1 \\ 0 & \text{for } m \neq 1 \end{cases}.$$

Compute

$$x^n := \underbrace{x * \ldots * x}_{n \text{ times}} \qquad \text{on } m \in \mathbb{N}:$$

$$x^n(m) = \sum_{\substack{m_1,\ldots,m_n \in \mathbb{N} \\ m_1+\ldots+m_n=m}} x(m_1)\ldots x(m_n) = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}.$$

It is natural to denote the convolution neutral element $\delta_e$ as $x^0$:

$$x^0(m) = \begin{cases} 1 & \text{for } m = 0 \\ 0 & \text{for } m \neq 0 \end{cases}.$$

Note that $\forall \alpha \in R[\mathbb{N}]$:

$$\alpha = \sum_{k=0}^{n} \alpha(k) x^k\,,$$

where $(r\beta)(m) := r\beta(m)$ for $r \in R$ and $\beta \in R[\mathbb{N}]$. Indeed,

$$\left(\sum_{k=0}^{n} \alpha(k)x^k\right)(m) = \sum_{k=0}^{n} \alpha(k)x^k(m) = \alpha(m).$$

The convolution multiplication of polynomials:

$$\begin{aligned}
\left(\left(\sum_{k=0}^{m} r_k x^k\right) * \left(\sum_{l=0}^{n} s_l x^l\right)\right)(q) &= \sum_{\substack{q_1+q_2=q\\ q_1,q_2\in\mathbb{N}}} \sum_{k=0}^{m}\sum_{l=0}^{n} \left(r_k x^k(q_1)\right)\left(s_l x^l(q_2)\right) \\
&= \sum_{k=0}^{m}\sum_{l=0}^{n} r_k s_l \sum_{\substack{q_1+q_2=q\\ q_1,q_2\in\mathbb{N}}} \left(x^k(q_1)x^l(q_2)\right) = \left(\sum_{k=0}^{m}\sum_{l=0}^{n} r_k s_l x^{k+l}\right)(q) \\
&= \left| \begin{array}{c} N := k+l \\ l = N-k \end{array} \right| = \left(\sum_{N=0}^{m+n} \left(\sum_{\substack{k+l=N\\ 0\le k\le m\\ 0\le l\le n}} r_k s_l\right) x^{k+l}\right)(q) \\
&= \left(\sum_{N=0}^{m+n} \left(\sum_{\substack{0\le k\le m\\ 0\le N-k\le n}} r_k s_{N-k}\right) x^N\right)(q) \\
&= \left(\sum_{N=0}^{m+n} \left(\sum_{\substack{0\le k,\, N-n\le k,\\ k\le m,\, k\le N}} r_k s_{N-k}\right) x^N\right)(q) \\
&= \left(\sum_{N=0}^{m+n} \left(\sum_{k=\max\{0,N-n\}}^{\min\{m,N\}} r_k s_{N-k}\right) x^N\right)(q)\,.
\end{aligned}$$

<u>An example</u>: $R = \mathbb{Z}$, $m = 2$, $n = 3$, $r_k = k$, $s_l = l^2$. Then

$$\begin{aligned}
\left(\sum_{k=0}^{2} kx^k\right) * \left(\sum_{l=0}^{3} l^2 x^l\right) &= \sum_{N=0}^{5} \left(\sum_{k=\max\{0,N-3\}}^{\min\{2,N\}} k(N-k)^2\right) x^N \\
&= \sum_{k=0}^{0} 0\cdot(0-0)x^0 + \sum_{k=0}^{1} k(1-k)^2 x + \sum_{k=0}^{2} k(2-k)^2 x^2 + \\
&\quad + \sum_{k=0}^{2} k(3-k)^2 x^3 + \sum_{k=1}^{2} k(4-k)^2 x^4 + \sum_{k=2}^{2} k(5-k)^2 x^5 \\
&= x^2 + (4+2)x^3 + (9+8)x^4 + 18x^5 = x^2 + 6x^3 + 17x^4 + 18x^5\,.
\end{aligned}$$

Computing directly, we obtain

$$\begin{aligned}
(x+2x^2)(x+4x^2+9^3) &= x^2 + 4x^3 + 9x^4 + 2x^3 + 8x^4 + 18x^5 \\
&= x^2 + 6x^3 + 17x^4 + 18x^5\,.
\end{aligned}$$

Evaluation at $r \in R$ is a map

$$R[\mathbb{N}] \ni \sum_{k=0}^{n} s_k x^k \overset{\mathrm{ev}_r}{\longmapsto} \sum_{k=0}^{n} s_k r^k \in R\,.$$

The polynomial function $f_\alpha$ of a polynomial $\alpha \in R[\mathbb{N}]$ is the map

$$R \ni r \overset{f_\alpha}{\longmapsto} f_\alpha(r) := \mathrm{ev}_r(\alpha) \in R\,.$$

2. For the additive group of integers $(\mathbb{Z}, +, 0)$, the group ring $R[\mathbb{Z}]$ is the ring of Laurent polynomials with coefficients in $R$. A Laurent polynomial $\alpha$ can be written as

$$\alpha = \sum_{k=-m}^{n} \alpha(k)x^k,$$

where $x^{-1}$ is defined by

$$x^{-1}(m) := \begin{cases} 1 & \text{for } m = -1 \\ 0 & \text{for } m \neq -1 \end{cases}.$$

Clearly, $x^{-1}$ is the inverse of $x$:

$$\left(x^{-1} * x\right)(m) = \sum_{\substack{k,l\in\mathbb{Z} \\ k+l=m}} x^{-1}(k)x(l) = \begin{cases} 1 & \text{for } m = 0 \\ 0 & \text{for } m \neq 0 \end{cases}.$$

3. Taking again the monoid $(\mathbb{N}, +, 0)$ but replacing $R[\mathbb{N}]$ with $R[[\mathbb{N}]] := \mathrm{Map}(\mathbb{N}, R)$, we obtain the ring of formal power series: $\sum_{k=0}^{\infty} \alpha_k x^k$.

# 2 Fields

**Definition 2.1.** *A non-zero commutative ring whose every non-zero element is invertible is called a field. The least natural number $n$ such that*

$$\underbrace{1 + \ldots + 1}_{n-\textit{times}} = 0$$

*is called the characteristic of a field. If it does not exist, we say that the characteristic is zero.*

Elementary observations:

1. A ring $(K, +, 0, \cdot, 1)$ is a field $\iff$ $(K \setminus \{0\}, \cdot, 1)$ is an abelian group.

2. The characteristic of a field is either zero or a prime number. Indeed, suppose the contrary. Then $n = kl$ with $1 < k, l < n$, so both $k \cdot 1$ and $l \cdot 1$ are non-zero, whence invertible. Therefore, $0 = n \cdot 1 = (kl) \cdot 1 = (k \cdot 1)(l \cdot 1)$, so $0 \cdot (l \cdot 1)^{-1}(k \cdot 1)^{-1} = 1$, i.e. $0 = 1$, which contradicts $(K \setminus \{0\}, \cdot, 1)$ being a group.

Examples of fields:

1. The field $\mathbb{Q}$ of rational numbers (more generally, a field of fractions).
2. The field $\mathbb{R}$ of real numbers (more generally, a metric closure of a field, e.g. the field $\mathbb{Q}_p$ of $p$-adic numbers).
3. The field $\mathbb{C}$ of complex numbers (more generally, the algebraic closure of a field).
4. $\mathbb{Q} + \sqrt{2}\,\mathbb{Q}$ (more generally, an algebraic extension of a field).
5. $\mathbb{Z}/p\mathbb{Z}$, where $p$ is a prime number (more generally, a finite field).

**Theorem 2.2.** *The ring $\mathbb{Z}/N\mathbb{Z}$ is a field if and only if $N$ is a prime number.*

*Proof.* If $N$ is not a prime number, then $N = kl$ with $1 < k, l < N$, so both $(k \cdot 1)$ and $(l \cdot 1)$ are non-zero and non-invertible. Hence $\mathbb{Z}/N\mathbb{Z}$ is not a field. Assume now that $N$ is a prime number. Since $\mathbb{Z}/N\mathbb{Z}$ is commutative. it suffices to show that

$$\forall\, k \in \{1, \ldots, N-1\}\; \exists\, l \in \{1, \ldots, N\} : kl = 1 \bmod N, \text{ i.e. } kl = mN + 1 \text{ for some } m \in \mathbb{N}.$$

Consider a map

$$f_k : \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z} : f_k(l \cdot 1) := (k \cdot 1)(l \cdot 1) = (kl) \cdot 1.$$

If $f_k(l \cdot 1) = f_k(l' \cdot 1)$, then $kl = kl' + mN$ for some $m$, i.e. $k(l - l') = mN$ for some $m$. Hence $l - l' = m'N$ for some $m'$ because $N$ is a prime number not dividing $k$. This means that $l \cdot 1 = l' \cdot 1$, so $f_k$ is injective. Now as $\mathbb{Z}/N\mathbb{Z}$ is finite, it is also surjective. Thus we proved that

$$\forall\, k \in \{1, \ldots, N-1\}\; \exists\, l \in \{1, \ldots, N\} : kl = 1 \bmod N, \text{ i.e. } kl = mN + 1 \text{ for some } m \in \mathbb{N}.$$

□

**Theorem 2.3.** *Let $K$ be a finite field. Then the number of elements in $K$ is $p^n$, where $p$ is the characteristic of $K$ and $n$ is a positive integer.*

*Proof.* Note first that $K$ is a vector space of $\mathbb{Z}/p\mathbb{Z}$:

$$\mathbb{Z}/p\mathbb{Z} \times K \longrightarrow K : (n \cdot 1, x) \longmapsto (n \cdot 1_K)x\,.$$

(The map is well-defined because $p$ is the characterstic of $K$.) Since $K$ is finite, it is finite dimensional over $\mathbb{Z}/p\mathbb{Z}$. As every element of $K$ can be uniquely written as $\sum_{i=1}^{n} \alpha_i e_i$, where $\{e_i\}_1^n$, is a basis of $K$, there are $p^n$ elements in $K$. □

The 4-element field $K_4$:

As a vector space over $\mathbb{Z}/2\mathbb{Z}$ and as an abelian group, $K_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We adopt the notation $0 := (0,0)$, $1 := (1,1)$, $a := (1,0)$, $b := (0,1)$. We need a multiplication table

| $\cdot$ | $a$ | $b$ |
|---|---|---|
| $a$ | | |
| $b$ | | |

.

As both $a$ and $b$ are invertible, the only values for this table are from $\{1, a, b\}$. We have $ab = 1$, as $ab \neq a$ and $ab \neq b$. Hence $a = a \cdot 1 = a(a+b) = a^2 + ab = a^2 + 1$, so $a^2 = a - 1 = -b = b$. Much in the same way, $b^2 = a$. Thus we obtain:

| $\cdot$ | $a$ | $b$ |
|---|---|---|
| $a$ | $b$ | $1$ |
| $b$ | $1$ | $a$ |

.

A quick direct check shows that $(K_4, \cdot, 1)$ is a monoid, s.t. $(K_4 \setminus \{0\}, \cdot, 1)$ is an abelian group. Much in the same way, one checks that the distributivity holds. Hence the above table defines a field.

Remarks on fields:

1. For every prime number $p$ and a positive integer $n$, there exists a unique field $K_{p^n}$ with $p^n$-many elements. If the characteristic of $K_{p^n}$ is a prime number $q$, then $q^m = p^n$ for some $m \in \mathbb{N} \setminus \{0\}$ so $q = p$ and $m = n$.

2. The characteristic of every finite field is positive, but not every field of positive characteristic is finite: the algebraic closure $\overline{\mathbb{Z}/p\mathbb{Z}}$ of $\mathbb{Z}/p\mathbb{Z}$ is infinite. Indeed, a finite field $F$ cannot be algebraically closed as the polynomial $x^0 + \prod_{\alpha \in F}(x - \alpha) \in F[\mathbb{N}]$ has no root in $F$.

# 3 Algebras

## 3.1 Definition, examples, remarks

The concept of an algebra unifies the notion of a ring with the notion of a vector space: an algebra is a vector space with a compatible ring structure.

**Definition 3.1.** *$(A, k, \cdot, *)$ is called an algebra over (the ground field) $k$ iff*

$(A_1)$ *$(A, k, \cdot)$ is a vector space over $k$,*

$(A_2)$ *$(A, +, 0, *)$ is a not necessarily unital ring with respect to the abelian group $(A, +, 0)$ that is a part of the vector space structure,*

$(A_3)$ *The algebra multiplication $A \times A \xrightarrow{*} A$ is bilinear over $k$:*

$$(\alpha x) * (\beta y) = (\alpha\beta) \cdot (x * y).$$

*An algebra is called unital iff $\exists\, 1 \in A : (A, +, 0, *, 1)$ is a ring.*

Examples of algebras:

1. Matrix algebras. For any positive integer $n$ and any field $k$, the set $M_n(k)$ of all $n \times n$ matrices with coefficients in $k$ is a unital algebra over $k$. The multiplication neutral element is the identity matrix $I_n$. The set $M_\infty(k)$ of all finite-size matrices with coefficients in $k$ is a non-unital algebra over $k$. Embedding $M_n(k)$ into $M_{n+1}(k)$ via
$$M \mapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}$$
allows us to think of $M_\infty(k)$ as the union $\bigcup_{n=1}^\infty M_n(k)$. Note that $\dim_k M_n(k) = n^2$ and $\dim_k M_\infty(k) = \infty$.

2. Function algebras. For any non-empty space $X$ and a field $k$, the set $\mathrm{Map}(X, k)$ is a unital algebra over $k$ with respect to the pointwise addition and multiplication. The multiplication neutral element is the constant function $x \mapsto 1$. If the set $X$ is an infinite set, then the set $\mathrm{Map}_{\mathrm{f.s}}(X, k)$ of all finitely supported maps is a non-unital algebra over $k$.

3. Matrix algebra with coefficients in algebras. If $A$ is a $k$-algebra and $n$ is a positive integer, then $M_n(A)$ is a $k$-algebra. It is unital if $A$ is unital. If $A = M_m(k)$, then $M_n(M_m(k)) = M_{nm}(k)$.

4. Algebra-valued function algebras. If $X$ is a non-empty set and $A$ is a $k$-algebra, then $\mathrm{Map}(X, A)$ is a $k$-algebra w.r.t. the pointwise addition, multiplication and the scalar multiplication. (Note that for any vector space $V$, the set $\mathrm{Map}(X, V)$ is again a vector space with the pointwise structure and $\dim_k \mathrm{Map}(X, V) = |X| \dim_k V$.) If $A$ is a unital algebra, then $\mathrm{Map}(X, A)$ is a unital algebra.

5. The above two cases might coincide.
$$M_n(\mathrm{Map}(X, k)) = \mathrm{Map}(X, M_n(k)).$$

6. Convolution algebra. If $A$ is a $k$-algebra and $G$ is a monoid, then $A[G]$ is a $k$-algebra. It is unital if $A$ is unital: $\delta_e \in A[G]$,
$$\delta_e(g) := \begin{cases} 1_A & \text{for } g = e \\ 0 & \text{for } g \neq e \end{cases},$$
is the multiplication neutral element. In particular, we can take $A$ to be a field $k$, and $G$ to be a group. Group algebras $k[G]$ are very important and well studied.

7. Algebras given by generators and relations: $k\langle x \rangle := k[x]$ (polynomials), $k[x]/\langle x^2 \rangle$ (truncated polynomials), $k\langle x, y \rangle / \langle xy - yx \rangle = k[x, y]$ (polynomials in two variables).

8. Coupling algebras with topology and analysis yields a plethora of fundamental examples of operator algebras: Banach algebras, C*-algebras, von Neumann algebras.

## 3.2 Unital vs non-unital algebras

Every non-unital algebra can be unitalized in many ways.

Minimal unitization:

Let $A$ be an algebra over a field $k$. The minimal unitization $A^+$ of $A$ is the vector space $A \oplus k$ with the following multiplication:

$$(A \oplus k) \times (A \oplus k) \xrightarrow{\ *\ } A \oplus k\,,$$

$$((a,\alpha),(b,\beta)) \longmapsto (a,\alpha) * (b,\beta) := (ab + \alpha b + \beta a, \alpha\beta)\,.$$

1. It is associative:

$$\begin{aligned}((a,\alpha) * (b,\beta)) * (c,\gamma) &= (ab + \alpha b + \beta a, \alpha\beta) * (c,\gamma)\\ &= (abc + \alpha bc + \beta ac + \alpha\beta c + \gamma ab + \alpha\gamma b + \beta\gamma a, \alpha\beta\gamma),\end{aligned}$$

$$\begin{aligned}(a,\alpha) * ((b,\beta) * (c,\gamma)) &= (a,\alpha) * (bc + \beta c + \gamma b, \beta\gamma)\\ &= (abc + \beta ac + \gamma ab + \beta\gamma a + \alpha bc + \alpha\beta c + \alpha\gamma b, \alpha\beta\gamma).\end{aligned}$$

2. It extends the multiplication in $A$: $(a,0) * (b,0) = (ab,0)$.

3. $(0,1)$ is the multiplication neutral element:

$$\begin{aligned}(a,\alpha) * (0,1) &= (a0 + \alpha 0 + a \cdot 1, \alpha \cdot 1) = (a,\alpha),\\ (0,1) * (a,\alpha) &= (0a + 1 \cdot a + \alpha 0, 1 \cdot \alpha) = (a,\alpha).\end{aligned}$$

4. If $A$ is already unital, then $A^+ \cong A \oplus k$, where the multiplication on the right-hand-side is component-wise: $(a,\alpha) \cdot (b,\beta) = (ab, \alpha\beta)$. Indeed,

$$A^+ \xrightarrow{f} A \oplus k, \quad f(a,\alpha) := (a + \alpha \cdot 1_A, \alpha)$$

   is a linear bijection satisfying $f(a,\alpha) \cdot f(b,\beta) = f((a,\alpha) * (b,\beta))$.

Examples of unitization:

1. Let $X = \left\{\frac{1}{n} \mid n \in \mathbb{N} \setminus \{0\}\right\}$. Define

$$C_0(X) := \left\{ f \in \text{Map}(X,\mathbb{R}) \;\middle|\; \lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0 \right\}.$$

   Then the minimal unitization $C_0(X)^+$ is

$$C(X) := \left\{ f \in \text{Map}(X,\mathbb{R}) \;\middle|\; \lim_{n\to\infty} f\left(\frac{1}{n}\right) \in \mathbb{R} \right\}.$$

Indeed,

$$F : C(X) \to C_0(X)^+, \quad F(f) := \left( f - \lim_{n\to\infty} f\left(\frac{1}{n}\right) \cdot 1_{\mathrm{Map}(X,\mathbb{R})} \, , \lim_{n\to\infty} f\left(\frac{1}{n}\right)\right)$$

is a linear bijection intertwining the multiplication. Note that

$$C_0(X) \cong \{f \in \text{Map}(X \cup \{0\}, \mathbb{R}) \mid f \text{ is continuous and } f(0) = 0\},$$
$$C(X) \cong \{f \in \text{Map}(X \cup \{0\}, \mathbb{R}) \mid f \text{ is continuous}\}.$$

2. Let

$$A := \{f \in \text{Map}(\mathbb{C}, M_n(k)) \mid f(x) = 0 \text{ for some } x \in \mathbb{C}\}\,.$$

Then

$$A^+ \cong \{f \in \text{Map}(\mathbb{C}, M_n(k)) \mid f(x) = \alpha I_n \text{ for some } \alpha \in k\} \cong A \oplus k.$$

Indeed, $A$ is a unital algebra with $1_A(z) = \begin{cases} 1 & z \neq x \\ 0 & z = x \end{cases}$ .

3. Let $A := x * k[\mathbb{N}]$. Then $A^+ \cong k[\mathbb{N}]$. Indeed,

$$F : k[\mathbb{N}] \longrightarrow A^+, \quad F(\alpha) := (\alpha - \alpha(0)x^0, \alpha(0))$$

is a linear bijection intertwining the multiplications:

$$\begin{aligned}\forall \alpha \in k[\mathbb{N}] : \alpha - \alpha(0)x^0 &= \sum_{i=0}^{n} \alpha(i)x^i - \alpha(0)x^0 = \sum_{i=1}^{n} \alpha(i)x^i \\ &= x * \left( \sum_{j=0}^{n-1} \alpha(j+1)x^j \right) \in x * k[\mathbb{N}]\,.\end{aligned}$$

(Note that $x * k[\mathbb{Z}] = k[\mathbb{Z}]$ because $\alpha = x * x^{-1} * \alpha$ .)

# 4 Ideals

Let $A$ be an algebra over a field $k$, and let $I$ be a vector subspace of $A$. The subspace $I$ is called an ideal of $A$ iff

$$\forall\, a \in A, v \in I : \quad av \in I \;\text{ and }\; va \in I$$

Elementary remarks:

1. For any algebra $A$, the subspace $I = 0$ or $I = A$ is an ideal of $A$. The zero ideal is called trivial, and the whole-algebra ideal is called improper Any ideal of $A$ that is not $A$ is called a proper ideal.

2. Every ideal is an algebra.

3. An ideal $I$ of a unital algebra $A$ is unital if and only if $I = A$. Indeed, if $A = I$, then $1 \in I$, and then $\forall\, a \in A$: $a \cdot 1 = a \in I$.

4. Ideals make sense already for rings: it suffices to replace a vector subspace with an abelian subgroup.

5. Sometimes one-sided ideals are considered:

   $I$ is a left ideal of $A$ iff $\forall\, a \in A, v \in I : av \in I$,

   $I$ is a right ideal of $A$ iff $\forall\, a \in A, v \in I : va \in I$.

A fundamental observation:

Let $I$ be any ideal of an algebra $A$. Then the quotient vector space $A/I$ is an algebra with respect to the multiplication inherited form $A$. Indeed, the induced multiplication

$$A/I \times A/I \longrightarrow A/I, \quad ([a],[b]) \longmapsto [ab]$$

is well defined because $(a+I)(b+I) = ab + Ib + aI + I^2 \subseteq ab + I$. (Associativity and distributivity are clear.)

Examples of ideals:

1. $A$ is always an ideal of its minimal unitization $A^+$:

$$(a,\alpha)(b,0) = (ab + 0a + \alpha b, \alpha 0) = (ab + \alpha b, 0) \in A,$$
$$(a,0)(b,\beta) = (ab + \beta a + 0b, 0\beta) = (ab + \beta a, 0) \in A.$$

2. Let $Y \subseteq X$, $Y \neq \emptyset$, and let $A$ be a $k$-algebra. Then $I := \{f \in \text{Map}(X,A) \mid f(Y) = \{0\}\}$ is an ideal of Map$(X,A)$ considered with the pointwise structure:

$$f(Y) = \{0\} \Rightarrow \forall\, y \in Y, g \in \text{Map}(X,A) : (gf)(y) = g(y)f(y) = g(y)0 = 0 \text{ and } (fg)(y) = 0.$$

3. $N\mathbb{Z}$ is an ideal in $\mathbb{Z}$.

4. $I := \{M \in M_n(k) \mid M_{in} = 0\, \forall\, i\}$, $n > 1$, is a left (but not right) ideal of $M_n(k)$:

$$(NM)_{in} = \sum_{k=1}^{n} N_{ik} M_{kn} = 0.$$

   To show that it is not a right ideal, starting from $n = 2$, take matrices of the form

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in I, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(k).$$

   Observe that

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \notin I.$$

   The case of $n > 2$ is analogous.

5. For any polynomial $\alpha \in k[\mathbb{N}]$, the vector space $\alpha * k[\mathbb{N}]$ is an ideal of $k[\mathbb{N}]$, because $k[\mathbb{N}]$ is a commutative ring.

6. Let $k[G]$ be a group algebra. Then $I := \{\alpha \in k[G] \mid \sum_{g\in G} \alpha(g) = 0\}$ is an ideal of $k[G]$. Indeed,

$$\sum_{g\in G}(\alpha * \beta)(g) = \sum_{g\in G}\sum_{\substack{h,k\in G\\ hk=g}} \alpha(h)\beta(k) = \sum_{g\in G}\sum_{h\in G}\alpha(h)\beta(h^{-1}g) = \sum_{(g,h)\in G\times G} \alpha(h)\beta(h^{-1}g).$$

Now, the map

$$\begin{aligned} G\times G &\longrightarrow G\times G,\\ (g,h) &\longmapsto (h^{-1}g,h), \end{aligned}$$

is a bijection with the inverse $(k,h) \mapsto (hk,h)$. Hence, changing variables form $(g,h)$ to $(k,h)$, we obtain

$$\sum_{(g,h)\in G\times G} \alpha(h)\beta(h^{-1}g) = \sum_{(k,h)\in G\times G} \alpha(h)\beta(k) = \sum_{h\in G}\alpha(h)\sum_{k\in G}\beta(k) = 0.$$

Much in the same way,

$$\sum_{g\in G}(\beta * \alpha)(g) = \sum_{k\in G}\beta(k)\sum_{h\in G}\alpha(h) = 0.$$

**Theorem 4.1.** *Let $k$ be a field. There are no proper non-zero ideals in $M_n(k)$ for any $n \in \mathbb{N}\setminus\{0\}$.*

*Proof.* Let $I \neq 0$ be an ideal of $M_n(k)$. Take $0 \neq M \in I$. Then there exist $i, j \in \{1,\ldots,n\}$ such that $M_{ij} \neq 0$. Next, recall that the set of all elementary matrices (matrix units) $E_{kl}$, $k,l \in \{1,\ldots n\}$, is a linear basis of $M_n(k)$. Hence,

$$M = \sum_{k,l\in\{1,\ldots,n\}} M_{kl}E_{kl}\,.$$

Also, for any $m \in \{1,\ldots,n\}$, we obtain:

$$\begin{aligned} E_{mi}ME_{jm} &= \sum_{k,l\in\{1,\ldots,n\}} M_{kl}E_{mi}E_{kl}E_{jm} = \sum_{k,l\in\{1,\ldots,n\}} M_{kl}E_{mi}\delta_{lj}E_{km} = \sum_{k\in\{1,\ldots,n\}} M_{kj}E_{mi}E_{km}\\ &= \sum_{k\in\{1,\ldots,n\}} M_{kj}\delta_{ik}E_{mm} = M_{ij}E_{mm}. \end{aligned}$$

Now, as $k$ is a field and $M_{ij} \neq 0$, we get $E_{mm} = (M_{ij})^{-1}E_{mi}ME_{jm}$. Consequently, as $I$ is an ideal, $I_n = \sum_{m=1}^{n} E_{mm} \in I$, so $I = M_n(k)$. This shows that there are no non-zero proper ideals in $M_n(k)$. □

**Corollary 4.2.** *There are no non-zero proper ideals in any field.*

# 5 Homomorphisms

## 5.1 Definition, examples, remarks

A homomorphism between algebraic objects is a map between them preserving their algebraic structure.

**Definition 5.1.** *A map $M \xrightarrow{f} N$ is called a homomorphism of*

- *monoids iff $M$ and $N$ are monoids, and*

$$\forall\, m_1, m_2 \in M: \quad f(m_1 m_2) = f(m_1) f(m_2), \quad f(e_M) = e_N.$$

  *Here $e_M$ and $e_N$ are the neutral elements of $M$ and $N$ respectively.*

- *groups iff $M$ and $N$ are groups, and*

$$\forall\, m_1, m_2 \in M: \quad f(m_1 m_2) = f(m_1) f(m_2).$$

- *rings (or fields) iff $M$ and $N$ are rings (or fields), and $f$ is a homomorphism of both their additive groups and multiplicative monoids.*
- *unital algebras iff $M$ and $N$ are unital algebras and $f$ is a linear ring homomorphism.*
- *algebras iff $M$ and $N$ are algebras, and $f$ is linear map satisfying*

$$\forall\, m_1, m_2 \in M: \quad f(m_1 m_2) = f(m_1) f(m_2).$$

**Definition 5.2.** *A homomorphism $f$ (in all the above cases) is called*

- *a monomorphism iff $f$ is injective,*
- *an epimorphism iff $f$ is surjective,*
- *an isomorphism iff $f$ is bijective.*

Warning: This is a simple-minded set-theoretical approach to these concepts. A general categorical approach is more subtle, and these terms might have a different meaning:

1. An isomorphism is a morphism $f$ with the two-sided inverse: there exists a morphism $g$, s.t. $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$. In topology, a morphism is a continuous map. But a continuous bijection need not be invertible as the inverse of a continuous bijection need not be continuous

$$[0,1) \xrightarrow{\exp} S^1: \quad t \longmapsto e^{2\pi i t}.$$

2. A map $X \xrightarrow{f} Y$ is surjective if and only if $g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$. Therefore, one says that a morphism $f$ is an epimorphism iff for morphisms $g_1$ and $g_2$ the above implication holds. Consider as morphisms homomorphisms of rings. Then the standard inclusion $\mathbb{Z} \xrightarrow{f} \mathbb{Q}$ is a ring homomorphism with the property that, for any ring homomorphisms $g_1, g_2 : \mathbb{Q} \to R$ the implication $g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$ holds despite $f$ not being surjective. Indeed, for any ring homomorphism $g : \mathbb{Q} \to R$

$$g\left(\frac{p}{q}\right) = pg(q^{-1}) = pg(q)^{-1} = p(qg(1))^{-1},$$

so it is completely determined by its value on $1 \in \mathbb{Q}$, which is $f(1)$.

3. A map $X \xrightarrow{f} Y$ is injective if and only if $f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$. Therefore, a morphism is called a monomorphism iff for morphisms $g_1$ and $g_2$ the above implication holds.

Elementary remarks:

1. Homomoprhisms with the same domain and target are called endomorphisms. Bijective endomorphisms are called automorphisms.

2. If $M \xrightarrow{f} N$ is a homomorphism of monoids and $\exists\, m^{-1} \in M$, then $f(m^{-1}) = f(m)^{-1}$. Indeed,

$$e_N = f(e_M) = f(mm^{-1}) = f(m)f(m^{-1}).$$

Much in the same way $e_N = f(m^{-1})f(m)$, so $f(m^{-1})$ is the unique two-sided inverse.

3. If $G \xrightarrow{f} H$ is a homomorphism of groups, then

$$e_H = f(e_G)f(e_G)^{-1} = f(e_Ge_G)f(e_G)^{-1} = f(e_G)f(e_G)f(e_G)^{-1} = f(e_G).$$

4. In our context, the inverse of a bijective homomorphis is a homomorphism:

$$f^{-1}(xy) = f^{-1}(f(f^{-1}(x))f(f^{-1}(y))) = f^{-1}(f(f^{-1}(x)f^{-1}(y))) = f^{-1}(x)f^{-1}(y).$$

Much in the same way we prove it for $+$ and the same for scalar multiplication.

5. If $A \xrightarrow{f} B$ is an algebra homomorphism, then

$$A^+ \xrightarrow{f^+} B^+, \quad f^+(a,\alpha) := (f(a),\alpha)$$

is a homomorphism of unital algebras. The multiplication and the linearity are obviously preserved, and $f(0,1) = (0,1)$ follows from $f(0) = 0$ (a homomorphism of groups automatically preserves the neutral element).

6. A homomorphism of fields is always injective. Indeed, suppose the contrary. Let $k \xrightarrow{f} k'$ be a homomorphism of fields with $f(x) = f(x')$ for $x \neq x'$. Then $f(x - x') = 0$. But $x - x' \neq 0$, so it is invertible. Hence

$$1 = f((x-x')(x-x')^{-1}) = f(x-x')f((x-x')^{-1}) = 0 \cdot f((x-x')^{-1}) = 0,$$

which is impossible in a field.

## 5.2 Homomorphisms from convolution rings

Recall that the evaluation of a polynomial at $r \in R$ is a map

$$R[\mathbb{N}] \ni \sum_{k=0}^{n} s_k x^k \overset{\mathrm{ev}_r}{\longmapsto} \sum_{k=0}^{n} s_k r^k \in R.$$

A polynomial function $f_\alpha$ of a polynomial $\alpha \in R[\mathbb{N}]$ is the map

$$R \ni r \overset{f_\alpha}{\longmapsto} f_\alpha(r) := \mathrm{ev}_r(\alpha) \in R.$$

Consider now the map

$$R[\mathbb{N}] \ni \alpha \overset{f}{\longmapsto} f_\alpha \in \mathrm{Map}(R, R).$$

Properties of $f$:

1. If $R$ is commutative, then $f$ is a homomorphism of rings, where $\mathrm{Map}(R, R)$ is a ring with respect to the pointwise structure. Indeed, for any $r \in R$,

$$f_{\alpha+\beta}(r) = \mathrm{ev}_r(\alpha + \beta) = \mathrm{ev}_r(\alpha) + \mathrm{ev}_r(\beta) = f_\alpha(r) + f_\beta(r) = (f_\alpha + f_\beta)(r).$$

   Also, $\forall r \in R:$

$$\begin{aligned} f_{\alpha*\beta}(r) &= \mathrm{ev}_r(\alpha * \beta) = \mathrm{ev}_r\left(\left(\sum_{k=0}^{m} \alpha(k)x^k\right) * \left(\sum_{l=0}^{n} \beta(l)x^l\right)\right) \\ &= \mathrm{ev}_r\left(\sum_{N=0}^{m+n}\left(\sum_{k=\max\{0,N-n\}}^{\min\{m,N\}} \alpha(k)\beta(N-k)\right)x^N\right) \\ &= \sum_{N=0}^{m+n}\left(\sum_{k=\max\{0,N-n\}}^{\min\{m,N\}} \alpha(k)\beta(N-k)\right)r^N \\ &= \left(\sum_{k=0}^{m} \alpha(k)r^k\right)\left(\sum_{l=0}^{n} \beta(l)r^l\right) = f_\alpha(r)f_\beta(r) = (f_\alpha f_\beta)(r). \end{aligned}$$

2. If $R \neq 0$ is finite, then $f$ is not injective because $R[\mathbb{N}]$ is infinite and $\mathrm{Map}(R, R)$ is finite.

3. A zero divisor is $r \in R$ such that $\exists\, s \in R \setminus \{0\}$: $rs = 0$ or $sr = 0$. For instance, in $\mathbb{Z}/6\mathbb{Z}$ the element $2$ is a zero divisor because $3 \neq 0$ and $2 \cdot 3 = 0$. Also, $1 \neq e = e^2 \in R$, then $e$ is a zero divisor because $1 - e \neq 0$ and $(1 - e)e = e - e^2 = e - e = 0$. In particular, $0$ is a (trivial) zero divisor. A commutative ring whose only zero divisor is $0$ is called an integral domain. If $R$ is an infinite integral domain (e.g. $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$), then $f$ is a monomorphism of rings.

4. In general, we cannot replace $R[\mathbb{N}]$ by $R[[\mathbb{N}]]$ as the domain of $f$. However, with the help of analysis, we can consider a restricted version of $f$. For instance,

$$\mathbb{R}[[\mathbb{N}]] \ni \alpha \longmapsto f_\alpha \in \mathrm{Map}((-r_\alpha, r_\alpha), \mathbb{R}).$$

Here $r_\alpha$ is the radius of convergence of $\alpha$. For

$$\alpha = \sum_{n=0}^{\infty} \frac{1}{n!} x^n\,, \quad \beta = \sum_{n=0}^{\infty} x^n\,, \quad \gamma = \sum_{n=0}^{\infty} (n+1)^n x^n\,,$$

the radii of convergence are $r_\alpha = \infty$, $r_\beta = 1$, $r_\gamma = 0$.

## 5.3 Substructures

**Definition 5.3.** *A submonoid, subgroup, subring, subfield, (unital) subalgebra of a monoid, group, field, (unital) algebra, respectively, is a subset $N$ with the properties*

- *the binary operation(s) restrict to it: $N \times N \to N$;*
- *$N$ contains the neutral element(s);*
- *with the inherited (restricted) structures, $N$ is a monoid, group, ring, field, (unital) algebra, respectively.*

Elementary remarks:

1. A subsomething of a subsomething of something is a subsomething of something.
2. $\mathbb{Z}$ is not a subfield of $\mathbb{Q}$ because the third property fails despite the first two being true.

**Proposition 5.4.** *Let $M \xrightarrow{f} N$ be a homomorphism of monoids, groups, rings, (unital) algebras, respectively. Then $f(M)$ is a submonoid, subgroup, subring, (unital) subalgebra of $N$, respectively.*

*Proof.* 1. $f(M)$ is a submonoid of $N$ because $e_N = f(e_M)$ and $f(m_1)f(m_2) = f(m_1 m_2)$.

2. $f(M)$ is a subgroup of $N$ because $f(g)^{-1} = f(g^{-1})$.
3. $f(M)$ is a subring of $N$ because it is an abelian subgroup by 2 and a submonoid by 1, and the distributivity laws hold for $f(M)$ as it is a subset of $N$.
4. $f(M)$ is a (unital) subalgebra of $N$ because it is a vector subspace and a (unital) subring of $N$, and the bilinearity of the multiplication in $f(M)$ holds because it holds in $N$.

□

Examples:

1. Every ideal $I$ of an algebra $A$ is a subalgebra of $A$.
2. $\mathbb{Q}$ is a subfield of $\mathbb{R}$ and $\mathbb{R}$ is a subfield of $\mathbb{C}$.

3. $SL_n(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$.

4. $\forall\, n \in \mathbb{N} : n\mathbb{N}$ is a submonoid of $\mathbb{N}$.

**Definition 5.5.** *Let $H$ be a subgroup of $G$. It is called* normal *iff*

$$\forall\, g \in G : \quad gH = Hg$$

*(for any $h \in H$ there is $h' \in H$ such that $gh = h'g$). In other words, $gHg^{-1} = H$ for all $g \in G$.*

## 5.4 Kernels

**Definition 5.6.** *Let $M \xrightarrow{f} N$ be a homomorphism of monoids. Then the* kernel *of $f$ is*

$$\ker f := \{m \in M \mid f(m) = e_N\}.$$

Elementary remarks:

1. For any homomorphism $M \xrightarrow{f} N$ of monoids, the kernel of $f$ is a submonoid of $M$. Indeed,

$$x, y \in \ker f \quad \Rightarrow \quad f(xy) = f(x)f(y) = 0,$$

so $xy \in \ker f$. Also, the neutral element $e_M$ of $M$ is in $\ker f$. Furthermore, if $f$ is a homomorphisms of groups, $x \in \ker f$ and $e_N$ is the neutral element of $N$, then

$$f(x^{-1}) = f(x)^{-1} = e_N^{-1} = e_N,$$

so $x^{-1} \in \ker f$. Hence $\ker f$ is a subgroup.

2. A homomorphism $M \xrightarrow{f} N$ of groups is injective if and only if $\ker f = \{e_M\}$, where $e_M$ is the neutral element of $M$. Indeed, if $\ker f \neq \{e_M\}$, then $f$ is not injective. Vice versa, suppose that $f$ is not injective. Then there exist $g, h \in G$ such that $g \neq h$ and $f(g) = f(h)$. Denote the neutral element of $N$ by $e_N$. It follows that $e_N = f(g)f(h)^{-1} = f(g)f(h^{-1}) = f(gh^{-1})$, so $gh^{-1} \in \ker f$ and $gh^{-1} \neq e_M$. Hence $\ker f \neq \{e_M\}$.

3. If $f$ is a homomorphism of rings or algebras, then $\ker f := \{m \in M \mid f(m) = 0\}$ is a subgroup or a vector subspace of $M$ that is also an ideal of $M$:

$$\begin{aligned} \forall\, x \in \ker f, y \in M : \quad & f(xy) = f(x)f(y) = 0f(y) = 0, \\ & f(yx) = f(y)f(x) = f(y)0 = 0, \end{aligned}$$

so $xy, yx \in \ker f$.

**Theorem 5.7.** *If $G \xrightarrow{f} H$ is a group homomorphism, then $\ker f$ is a normal subgroup of $G$, and the induced map $G/\ker f \ni [g] \mapsto f(g) \in f(G)$ is a group isomorphism.*

**Lemma 5.8.** *If $K$ is any subgroup of $G$, then $gh^{-1} \in K$ defines an equivalence relation.*

*Proof.* We check the reflexivity, symmetry, and transitivity of the relation:

$$gg^{-1} = e_G \in K, \quad gh^{-1} \in K \Rightarrow (gh^{-1})^{-1} = hg^{-1} \in K,$$
$$(gh^{-1} \in K \text{ and } hk^{-1} \in H) \Rightarrow gh^{-1}hk^{-1} = gk^{-1} \in K.$$

Here $e_G$ stands for the neutral element of $G$. □

**Lemma 5.9.** *If $N$ is a normal subgroup of $G$, then $G/N := \{[g] \mid g \in G\}$, where $h \in [g]$ iff $gh^{-1} \in N$, is a group.*

*Proof.* The induced multiplication

$$G/N \times G/N \longrightarrow G/N : \quad ([g],[h]) \longmapsto [gh],$$

is well defined:

$$n_1, n_2 \in N,\ g = n_1k,\ h = n_2l \quad \Rightarrow \quad gh = n_1kn_2l = n_1n_2'kl,$$

where $n_2' \in N$, and the last step holds by the by normality of $N$. It is immediate that the induced multiplication is associative, enjoys the neutral element, and has any element invertible. □

**Lemma 5.10.** *The kernel* $\ker f$ *is a normal subgroup.*

*Proof.* If $x \in \ker f$ and $e_H$ is the neutral element of $H$, then, for any $g \in G$,

$$f(gxg^{-1}) = f(g)f(x)f(g)^{-1} = f(g)e_Hf(g)^{-1} = f(g)f(g)^{-1} = e_H,$$

so $g \ker fg^{-1} \subseteq \ker f$. The equality $g\ker fg^{-1} = \ker f$ follows from the fact $x = gg^{-1}xgg^{-1}$ and $g^{-1}xg = g^{-1}x(g^{-1})^{-1} \in \ker f$. □

*Proof.* (of Theorem 5.7) We already know that $G/\ker f$ and $f(G)$ are groups. Note now that the induced map

$$G/\ker f \xrightarrow{\bar{f}} f(G), \quad \bar{f}([g]) := f(g),$$

is well defined: if $x \in \ker f$, then

$$f(xg) = f(x)f(g) = e_Hf(g) = f(g).$$

It is also a group homomorphism because

$$\bar{f}([g][h]) = \bar{f}([gh]) = f(gh) = f(g)f(h) = \bar{f}([g])\bar{f}([h]).$$

Next, $\bar{f}$ is injective because $\ker \bar{f} = \{[e_G]\}$:

$$\bar{f}([g]) = e_H \iff f(g) = e_H \iff g \in \ker f \iff ge_G^{-1} \in \ker f \iff [g] = [e_G].$$

Finally, by the definition of the image of a map, all maps are surjective onto their images, and $\bar{f}(G/\ker f) = f(G)$, so $\bar{f}$ is an isomorphism of groups. □

## 5.5 Preimages

**Definition 5.11.** *Let $X \xrightarrow{f} Y$ be a map, and let $B \subseteq Y$. Then the preimage of $B$ under $Y$ is*

$$\boxed{f^{-1}(B) := \{x \in X \mid f(x) \in B\}}.$$

Elementary remarks:

1. If $f$ is a bijection, then the preimage under $f$ is the image under its inverse:
$$f(x) \in B \iff x = f^{-1}(f(x)) \in f^{-1}(B).$$

2. If $M \xrightarrow{f} N$ is a homomorphism of monoids, groups, rings, (unital) algebras, respectively, and $L$ is a submonoid, subgroup, subring, (unital) subalgebra, respectively, of $N$, then $f^{-1}(L)$ is a submonoid, subgroup, subring, (unital) subalgebra, respetively, of $M$.

3. If $M \xrightarrow{f} N$ is homomorphism of monoids, then $\ker f = f^{-1}(e_N)$, where $e_N$ is the neutral element of $N$.

4. Let $A \xrightarrow{f} B$ be a homomorphism of algebras, and let $I$ be an ideal of $B$. Then $f^{-1}(I)$ is an ideal of $A$:
$$\forall\, x \in A, y \in f^{-1}(I): \quad \begin{array}{l} f(xy) = f(x)f(y) \in I, \\ f(yx) = f(y)f(x) \in I. \end{array}$$
However, the image of an ideal need not be an ideal. Indeed, consider a non-unital algebra homomorphism $M_n(k) \xrightarrow{f} M_{n+1}(k)$ given by
$$M \longmapsto \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}.$$
Then $M_n(k)$ is an ideal of $M_n(k)$, and $f(M_n(k))$ is a proper non-zero subalgebra of $M_{n+1}(k)$, so it cannot be an ideal of $M_{n+1}(k)$ as $M_{n+1}(k)$ has no proper non-zero ideals.

Examples:

1. Let $A \xrightarrow{f} B$ be a homomorphism of algebras, and $A^+ \xrightarrow{f^+} B^+$ its minimal unitization. Then $B$ is an ideal of $B^+$, and $(f^+)^{-1}(B) = A$:
$$f^+(a, \alpha) = (f(a), \alpha) \in B \iff \alpha = 0.$$

2. Let $X \xrightarrow{F} Y$ be a map, and let $\mathrm{Map}(X, k)$ and $\mathrm{Map}(Y, k)$ be $k$-algebras with the pointwise structure. Then $\mathrm{Map}(Y, k) \xrightarrow{F^*} \mathrm{Map}(X, k)$, $F^*(g) := g \circ F$, is an algebra homomorphism (called the pullback of $F$). Furthermore, let $\emptyset \neq A \subseteq X$. Then
$$I_A := \{f \in \mathrm{Map}(X, k) \mid f(A) = 0\}$$
is an ideal of $\mathrm{Map}(X, k)$, and
$$(F^*)^{-1}(I_A) = I_{F(A)} := \{g \in \mathrm{Map}(Y, k) \mid g(F(A)) = 0\}.$$
Indeed,
$$g \in (F^*)^{-1}(I_A) \iff F^*(g) \in I_A \iff (g \circ F)(A) = 0 \iff g(F(A)) = 0 \iff g \in I_{F(A)}.$$

3. Consider the group homomorphism $\mathbb{Z} \xrightarrow{f} \mathbb{Z}$, $f(x) := 2x$, and the subgroup $3\mathbb{Z}$. Then $f^{-1}(3\mathbb{Z}) = 3\mathbb{Z}$ because $2x \in 3\mathbb{Z} \iff x \in 3\mathbb{Z}$.

## 5.6 Cokernels

If $H$ is a subgroup of $G$, then we can always consider the quotient $G/H$. However, to make $G/H$ a group, we need $H$ to be a normal subgroup. To ensure that it is so, let us assume that $G$ is abelian. Then all its subgroups are normal: $gHg^{-1} = gg^{-1}H = eH = H$.

**Definition 5.12.** *Let $M \xrightarrow{f} N$ be a homomorphism of abelian groups. The* <u>*cokernel*</u> *of $f$ is the abelian group*

$$\boxed{\operatorname{coker} f := N/f(M)}.$$

<u>Elementary remarks</u>:

1. A homomorphism $M \xrightarrow{f} N$ of abelian groups is surjective if and only if $\operatorname{coker} f = 0$:

   $\operatorname{coker} f = 0 \iff \forall\, n \in N : [n] = [0] \iff \forall\, n \in N : n - 0 = n \in f(M) \iff f$ is surjective.

2. If $M \xrightarrow{f} N$ is a homomorphism of vector spaces, rings, (unital) algebras, respectively, then $\operatorname{coker} f$ is defined with respect to the additive group structure. In the case of vector spaces or (unital) algebras, $\operatorname{coker} f$ is a vector space. However, in the case of rings, $\operatorname{coker} f$ is only an abelian group. Indeed, $\mathbb{Z} \ni m \overset{f}{\mapsto} (m, m) \in \mathbb{Z} \oplus \mathbb{Z}$ is a homomorphism of rings, but there is no induced multiplication on $\operatorname{coker} f$: $[(m, n)][(m', n')] := [(mm', nn')]$ but $[(m, n)] = [(m, n) + (k, k)]$ and $[(m', n')] = [(m', n') + (l, l)]$, so we should have

   $$\begin{aligned} [(mm', nn')] &= [(m + k, n + k)(m' + l, n' + l)] \\ &= [(mm' + km' + ml + kl, nn' + nl + kn' + kl)] \\ &= [(mm' + k(m' - n'), nn' + l(n - m))]. \end{aligned}$$

   Hence $[(k(m' - n'), l(m - n))] = 0$, i.e. $k(m' - n') = l(m - n)\ \forall\, k, l, m, n, m', n' \in \mathbb{Z}$, which is clearly false for $k = l = m = n = 1$ and $m' = 3$, $n' = 2$.

3. Consider the linear map

   $$\mathbb{C}^2 \ni (x, y) \overset{f}{\mapsto} (x + y, -x - y) \in \mathbb{C}^2.$$

   Then $\operatorname{coker} f \cong \mathbb{C}$ via

   $$\operatorname{coker} f \ni [(a, b)] \overset{g}{\mapsto} a + b \in \mathbb{C}.$$

   Indeed, $g$ is well defined:

   $$g([(a, b) + (m, -m)]) = a + m + b - m = a + b = g([a, b)]).$$

   Furthermore, $g$ is linear (because it is induced by a linear map), surjective and injective:

   $$g([(a, b)]) = 0 \iff a + b = 0 \iff (a, b) = (a, -a) \iff [(a, b)] = 0.$$

## 5.7 Exact sequences

**Definition 5.13.** *A sequence of homomorphisms*

$$\ldots \longrightarrow M_n \xrightarrow{f_n} M_{n+1} \xrightarrow{f_{n+1}} M_{n+2} \longrightarrow \ldots$$

*of abelian groups is called exact iff*

$$\boxed{\forall\, n \in \mathbb{Z}: \quad f_n(M_n) = \ker f_{n+1}\,.}$$

*A short exact sequence is an exact sequence of the form:*

$$\boxed{0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0\,.}$$

Elementary remarks:

1. If $(f_n)_{n\in\mathbb{Z}}$ is an exact sequence, then

$$\forall\, n \in \mathbb{Z}: \quad f_{n+1} \circ f_n = 0\,.$$

2. A sequence
   - $0 \to K \xrightarrow{f} M$ is exact $\iff$ $f$ is injective,
   - $M \xrightarrow{f} L \to 0$ is exact $\iff$ $f$ is surjective,
   - $0 \to M \xrightarrow{f} N \to 0$ is exact $\iff$ $f$ is bijective.

3. Every exact sequence can be factored to short exact sequences:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \ker f_n & \longrightarrow & M_n & \xrightarrow{\widetilde{f_n}} & f_n(M_n) & \longrightarrow & 0 & & & & \\
 & & & & & & \| & & & & & & \\
 & & & & 0 & \longrightarrow & \ker f_{n+1} & \longrightarrow & M_{n+1} & \xrightarrow{\widetilde{f}_{n+1}} & f_{n+1}(M_{n+1}) & \longrightarrow & 0\,.
\end{array}$$

Here $\widetilde{f_k}$ stands for the corestriction of $f_k$: $\widetilde{f_k}(x) := f_k(x)$ for all $x \in M_k$.

4. If $0 \to K \xrightarrow{g} M \xrightarrow{f} N$ is an exact sequence, then the corestriction $K \xrightarrow{\widetilde{g}} \ker f$ is an isomorphism:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & \xrightarrow{g} & M & \xrightarrow{f} & N \\
 & & & \searrow^{\widetilde{g}} & \uparrow & & \\
 & & & & g(K) = \ker f & &
\end{array} \quad .$$

5. If $M \xrightarrow{f} N \xrightarrow{g} L \to 0$ is an exact sequence, then the induced map $\mathrm{coker} f \xrightarrow{\bar{g}} N$ is an isomorphism:

$$\begin{array}{ccccccc}
M & \xrightarrow{f} & N & \xrightarrow{g} & L & \longrightarrow & 0 \\
 & & \downarrow & \nearrow_{\bar{g}} & & & \\
 & & N/f(M) = \mathrm{coker} f & & & &
\end{array} \quad .$$

Indeed, it is well defined because, $\forall\, m \in M :\ f(m) \in \ker g$, so

$$\bar{g}([n + f(m)]) = g(n) + g(f(m)) = g(n) = \bar{g}([n]),$$

it is surjective because $g$ is surjective, and it is injective because

$$\bar{g}([n]) = 0 \iff g(n) = 0 \iff n \in \ker g = f(M) \iff [n] = 0.$$

6. If $0 \to K \to M \to N \to 0$ is a short exact sequence of vector spaces, and $\dim M < \infty$, then $\dim M = \dim K + \dim N$.

7. If $I$ is an ideal of a $k$-algebra $A$, then $0 \to I \to A \to A/I \to 0$ is a short exact sequence. For instance,
$$0 \longrightarrow \{f \in C(S^2) \mid f(x_0) = 0\} \longrightarrow C(S^2) \xrightarrow{\mathrm{ev}_{x_0}} \mathbb{C} \longrightarrow 0\,.$$

8. The sequence $0 \to H \to H \oplus G \to G \to 0$ is clearly a short exact sequence. A different type of a short exact sequence is the sequence
$$0 \longrightarrow n\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0\,.$$

**Definition 5.14.** *A short exact sequence $0 \to K \to M \to N \to 0$ is called split iff there exists a homomorphism $s : N \to M$ such that $f \circ s = \mathrm{id}$.*

**Theorem 5.15.** *Let $0 \to K \xrightarrow{g} M \xrightarrow{f} N \to 0$ be an exact sequence of abelian groups. Then the following are equivalent:*

1. *There exists a homomorphism $s_f : N \to M$ such that $f \circ s_f = \mathrm{id}$,*
2. *There exists a homomorphism $s_g : M \to K$ such that $s_g \circ g = \mathrm{id}$,*
3. *There exists a subgroup $M' \subseteq M$ such that $M = \ker f \oplus M' = g(K) \oplus M'$.*

**Theorem 5.16.** *A short exact sequence $0 \to K \to M \to N \to 0$ of vector spaces and linear maps always splits.*

Counterexample: The short exact sequence

$$0 \to n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

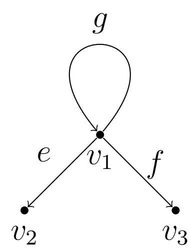
does not split for any $n \in \mathbb{N} \setminus \{0, 1\}$.

# 6 Graphs (quivers)

**Definition 6.1.** *A graph is a quadruple $E := (E^0, E^1, s, t)$, where:*

- $E^0$ *is the set of* vertices*,*
- $E^1$ *is the set of* edges *(arrows),*
- $E^1 \xrightarrow{s} E^0$ *is the* source *map assigning to each edge its beginning,*
- $E^1 \xrightarrow{t} E^0$ *is the* target *(range) map assigning to each edge its end.*

For instance, consider the following graph




.

Here

$$\begin{aligned} s(e) &= v_1, & t(e) &= v_2, \\ s(f) &= v_1, & t(f) &= v_3, \\ s(g) &= v_1, & t(g) &= v_1. \end{aligned}$$

Elementary remarks:

1. The maps $s$ and $t$ need not be injective nor surjective.
2. If both $E^0$ and $E^1$ are empty, we call $E$ the empty graph. The set $E^1$ might always be empty, but $E^0$ must not be empty if $E^1$ is not empty: every edge must have its beginning and its end.
3. $E^0$ and $E^1$ might be infinite (usually, at most countable).

## 6.1 Paths

**Definition 6.2.** *Let $E$ be a graph. A* finite path *in $E$ is a finite tuple $p_n := (e_1, \ldots, e_n)$ of edges satisfying*

$$t(e_1) = s(e_2), \quad t(e_2) = s(e_3), \quad \ldots, \quad t(e_{n-1}) = s(e_n).$$

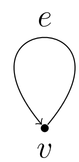
*The beginning $s(p_n)$ of $p_n$ is $s(e_1)$ and the end $t(p_n)$ of $p_n$ is $t(e_n)$. If $s(p_n) = t(p_n)$, we call $p_n$ a* loop*. An* infinite path *is a sequence $(e_i)_{i\in\mathbb{N}}$ of edges satisfying*

$$\forall\, i \in \mathbb{N} : \quad t(e_i) = s(e_{i+1}).$$

**Definition 6.3.** *The* length *of a path is the size of the tuple. Every edge is a path of length* 1*. Vertices are considered as finite paths of length* 0*. The length of an infinite path is infinity.*

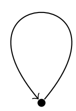
Elementary remarks:

1. The space FP$(E)$ of all finite paths in $E$ (vertices included) might be infinite even if $E$ is a finite graph (both $E^0$ and $E^1$ are finite):




$$E^0 = \{v\}, \quad E^1 = \{e\}, \quad \text{FP}(E) = \{v, e, (e, e), (e, e, e), \ldots\}.$$

2. Examples of infinite paths:



winding around infinitely many times,



marching off to infinity,

or a combination of the above cases.

**Theorem 6.4.** *Let $E$ be a finite graph. Then* $\mathrm{FP}(E)$ *is finite if and only if there are no loops in* $E$.

*Proof.* If there is a loop in $E$, then we have paths of arbitrary length, so there are infinitely many of them:

$$e_1, \quad (e_1, e_2), \quad \ldots, \quad (e_1, \ldots, e_n), \quad (e_1, \ldots, e_n, e_1), \quad \text{etc.}$$

Vice versa, if there are no loops, then edges in any path $(e_1, e_2, \ldots, e_n)$ cannot repeat themselves:

$$e_i = e_j \Rightarrow i = j.$$

Indeed, suppose the contrary: $e_i = e_j$ for $i < j$. Then

$$s(e_i) = s(e_j) = t(e_{j-1}),$$

so the path $p_{ij} := (e_i, \ldots, e_{j-1})$ is a loop:

$$s(p_{ij}) = s(e_i) = t(e_{j-1}) = t(p_{ij}),$$

which contradicts our assumption of not having loops.

Therefore, the length of the longest possible path in $E$ is at most the number $N$ of all edges. This yields the finite decomposition

$$\text{FP}(E) = \text{FP}_0(E) \cup \text{FP}_1(E) \cup \ldots \cup \text{FP}_N(E),$$

where FP$_k(E)$ is the space of all paths in $E$ of length $k$. Furthermore, the sets FP$_0(E) = E^0$ and FP$_1(E) = E^1$ are finite by assumption. To construct a path of length $k$, first we must choose $k$

different edges from the set of $N$ edges. We can do it in $\binom{N}{k}$ many ways. Then we can order these $k$ edges into a path in at most $k!$ different ways, so there are at most

$$k!\binom{N}{k} = \frac{N!}{(N-k)!}$$

many paths of length $k$.

Summarizing, FP$(E)$ is a finite union of finite sets, so it is finite. □

The estimate of the number of paths of length $k$ used in the above proof is far from optimal. Our goal now is to find the optimal estimate, i.e. the estimate for which there exists a graph having exactly as many paths as allowed by the estimate.

**Definition 6.5.** *Let $E$ be a graph, and let $p_n := (e_1, \ldots, e_n)$ be a finite path of length at least one. A subpath $q_k$ of $p_n$ is a path $(e_i, e_{i+1}, \ldots, e_{i+k})$, where $i \in \{1, \ldots, n\}$ and $k \in \{0, \ldots, n-i\}$. If $(e_n)_{n\in\mathbb{N}}$ is an infinite path, then any $(k+1)$-tuple $(e_i, \ldots, e_{i+k})$, for any $i \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$, is a subpath of $(e_n)_{n\in\mathbb{N}}$. Every source and every target of each edge of a path (finite or infinite) is viewed as a subpath of length zero.*

**Theorem 6.6.** *Let $E$ be any graph. If there exists a path $p$ (finite or infinite) whose edges can be rearranged (permuted) into a path, then there exists a loop in $E$.*

*Proof.* Let $S$ be a subset of $\mathbb{N}$ containing at least two elements, and let $\sigma : S \to S$ be a bijection that is not the idenity. Since $\sigma \neq \mathrm{Id}$, there exist the smallest $j \in S$ such that $\sigma(j) \neq j$. As $\sigma$ is bijective, $\sigma(j) > j$. Indeed, if $j$ is the smallest element of $S$, we are done. If there is $i < j$, then $\sigma(j) \neq \sigma(i) = i$, so $\sigma(j) > j$. Furthermore, $\sigma^{-1}(j) \neq j$. If $\sigma^{-1}(j) < j$, then we get a contradiction: $j = \sigma(\sigma^{-1}(j)) = \sigma^{-1}(j) < j$. Therefore, also $\sigma^{-1}(j) > j$.

Next, let $p := (e_1, ..., e_n, ...)$ or $p := (e_1, ..., e_n)$. Then let $S := \mathbb{N}$ or $S := \{1, ..., n\}$, respectively. Suppose now that $p_\sigma := (e_{\sigma(1)}, ..., e_{\sigma(n)}, ...)$ or $p_\sigma := (e_{\sigma(1)}, ..., e_{\sigma(n)})$ is again a path for a bijection $\sigma$ as above. Then $(e_j, \ldots, e_{\sigma^{-1}(j)})$ is a subpath of $p$, so $(e_{\sigma(j)}, ..., e_j)$ is a subpath of $p_\sigma$. Combining the latter path with the path $(e_j, e_{j+1}, \ldots, e_{\sigma(j)-1}, e_{\sigma(j)})$, we obtain a loop:

$$(e_{\sigma(j)}, ..., e_j, e_{j+1}, \ldots, e_{\sigma(j)-1}).$$

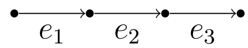
Note that, if $\sigma(j) = j+1$, then the path $(e_{\sigma(j)}, ..., e_j) = (e_{j+1}, ..., e_j)$ is already a loop. □

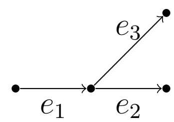
**Corollary 6.7.** *If $E$ is a graph with $N$ edges and no loops, then there are at most $\binom{N}{k}$ different paths of length $1 \leq k \leq N$.*

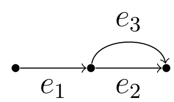
*Proof.* No loops in $E$ implies that edges cannot repeat themselves in any path, so one needs to choose $k$ different edges from $N$ edges. By the above proposition, there is at most one way these $k$ different edges can form a path of length $k$. □

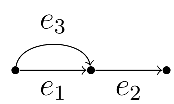
In any graph with $N \geq 1$ edges, there are exactly $\binom{N}{1} = N$ paths of length one, i.e. edges. There is a graph

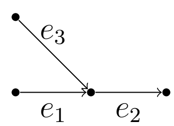

with $N$ edges and no loops with exactly $\binom{N}{N} = 1$ path of length $N$. However, there is no graph with 3 edges and no loops and $\binom{3}{2} = 3$ different paths of length 2:

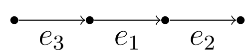


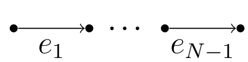




There are at most 2 different paths of lenght 2.

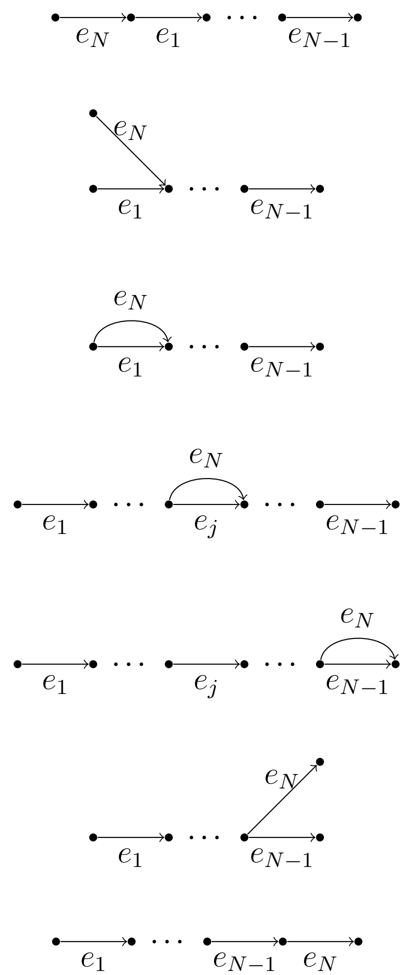
**Proposition 6.8.** *Let $E$ be a graph with $N \geq 2$ edges and no loops. Then there are at most two different paths of length $N-1$.*

*Proof.* A path of length $k$ must be of the form


so, if we have at least one path of length $N-1$, our graph must be of the form



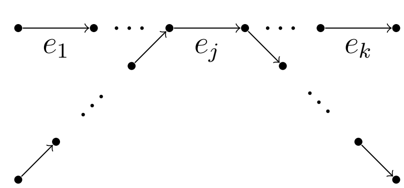

and $e_N$ attached somewhere. The only attachment possibilities increasing the number of paths of length $N-1$ are:



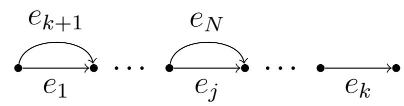

In each of the above cases, we have exactly two different paths of length $N-1$. $\square$

**Lemma 6.9.** *Let $E$ be a graph with $N \geq 2$ edges and no loops. Assume that $N \geq k \geq N-k$. Then there are at most $2^{N-k}$ different paths of length $k$ in $E$ and the bound is optimal.*

*Proof.* One can always construct a graph with a path $p_1$ of length $k$. Then there remain precisely $N-k$ many edges that can be used to construct more paths. Call the set of all these edges $F^1$. Any path of length $k$ is composed out of $l$ edges in $F^1$ and $k-l$ edges from the path $p_1$. For instance:




Any such a path is uniquely determined by the choice of $l$ edges from $F^1$ because there is always only one way in which edges from the path $p_1$ can connect disconnected subpaths composed from edges in $F^1$ and edges in a path cannot be rearranged. This gives at most $\binom{N-k}{l}$ possibilities for having paths of length $k$ with $l$ edges from $F^1$. As $l$ can vary from $0$ to $N-k$, there are at most $\sum_{l=0}^{N-k} \binom{N-k}{l} = 2^{N-k}$ different paths of length $k$. The bound is optimal because the graph




has exactly $2^{N-k}$ edges of length $k$. □

**Theorem 6.10.** *Let $E$ be a graph with $N \geq 2$ edges and no loops, and let $1 \leq k \leq N =: nk + r$, $0 \leq r \leq k-1$. Then there are at most*

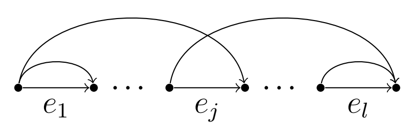
$$\boxed{P_k^N := (n+1)^r n^{k-r}}$$

*different paths of length $k$ and the bound is optimal.*

**Remark 6.11.** For $k > N - k$, we have $N = 1 \cdot k + (N-k)$, so $P_N(k) = 2^{N-k} \cdot 1^{k-(N-k)} = 2^{N-k}$. Also, if $k = N - k$, then $N = 2 \cdot k + 0$, so $P_N(k) = 3^0 \cdot 2^k = 2^{N-k}$. Hence the preceding proposition proves the theorem for $k \geq N - k$.

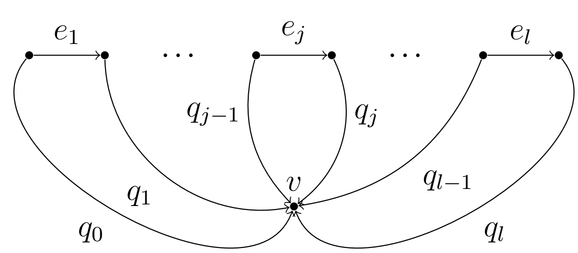
*Proof.* Our first step is to transform the graph $E$ into a graph $E_1$ with the same amount of edges but with all vertices on its longest path $p_1$. We need to show that we can always do this without introducing loops or decreasing the amount of different paths of length $k$. Clearly, we can first remove all vertices in $E^0$ that are not in $s(E^1) \cup t(E^1)$. This way we end up with finitely many vertices. Furthermore, we identify unrelated vertices. In any graph, we call a pair of vertices unrelated iff there is no path between them. If our graph admits a pair of unrelated vertices, then we can choose such a pair and identify the vertices. We repeat the procedure until there are no unrelated vertices. We call the thus obtained graph $E_1$.

**Lemma 6.12.** *$E_1$ is a graph with $N$ edges, no loops and all vertices on its longest path $p_1$:*



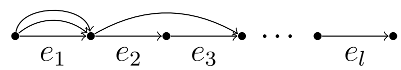

*It admits at least as many different paths of length $k$ as $E$.*

*Proof.* If identifying two vertices $v_1$ and $v_2$ introduces a loop, then breaking them apart destroys the loop. Hence the identified $v_1$ and $v_2$ are on the loop, so there was a path from $v_1$ to $v_2$ or the other way around, which means that $v_1$ and $v_2$ were not unrelated. It follows that identifying unrelated vertices introduces no loops. Next, suppose that all vertices are related but that there is a vertex $v$ that is not on the path $p_1$:

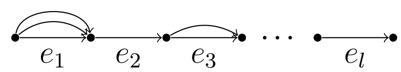


The path $q_0$ must go from $s(e_1)$ to $r$ as otherwise $p_1$ would not be the longest path. Furthermore, the fact that $p_1$ is of maximal length forces adjacent paths to have the same orientation. Hence all these paths, like $q_0$, must end in $v$. However, $q_l$ ending in $v$ contradicts the maximality of the length of $p_1$. Finally, $E_1$ has obviously at least as many paths of length $\geq 1$ as $E$ because identifying vertices can only increase the number of such paths. □

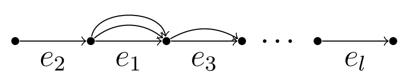
We can assume that the length of $p_1$ is $l \geq k$ as otherwise there are no paths of length $k$. Our next step is to transform $E_1$ into a graph $E_2$ will all edges that start in $s(e_1)$ ending in $t(e_1)$:




If we have an edge starting in $s(e_1)$ but ending in $t(e_i)$, $i > 1$, then we shift the beginning of such an edge to $s(e_i)$. As there are no edges ending in $s(e_1)$, we do not loose any paths this way. Now we transform $E_2$ into $E_3$ by shifting the beginnings of edges from $s(e_2)$ to $t(e_j)$, $j > 2$, to $s(e_3)$:



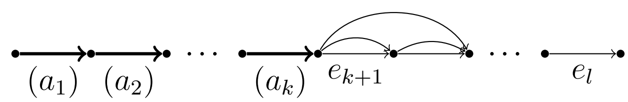

This time possibly we loose the paths of length $k$ that started in $s(e_1)$ and involved the just shifted edges, we possibly gain paths of length $k$ that start in $s(e_2)$ and involve the shifted edges. Let $a_i$ denote the number of edges starting at $s(e_i)$. Then, if the shifted edges are the first of $x$ different paths of length $k-1$, we loose $a_1 \cdot x$ paths of length $k$ but gain $a_2 \cdot x$ paths of length $k$. To ensure that we gain at least as much as we loose, we transform $E_3$ to $E_4$ by switching places of the edges from $s(e_1)$ to $t(e_1)$ with the edges from $s(e_2)$ to $t(e_2)$, if $a_1 > a_2$:




As the number of paths of length $k$ beginning with a shifted edge is unchanged, and the number of paths of length $k$ with a shifted edge as the second edge is not decreased, the number of paths of length $k$ involving a shifted edge does not decrease. Now we have to make sure that transforming $E_2$ to $E_4$ we did not decrease the number of all paths of length $k$ not involving the shifted edges.

If $k = 1$, we are done because in any graph $E$ with $N$ edges we have $P_N(1) = N$. If $k \geq 2$, then any path of length $k$ in $E_2$ that does not involve any shifted edge and that starts in $s(e_1)$ must involve edges from $s(e_1)$ to $t(e_1)$ and from $s(e_2)$ to $t(e_2)$, so the number of paths of length $k$ not involving the shifted edges and starting at the leftmost vertex is the same in $E_2$ as in $E_4$ even if $a_1 > a_2$ and we made the switch: $a_1 a_2 y = a_2 a_1 y$, where $y$ is the number of paths of length $k-2$ not involving the shifted edges and starting at $s(e_3)$. (In the case $l = k = 2$, we take $y = 1$.) Next, concerning the number of paths of length $k$ starting at the second vertex from the left and not involving the shifted edges, it does not decrease when moving from $E_2$ to $E_4$ as we have at least as many edges going from the second to the third vertex and exactly as many paths not involving the shifted edges starting at the third vertex in $E_2$ as in $E_4$. Finally, the number of paths of length $k$ not involving the shifted edges and starting at the third or further vertex is unaffected when going from $E_2$ to $E_4$.

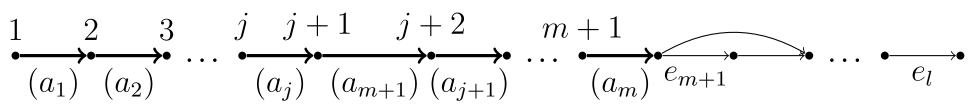
We can continue this $E_2$-$E_3$-$E_4$ procedure until we obtain a graph $F_k$ whose all edges emitted from first $k$ vertices end in the consecutive vertex and with the number of edges satisfying the inequalities $a_1 \leq a_2 \leq \ldots \leq a_k$:




Indeed, take $m < k$ and apply the $E_2$-$E_3$-$E_4$ procedure to the graph $F_m$ defined as $F_k$ but with $k$ replaced by $m$. Assume that $a_j \leq a_{m+1} \leq a_{j+1}$ for some $j$. Then we move the beginning of any edge starting at the $(m+1)$ vertex and ending at the $(m+3)$ vertex or further to the $(m+2)$ vertex. Next, we implement the swap of edges:

$$(m+1) \mapsto (j+1), \quad (j+1) \mapsto (j+2), \quad \ldots \quad , \quad m \mapsto (m+1),$$

and obtain:




If the shifted edges were the first edges of $x_1$ paths of length $k-1$, $x_2$ of length $k-2$, ..., and $x_{m+1}$ of length $k-(m+1)$, respectively, then by shifting the edges we lost

$$L := a_m x_1 + a_{m-1} a_m x_2 + \ldots + a_1 \ldots a_m x_m$$

paths of length $k$, but, due to the re-ordering procedure, we gained

$$G := a_m x_1 + a_{m-1} a_m x_2 + \ldots + a_{j+1} \ldots a_m x_{m-j} + a_{m+1} a_{j+1} \ldots a_m x_{m-j+1} \\ + \ldots a_2 \ldots a_{m+1} a_{j+1} \ldots a_m x_m + a_1 a_2 \ldots a_{m+1} a_{j+1} \ldots a_m x_{m+1}.$$

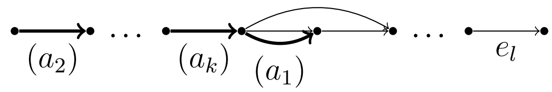
The first $m-j$ terms of $L$ are the same as in $G$, the next $j$ terms of $L$ are the same as in $G$, the next $j$ terms in $L$ are no bigger then they are in $G$, and the last term in $G$ does not appear in $L$. Thus, applying the $E_2$-$E_3$-$E_4$ procedure, we did not decrease the amount of paths of length $k$ involving the shifted edges.

Concerning the paths of length $k$ not involving the shifted edges, the re-arrangement procedure does not change the amount of paths starting at the first vertex, does not decrease the amount of paths starting at the vertices 2, ..., $m$, and does not change the amount of paths starting at $m+1$ or further.

We cannot apply the $E_2$-$E_3$-$E_4$ procedure any further because, if $a_1 > a_{k+1}$, swaping the edges starting at the first vertex with the edges starting at the $(k+1)$ vertex will decrease the amount of paths of length $k$ beginning at the first vertex:

$$a_1 a_2 \ldots a_k > a_{k+1} a_2 \ldots a_k \, .$$

Now we need to make a move decreasing the number of vertices. We identify the first vertex with the $(k+1)$ vertex and shift all edges from the first vertex to the second vertex to become edges from the $(k+1)$ to $(k+2)$ vertex:




We call the thus obtained new graph $G_l$.

Let $G$ be a graph with finitely many totaly ordered (by paths) vertices, and $N$ edges. Denote by $P_k^N(G,m)$ the number of paths in $G$ of length $k>1$ that start at the $m$ vertex. Furthermore, let $y_m$ denote the number of all paths in $F_k$ of length $1\le m\le k$ starting at the $(k+1)$ vertex. Then there are

$$\sum_{m=1}^{l-k+1} P_k^N(F_k,m)$$

many paths of length $k$ in $F_k$ and

$$\sum_{m=1}^{l-k} P_k^N(G_l,m)$$

many paths of length $k$ in $G_l$. For the first sum, we have

$$P_k^N(F_k,1)=a_1\dots a_k,\qquad P_k^N(F_k,2)=a_2\dots a_k y_1,\qquad \dots\quad ,$$
$$P_k^N(F_k,k)=a_k y_{k-1},\qquad P_k^N(F_k,k+1)=y_k\,.$$

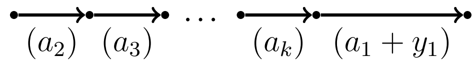
For the second sum, we have

$$P_k^N(G_l,1)=a_2\dots a_k(a_1+y_1),\qquad P_k^N(G_l,2)\ge a_3\dots a_k y_2=P_k^N(F_k,3),\qquad \dots\quad ,$$
$$P_k^N(G_l,k-1)\ge a_k y_{k-1}=P_k^N(F_k,k),\qquad P_k^N(G_l,k)\ge y_k=P_k^N(F_k,k+1).$$

Consequently,

$$\begin{aligned}
\sum_{m=1}^{l-k+1} P_k^N,(F_k,m) &= P_k^N(G_l,1)+\sum_{m=3}^{l-k+1} P_k^N(F_k,m)\\
&= P_k^N(G_l,1)+\sum_{m=3}^{k+1} P_k^N(F_k,m)+\sum_{m=k+2}^{l-k+1} P_k^N(F_k,m)\\
&= P_k^N(G_l,1)+\sum_{m=2}^{k} P_k^N(F_k,m+1)+\sum_{m=k+2}^{l-k+1} P_k^N(G_l,m-1)\\
&= P_k^N(G_l,1)+\sum_{m=2} P_k^N(F_k,m+1)+\sum_{m=k+1}^{l-k} P_k^N(G_l,m)\\
&\le P_k^N(G_l,1)+\sum_{m=2}^{k} P_k^N(G_l,m)+\sum_{m=k+1}^{l-k} P_k^N(G_l,m)\\
&= \sum_{m=1}^{l-k} P_k^N(G_l,m).
\end{aligned}$$

Thus there are at least as many paths of length $k$ in $G_l$ as there are in $F_k$. If $l=k+1$, we have the desired thick path:




Otherwise we repeat the $E_2$-$E_3$-$F_k$-$G_l$ procedure decreasing the amount of vertices by one but not decreasing the amount of paths of legth $k$.

All this shows that we can always transform our graph into a graph with totally ordered $(k+1)$ vertices that are on a path of length $k$ without changing the amount $N$ of all edges and without decreasing the number of paths of length $k$. In such a graph, if there are still edges that begin and end not in consecutive vertices, they do not contribute to paths of length $k$, so we can re-attach them so that they begin and end in consecutive vertices.

Now, the final step is to prove that given a thick path c with differences between numbers of edges bigger than one, we can evenly re-distribute the edges increasing the number of paths of length $k$ to the bound $P_N(k)$.

If there are any two indices $i \neq j$ such that $b_i - b_j > 1$, then we define

$$b'_n := \begin{cases} b_n & \text{for } n \neq i, j \\ b_n - 1 & \text{for } n = i \\ b_n + 1 & \text{for } n = j \end{cases}$$

and compute

$$\begin{aligned} \prod_{n=1}^{k} b'_n &= \left( \prod_{n \in \{1,\dots,k\} \setminus \{i,j\}} b_n \right) (b_i - 1)(b_j + 1) \\ &= \left( \prod_{n \in \{1,\dots,k\} \setminus \{i,j\}} b_n \right) (b_i b_j + b_i - b_j - 1) \\ &= \prod_{n=1}^{k} b_n + \left( \prod_{1,\dots,k \setminus \{i,j\}} b_n \right) (b_i - b_j - 1) > \prod_{n=1}^{k} b_n \,. \end{aligned}$$

We can repeat this procedure until there is no pair of indices $i \neq j$ with the property $b_i - b_j - 1$. Thus we arrive at a graph with $s$ pairs of consecutive vertices joined by $(b+1)$ and $(k-s)$ pairs joined by $b$ edges. Hence

$$s(b+1) + (k-s)b = s + kb = N.$$

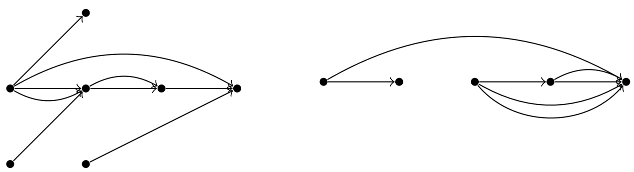
Therefore, if $0 \leq s \leq k$, then $s = r$ and $b = n$. If $s = k$, $r = 0$ and $n = b + 1$. The number of all paths of length $k$ is

$$(b+1)^s b^{k-s} = (n+1)^r n^{k-r} = P_k^N.$$

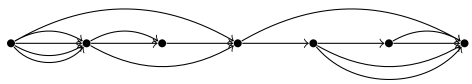
□

<u>An example:</u>

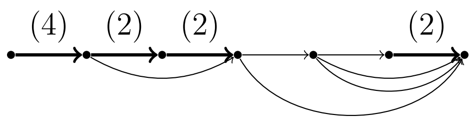
We take a graph $E$ with $N = 16$ edges, and ask about the number of all 3-paths.



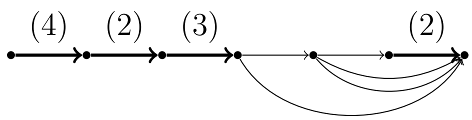
$P_3^{16}(E) = 6, l = 3,$



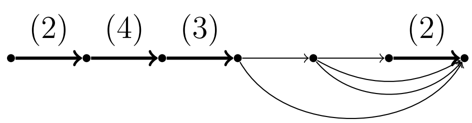
$P_3^{16}(E_1) = 31, l = 6,$




$P_3^{16}(E_2) = 43, l = 6,$

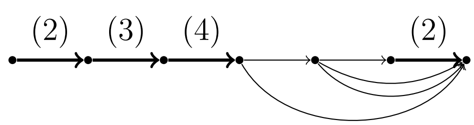



$P_3^{16}(E_3) = 47, l = 6,$

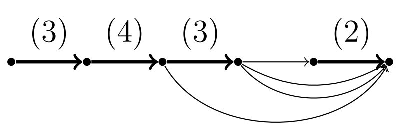



$P_3^{16}(E_4) = 59, l = 6.$

Now, we repeat the $E_2$-$E_3$-$E_4$ procedure to obtain $F_3$:

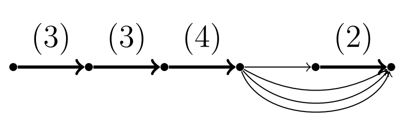



$P_3^{16}(F_3) = 62, l = 6,$

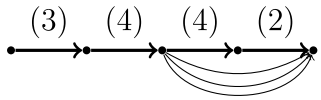



$P_3^{16}(G_5) = 90, l = 5.$

Applying again the $E_2$-$E_3$-$E_4$ procedure, yields:

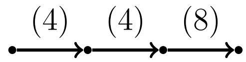



Next, repeating the $F$-$G$-procedure, we obtain $G_4$:



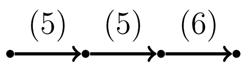

$P_3^{16}(G_4) = 116$, $l = 4$.

We still need to apply the $E_2$-$E_3$-$E_4$-$F$-$G$ procedure to obtain $G_3$:




$P_3^{16}(G_3) = 128$, $l = 3$.

The final equal-distribution procedure provides us with an optimal graph $M$ maximizing the number of $k$-paths and reaching the bound:




$P_3^{16}(M) = 150$, $l = 3$.

It agrees with the theorem: $N = nk + r$, $16 = 5 \cdot 3 + 1$,

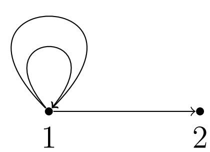
$$P_3^{16} = (5+1)^1 5^{3-1} = 6 \cdot 5 \cdot 5 = 150.$$

## 6.2 Adjacency matrices

**Definition 6.13.** *Let $E$ be a finite graph. The adjaceny matrix $A(E)$ of the graph $E$ is the square matrix whose entries are labelled by the pairs of vertices and each $(v, w)$-entry equals the number of edges that start at $v$ and end at $w$.*

Examples:

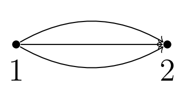
1. Consider graph $E$:

   Then,

   $$A(E) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

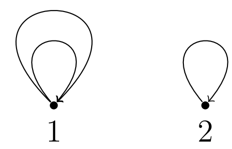
2. Consider graph $E$:

   Then,

   $$A(E) = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}.$$

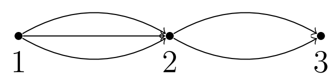
3. Consider graph $E$:



   Then,

$$A(E) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

4. Consider graph $E$:



   Then,

$$A(E) = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

In a finite graph $E$ with $N$ edges, consider all paths of length $k$ starting at a vertex $v$ and ending at a vertex $w$. If $k_1 + k_2 = k$ and $k_1, k_2 \in \mathbb{N} \setminus \{0\}$, then each path $p$ with $s(p) = v$ and $t(p) = w$ decomposes into a path $p_1$ of length $k_1$ with $s(p_1) = v$, $t(p_1) = u$, and a path $p_2$ of length $k_2$ with $s(p_2) = u$, $t(p_2) = w$. Hence the number $N_k(v, w)$ of all $k$-paths from $v$ to $w$ equals

$$N_k(v, w) = \sum_{u \in E^0} N_{k_1}(v, u) N_{k_2}(u, w).$$

Thus we have shown:

**Proposition 6.14.** *Let $E$ be a finite graph, and let $A_l(E)$ be a generalized adjacency matrix whose entries count the number of all $l$-paths between vertices. Then, $\forall\ k_1, k_2 \in \mathbb{N} \setminus \{0\}$:*

$$A_{k_1+k_2}(E) = A_{k_1}(E) A_{k_2}(E).$$

**Corollary 6.15.** $\forall\ n \in \mathbb{N} \setminus \{0\} : A_n(E) = A(E)^n$.

*Proof.* The statement holds for $n = 1$, and taking $k_1 = n$ and $k_2 = 1$ in Proposition 6.14 proves the inductive step. □

**Corollary 6.16.** *A finite graph $E$ has no loops if and only if its adjacency matrix is nilpotent.*

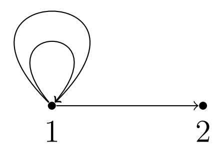
*Proof.* The finite graph $E$ has no loops $\iff$ $FP(E)$ is finite. The latter is equivalent to the existence of a longest path. Indeed, if there is no longest path, then there are paths of all lengths, so $FP(E)$ is infinite. Vice versa, if $FP(E)$ is infinite, then there is a loop, so there is no longest path.

Next, if the length of a longest path is $l$, then $A(E)^{l+1} = 0$, so $A(E)$ is nilpotent. Vice versa, if $A(E)$ is nilpotent, then there exists $n$ such that $A(E)^n = 0$. Hence there are no paths of length $\geq n$, so there exists a longest path. □

**Corollary 6.17.** *The number of all $k$-paths is given by $FP_k(E) = \sum_{v,w \in E^0} \left(A(E)^k\right)_{vw}$.*

<u>Examples:</u>

1. Consider graph $E$:

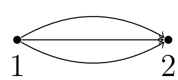


Then,

$$A(E)^k = \begin{bmatrix} 2^k & 2^{k-1} \\ 0 & 0 \end{bmatrix},$$

so there are $2^k + 2^{k-1} = 3 \cdot 2^{k-1}$ many $k$-paths in $E$.

2. Consider graph $E$:

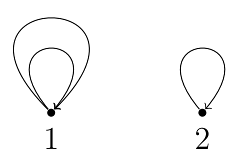


Then,

$$A(E)^2 = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so there are no paths longer than 1.

3. Consider graph $E$:

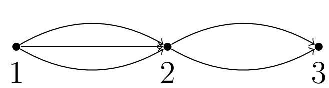


Then,

$$A(E)^k = \begin{bmatrix} 2^k & 0 \\ 0 & 1 \end{bmatrix},$$

so there are $2^k + 1$ many $k$-paths in $E$.

4. Consider graph $E$:



Then,

$$A(E)^2 = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so there are 6 paths of length 2.

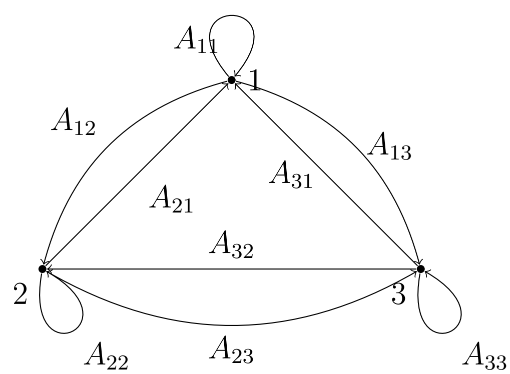
Note that there finitely many graphs with $N$ edges and whose all vertices emit or receive at least one edge. Indeed, for any such graph $E$, $E^0 = \{1, 2, \ldots, m\}$, $m \leq 2N$, $E^1 = \{1, 2, \ldots, N\}$, $s \subseteq E^1 \times E^0$, $t \subseteq E^1 \times E^0$, so the number of all such graphs is limited by $2N \cdot 2^{Nm} \cdot 2^{Nm} = N \cdot 2^{2Nm+1}$.

**Corollary 6.18.** *Let $\mathcal{E}_N$ denote the set of all graphs with $N$ edges, no loops, and whose all vertices emit or receive at least one edge. Then, $\forall\ k \in \{1, \ldots, N\}$*

$$\max_{E \in \mathcal{E}_n} \left\{ \sum_{v,w \in E^0} \left( A(E)^k \right)_{vw} \right\} = (n+1)^r n^{k-r},$$

*where $N =: nk + r$ with $r \in \{0, 1, \ldots, k-1\}$.*

Next, observe that, if $A(E) \in M_n(\mathbb{N})$, then it is the adjacency matrix of the graph $E(A)$ with $E(A)^0 = \{1, \ldots, n\}$ and $E(A)^1 = \bigcup_{i,j \in E^0} E_{ij}$, where $E_{ij}$ is the set of $A_{ij}$-many edges from $i$ to $j$. For instance, for $n = 3$, we have




A vertex $i$ of the graph $E(A)$ emits or receives at least one edge if and only if

$$\sum_{k=1}^{n} A_{ik} + A_{ki} > 0.$$

Note also that $A(E(M)) = M$ and $E(A(G)) = G$. Therefore, as no loops in $E$ means that $A(E)$ is nilpotent, we can reformulate the foregoing corollary as follows:

**Corollary 6.19.** *Let*

$$\mathcal{A}_N := \left\{ A \in \bigcup_{k \in \mathbb{N} \setminus \{0\}} M_k(\mathbb{N}) \mid \textit{satisfying } 1, 2, 3 \textit{ below} \right\}.$$

1. *$A$ is nilpotent ($E$ has no loops),*
2. $\sum_{\text{all } i,j} A_{ij} = N$ *($E$ has $N$ edges),*
3. $\forall\ i : \sum_{\text{all } j} A_{ij} + A_{ji} > 0$ *(each vertex of $E$ emits or receives).*

*Then,* $\forall\ k \in \{1, \ldots, N\}$*:*

$$\max_{A \in \mathcal{A}_n} \left\{ \sum_{\text{all } i,j} \left(A^k\right)_{ij} \right\} = (n+1)^r n^{k-r},$$

*where* $N =: nk + r$ *with* $r \in \{0, 1, \ldots, k-1\}$.

**Conjecture 6.20.** *Let* $N$ *be a non-negative real number, and let*

$$\mathcal{A}_N := \left\{ A \in \bigcup_{k \in \mathbb{N} \setminus \{0\}} M_k(\mathbb{R}_{\geq 0}) \mid \textit{satisfying } 1, 2, 3 \textit{ above} \right\}.$$

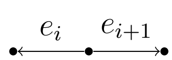
*Then,* $\forall\ k \in \{1, \ldots, [N]\}$*:*

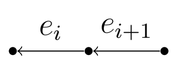
$$\sup_{A \in \mathcal{A}_N} \left\{ \sum_{\text{all } i,j} \left(A^k\right)_{ij} \right\} = \left(\frac{N}{k}\right)^k.$$

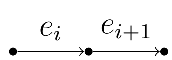
*Here* $[N]$ *stands for the integer part of* $N$.

## 6.3 The structure of graphs

**Definition 6.21.** *Let* $E$ *be a graph. An* <u>*undirected finite path*</u> *in* $E$ *is a finite sequence of edges* $(e_1, \ldots, e_n)$ *satisfying at least one of the* $4$ *equalities:*

1. $s(e_i) = s(e_{i+1})$, 

2. $s(e_i) = t(e_{i+1})$, 

3. $t(e_i) = s(e_{i+1})$, 

4. $t(e_i) = t(e_{i+1})$. 


*for all* $i \in \{1, \ldots, n-1\}$.

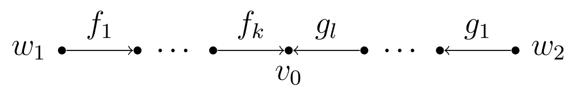
*An* <u>*undirected infinite path*</u> *in* $E$ *is an infinite sequence* $(e_1, \ldots, e_n, \ldots)$ *satisfying at least one of the above* $4$ *equalities for all* $i \in \mathbb{N} \setminus \{0\}$.

**Definition 6.22.** *We say that a finite graph is connected* $E$ *is* <u>*connected*</u> *iff for any pair of vertices* $(v, w)$*,* $v \neq w$*, there exists an* <u>*undirected*</u> *finite path* $p = (e_1, \ldots, e_n)$ *between* $v$ *and* $w$*: (*$s(e_1) = v$ *or* $t(e_1) = v$*) and (*$t(e_n) = w$ *or* $s(e_n) = w$*).*

**Definition 6.23.** *A vertex* $v$ *in a graph* $E$ *is called a* <u>*sink*</u> *iff* $s^{-1}(v) = \emptyset$.

**Proposition 6.24.** *If* $E$ *is a graph with finitely many edges, no loops, and exactly one sink, then* $E$ *is connected.*

*Proof.* Denote the sink by $v_0$. If it is the only vertex of $E$, then $E$ is connected. If there is $v_1 \neq v_0$, then there exists a path from $v_1$ to $v_0$. Indeed, as $v_0$ is the unique sink, $v_1$ emits an edge $e_1$. Consider any path $p_n := (e_1, \ldots, e_n)$, e.g. $p_1 = e_1$. If $s(p_n) \neq v_0$, then, sd $v_0$ is the unique sink, $s(p_n)$ emits $e_{n+1}$ yielding the path $p_{n+1} := (e_1, \ldots, e_n, e_{n+1})$ which is longer than $p_n$. Hence, if no path starting at $v_1$ terminates at $v_0$, we can have paths of arbitrary lengths, which is impossible because $E$ has finitely many edges and no loops. Therefore, there is a path from $v_1$ to $v_0$. Now, take any pair of distinct vertices in $E$. If one of them is $v_0$, then they are connected by a path. If $w_1 \neq v_0$, $w_2 \neq v_0$, $w_1 \neq w_2$, then there is a path $q_1 := (f_1, \ldots, f_k)$ from $w_1$ to $v_0$ and a path $q_2 := (g_1, \ldots, g_l)$ from $w_2$ to $v_0$. They combine into the undirected path $(f_1, \ldots, f_k, g_l, \ldots, g_1)$ from $w_1$ to $w_2$, so $E$ is connected.




□

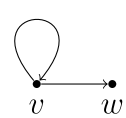
Question: What is the maximal number of all finite paths in a graph with $N$ edges, no loops, and exactly one sink?

**Definition 6.25.** *Let $E$ be a graph. A subset $H \subseteq E^0$ is called hereditary iff any finite edge starting at $v \in H$ ends at $w \in H$.*

Note that in the above definition one can replace the word "edge" by the word "path". Indeed, if $H$ is path-hereditary, then it is, in particular edge-hereditary. Also, if $H$ is not path hereditary, then there exists a path starting in $H$ and ending outside of $H$. Such a path must contain an edge starting at $H$ and ending outside of $H$, so it is also not edge-hereditary. This proves the equivalence of these two definitions.

Examples:

1. In any graph $E$, $\emptyset$ and $E^0$ are hereditary.
2. Consider a graph $E$:




Then, $\{w\}$ is hereditary and $\{v\}$ is not hereditary.

**Definition 6.26.** *Let $E = (E^0, E^1, s, t)$ be a graph. $F = (F^0, F^1, s_F, t_F)$ is a subgraph of $E$ iff $F^0 \subseteq E^0$, $F^1 \subseteq E^1$, and*

$$\forall\, e \in F^1 : \quad s_F(e) = s(e) \in F^0, \quad t_F(e) = t(e) \in F^0.$$

**Proposition 6.27.** *Let $E = (E^0, E^1, s, t)$ be a graph, and let $H \subseteq E^0$. Set $F^0 := E^0 \setminus H$ and $F^1 := E^1 \setminus t^{-1}(H)$. Then the formulas $s_F(e) = s(e)$, $t_F(e) = t(e)$, for any $e \in F^1$, define a subgraph of $E$ if and only if $H$ is hereditary.*

*Proof.* Note first that, if $e \in F^1 := E^1 \setminus t^{-1}(H)$, then $t_F(e) = t(e) \in E^0 \setminus H =: F^0$. Hence, $t : E^1 \to E^0$ always restricts-corestricts to $t_F : F^1 \to F^0$.

Assume now that $H$ is hereditary. Then, if $e \in E^1$ and $s(e) \in H$, we have $t(e) \in H$. Hence

$$e \in F^1 := E^1 \setminus t^{-1}(H) \iff t(e) \notin H \Rightarrow s(e) \notin H \iff s(e) \in F^0 := E^0 \setminus H.$$

Therefore, $s : E^1 \to E^0$ restricts-corestricts to $s_F : F^1 \to F^0$. Vice versa, assume that $s$ restricts-corestricts to $s_F$. Then

$$t(e) \notin H \iff e \in F^1 \Rightarrow s(e) \in F^0 \iff s(e) \notin H,$$

so $s(e) \in H \Rightarrow t(e) \in H$, i.e. $H$ is hereditary. $\square$

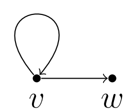
**Definition 6.28.** *Let $E$ be a graph. A subset $H \subseteq E^0$ is called saturated iff*

$$\nexists\, v \in E^0 \setminus H\colon\ 0 < |s^{-1}(v)| < \infty \text{ and } t(s^{-1}(v)) \subseteq H.$$

Examples:

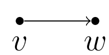
1. In any graph $E$, $\emptyset$ and $E^0$ are saturated.

2. Consider a graph $E$:




   Then all subsetes of $E^0$ are saturated.

3. Consider a graph $E$:




   Then $\{w\}$ is not saturated but it is hereditary.

4. Consider a graph $E$:


   Then $\{w\}$ is both saturated and hereditary.

## 6.4 Homomorphisms of graphs

**Definition 6.29.** *A homomorphism from a graph $E := (E^0, E^1, s_E, t_E)$ to a graph $F := (F^0, F^1, s_F, t_F)$ is a pair of maps*

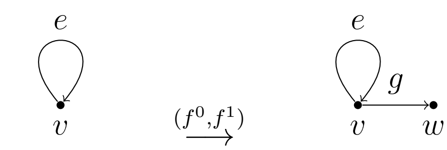
$$(f^0 : E^0 \to F^0, f^1 : E^1 \to F^1)$$

*satisfying the conditions:*

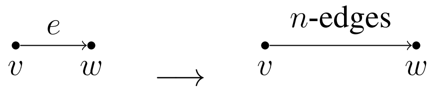
$$s_F \circ f^1 = f^0 \circ s_E\,, \qquad t_F \circ f^1 = f^0 \circ t_E\,.$$

Examples:

1. Inclusions of subgraphs $E^0 \overset{f^0}{\hookrightarrow} F^0$, $E^1 \overset{f^1}{\hookrightarrow} F^1$, e.g.:

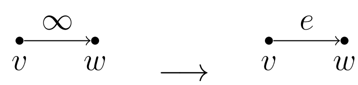



$$f^0(v) = v, \qquad f^1(e) = e, \qquad \text{or}$$

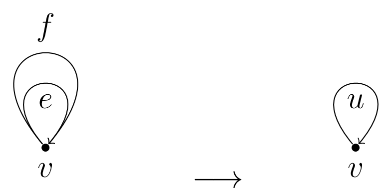



$$f^0(v) = v, \qquad f^0(w) = w, \qquad f^1(e) = e_1\,.$$

2. Collapsing edges between the same vertices to one edge, e.g.

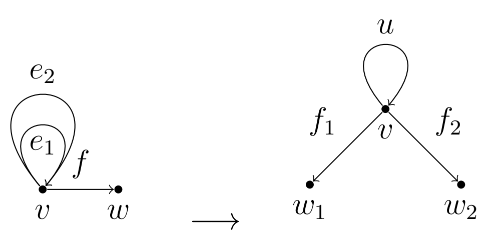



$$f^0(v) = v, \qquad f^0(w) = w, \qquad f^1(e_i) = e, \qquad \text{or}$$




$$f^1(e) = f^1(f) = u, \qquad f^0(v) = v.$$

3. A combination of both, e.g.




$$f^0(v) = v, \quad f^0(w) = w_1\,, \quad f^1(e_1) = f^1(e_2) = u, \quad f^1(f) = f_1\,.$$

From the graph-algebra point of view, of particular interest are injective graph homomorphisms $(f^0, f^1) : E \to F$ (both $f^0$ and $f^1$ injective) satisfying certain conditions.

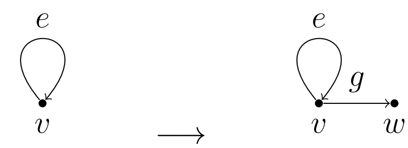
**Definition 6.30.** *We call an injective homomorphism of graphs $(f^0, f^1) : E \to F$ an admissible inclusion iff it satisfies the following conditions:*

*1.* $F^0 \setminus f^0(E^0)$ *is hereditary and saturated,*

*2.* $f^1(E^1) = t_F^{-1}(f^0(E^0))$.

Examples:

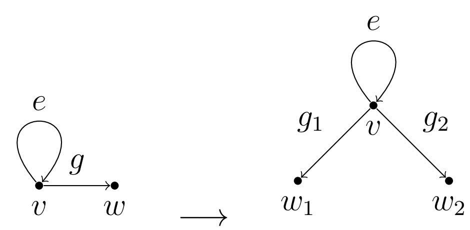
1.




$F^0 \setminus f^0(E^0) = \{w\}$ is hereditary and saturated. Also, $f^1(\{e\}) = \{e\}$ and

$$t_F^{-1}(f^0(E^0)) = t_F^{-1}(\{v\}) = \{e\}.$$

2.



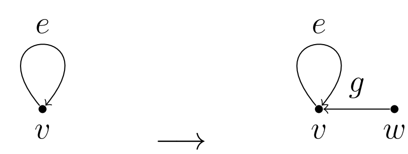

$$f^0(v) = v, \quad f^0(w) = w_1\,, \quad f^1(e) = e, \quad f^1(g) = g_1\,.$$

$F^0 \setminus f^0(E^0) = \{w_2\}$ is hereditary and saturated. Also, $f^1(\{e, g\}) = \{e, g_1\}$ and

$$t_F^{-1}(f^0(E^0)) = t_F^{-1}(\{v, w_1\}) = \{e, g_1\}.$$

Counterexamples:

1.

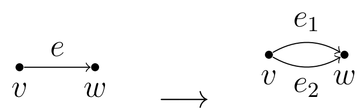



$$f^0(v) = v, \quad f^1(e) = e.$$

$F^0 \setminus f^0(E^0) = \{v, w\} \setminus \{v\} = \{w\}$ is saturated but not hereditary. Also,

$$t_F^{-1}(f^0(E^0)) = t_F^{-1}(\{v\}) = \{e, g\} \neq \{e\} = f^1(E^1).$$

2.




$$f^0(v) = v, \quad f^0(w) = w, \quad f^1(e) = e_1\,.$$

$F^0 \setminus f^0(E^0) = \{v, w\} \setminus \{v, w\} = \emptyset$ is hereditary and saturated. But

$$t_F^{-1}(f^0(E^0)) = F^1 = \{e_1, e_2\} \neq \{e_1\} = f^1(E^1).$$

The intersection of graphs:

Let $F$ and $G$ be graphs. Assume that $s_F$ and $t_F$ agree, respectively, with $s_G$ and $t_G$ on $F^1 \cap G^1$. Then we can define the intersection graph

$$F \cap G := (F^0 \cap G^0, F^1 \cap G^1, s_\cap, t_\cap),$$

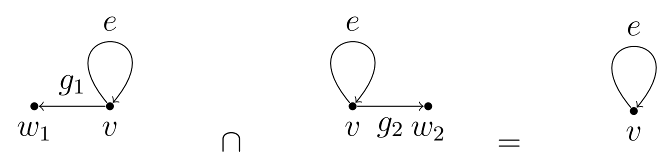
where $s_\cap, t_\cap : F^1 \cap G^1 \to F^0 \cap G^0$,

$$\forall\, e \in F^1 \cap G^1 : \quad s_\cap(e) = s_G(e) = s_F(e), \quad t_\cap(e) = t_G(e) = t_F(e).$$

$F \cap G$ is, clearly, a subgraph of both $F$ and $G$. We say that the intersection is admissible iff both inclusions $F \cap G \hookrightarrow F$ and $F \cap G \hookrightarrow G$ are admissible inclusions.

Examples:

1.

$F \cap G \quad =$ 

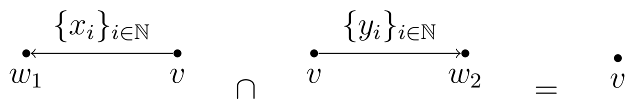

$$F^0 \cap G^0 = \{v\}, \quad F^1 \cap G^1 = \{e\}, \quad s_\cap(e) = t_\cap(e) = v.$$

The intersection is admissible because:

(a) The subset $F^0 \setminus (F^0 \cap G^0) = \{v, w_1\} \setminus \{v\} = \{w_1\}$ is hereditary and saturated in $F$, and the subset $G^0 \setminus (F^0 \cap G^0) = \{v, w_2\} \setminus \{v\} = \{w_2\}$ is hereditary saturated in $G$;

(b) $t_F^{-1}(F^0 \cap G^0) = t_F^{-1}(\{v\}) = \{e\} = F^1 \cap G^1$ and $t_G^{-1}(F^0 \cap G^0) = t_G^{-1}(\{v\}) = F^1 \cap G^1$.

2.

$F \cap G \quad =$ 


is admissible becase:

(a) $F^0 \setminus (F^0 \cap G^0) = \{w_1, v\} \setminus \{v\} = \{w_1\}$ is hereditary and saturated in $F$, and

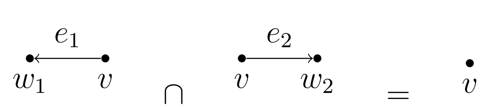
$$G^0 \setminus (F^0 \cap G^0) = \{w_2, v\} \setminus \{v\} = \{w_2\}$$

is hereditary and saturated in $G$;

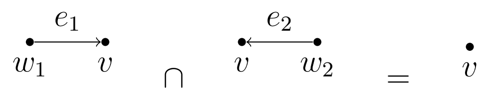
(b) $t_F^{-1}(F^0 \cap G^0) = t_F^{-1}(\{v\}) = \emptyset = t_G^{-1}(\{v\}) = t_G^{-1}(F^0 \cap G^0)$ and $F^1 \cap G^1 = \emptyset$.

Counterexamples:

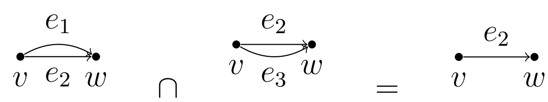
1.

$F \cap G \quad =$ 


is not admissible because $F^0 \setminus (F^0 \cap G^0) = \{w_1, v\} \setminus \{v\} = \{w_1\}$ is not saturated. (It is hereditary.)

2. $F \cap G =$ 


is not admissible because $F^0 \setminus (F^0 \cap G^0) = \{w_1, v\} \setminus \{v\} = \{w_1\}$ is not hereditary. (It is saturated.)

3. $F \cap G =$ 


is not admissible because

$$t_F^{-1}(F^0 \cap G^0) = t_F^{-1}(\{v, w\}) = \{e_1, e_2\} \neq \{e_2\} = F^1 \cap G^1.$$

However, both $F^0 \setminus (F^0 \cap G^0)$ and $G^0 \setminus (F^0 \cap G^0)$ are empty, so they are hereditary and saturated.

The union of graphs:

Let $F$ and $G$ be graphs. Again, assume that $s_F$ and $t_F$ agree, respectively, with $s_G$ and $t_G$ on $F^1 \cap G^1$. Then we can define the union graph

$$F \cup G := (F^0 \cup G^0, F^1 \cup G^1, s_\cup, t_\cup),$$

where $s_\cup, t_\cup : F^1 \cup G^1 \to F^0 \cup G^0$,

$$\forall\, x \in F^1 \cup G^1 : \quad s_\cup = \begin{cases} s_F(x) & \text{for } x \in F^1 \\ s_G(x) & \text{for } x \in G^1 \end{cases},$$

$$\forall\, x \in F^1 \cup G^1 : \quad t_\cup = \begin{cases} t_F(x) & \text{for } x \in F^1 \\ t_G(x) & \text{for } x \in G^1 \end{cases}.$$

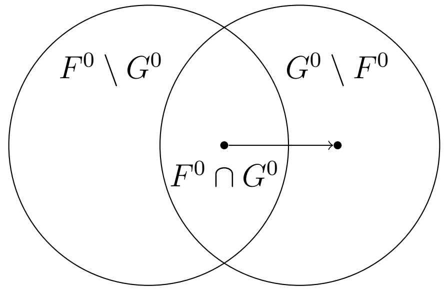
Note that $F$ and $G$ are subgraphs of $F \cup G$. We say that the union is admissible iff both the inclusions $F \hookrightarrow F \cup G$ and $G \hookrightarrow F \cup G$ are admissible.

**Lemma 6.31.** *Let $F$ and $G$ be graphs whose source and target maps agree, respectively on $F^1 \cap G^1$. Then, if the intersection graph $F \cap G$ is admissible, so is the union graph $F \cup G$.*

*Proof.* 1. $(F^0 \cup G^0) \setminus F^0 = G^0 \setminus F^0 = G^0 \setminus (F^0 \cap G^0)$. Since $F \cap G \hookrightarrow G$ is admissible, $G^0 \setminus (F^0 \cap G^0)$ is hereditary and saturated in $G$.

We need to show that $G^0 \setminus (F^0 \cap G^0)$ is hereditary and saturated in $F \cup G$. To this end, consider $p := (e_1, \ldots, e_n) \in FP(F \cup G)$ such that $s_\cup(p) \in G^0 \setminus (F^0 \cap G^0)$. Then $s_\cup(p) = s_\cup(e_1) \notin F^0$, so $e_1 \notin F^1$, whence $e_1 \in G^1$, so $s_\cup(e_1) = s_G(e_1)$. As $G^0 \setminus (F^0 \cap G^0)$ is hereditary in $G$, $s_\cup(e_2) = t_\cup(e_1) = t_G(e_1) \in G^0 \setminus (F^0 \cap G^0)$. Repeating this reasoning for all $\{e_i\}_{i=2}^n$, we conclude that $t_\cup(p) = t_\cup(e_n) = t_G(e_n) \in G^0 \setminus (F^0 \cap G^0)$, so $G^0 \setminus (F^0 \cap G^0)$ is hereditary in $F \cup G$.

Next, to establish that $G^0 \setminus (F^0 \cap G^0)$ is saturated in $F \cup G$, we consider all elements in $F^0 \cup G^0$ that emit an edge. Note first that any edge ending at a vertex in $G^0 \setminus (F^0 \cap G^0)$ must begin at a vertex in $G^0$, so we only need to consider vertices in $F^0 \cap G^0$:




Also, if $v$ is a sink in $G$ but not in $F \cup G$, it emits an edge ending at a vertex outside of $G^0 \setminus (F^0 \cap G^0)$, so we can disregard it. Furthermore, since $s_G^{-1}(\{v\}) \subseteq s_\cup^{-1}(\{v\})$, the finiteness of $s_\cup^{-1}(\{v\})$ implies the finiteness of $s_G^{-1}(\{v\})$. Hence we only need to consider vertices in $F^0 \cap G^0$ that are no sinks in $G$, that are finite emitters in $G$, and that emit all their edges to $G^0 \setminus (F^0 \cap G^0)$. For all such vertices, we have

$$\{t_\cup(e) \mid e \in s_\cup^{-1}(\{v\})\} = \{t_G(e) \mid e \in s_G^{-1}(\{v\})\}$$

because $t_\cup(e) \in G^0 \setminus (F^0 \cap G^0)$ implies that $e \in G^1$. Finally, such vertices do not exist by the saturation property of $G^0 \setminus (F^0 \cap G^0)$ in $G$, so $G^0 \setminus (F^0 \cap G^0)$ is saturated in $F \cup G$.

A symmetric argument proves that $F^0 \setminus (F^0 \cap G^0)$ is hereditary and staurated in $F \cup G$.

2. First, taking an advantage of the admissibility of $(F \cap G) \subseteq G$, we compute

$$t_\cup^{-1}(F^0) \setminus F^1 = (G^1 \setminus F^1) \cap t_G^{-1}(F^0 \cap G^0) = (G^1 \setminus F^1) \cap (F^1 \cap G^1) = \emptyset.$$

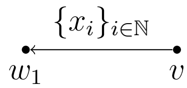
Therefore, as $F^1 \subseteq t_\cup^{-1}(F^0)$, we conclude that $F^1 = t_\cup^{-1}(F^0)$. Much in the same way, one shows that $t_\cup^{-1}(G^0) = G^1$.

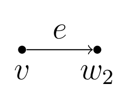
□

Remark: The opposite implication:

$$F \hookrightarrow F \cup G \hookleftarrow G \quad \text{is admissible} \quad \Rightarrow \quad F \hookleftarrow F \cap G \hookrightarrow G \quad \text{is admissible},$$

is not true:

$F :=$ 

$G :=$ 


$$F \cap G = (\{v\}, \emptyset, \emptyset, \emptyset), \qquad F \cup G = (\{w_1, v, w_2\}, \{x_i\}_{i\in\mathbb{N}} \cup \{e\}, s_\cup, t_\cup).$$

Let us check first that $F \hookrightarrow F \cup G$ is admissible. The set

$$(F^0 \cup G^0) \setminus F^0 = G^0 \setminus (F^0 \cap G^0) = \{v, w_2\} \setminus \{v\} = \{w_2\}$$

is hereditary in $F \cup G$ because $w_2$ is a sink in $F \cup G$. It is also saturated in $F \cup G$ because $w_1$ is a sink in $F \cup G$ and $v$ is an infinite emitter in $F \cup G$.

Next,

$$t_{\cup}^{-1}(F^0) = t_{\cup}^{-1}(\{v, w_1\}) = \{x_i\}_{i\in\mathbb{N}} = F^1.$$

Hence $F \hookrightarrow F\cup G$ is admissible. For the inclusion $G \hookrightarrow F \cup G$, consider the set

$$(F^0 \cup G^0) \setminus G^0 = F^0 \setminus (F^0 \cap G^0) = \{w_1, v\} \setminus \{v\} = \{w_1\}.$$

It is hereditary in $F \cup G$ because $w_1$ is a sink in $F \cup G$. It is also saturated in $F \cup G$ beacuse $w_2$ is a sink in $F \cup G$. Finally,

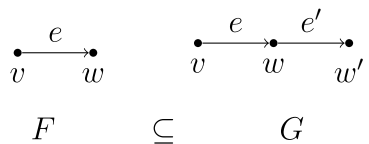
$$t_{\cup}^{-1}(G^0) = t_{\cup}^{-1}(\{v, w_2\}) = \{e\} = G^1.$$

Thus we have shown that the union $F \cup G$ is admissible. On the other hand, the intersection $F \cap G$ is not admissible because the set $G^0 \setminus (F^0 \cap G^0) = \{w_2\}$ is not saturated in $G$:

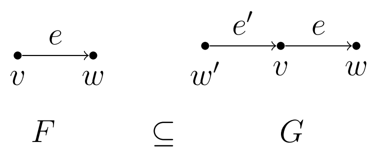
$$v \notin \{w_2\} \quad \text{and} \quad \{t_G(x) \mid x \in s_G^{-1}(\{v\})\} = \{w_2\}.$$

Elementary observations:

1. The properties of being hereditary and saturated are not preserved by the inclusion of graphs:

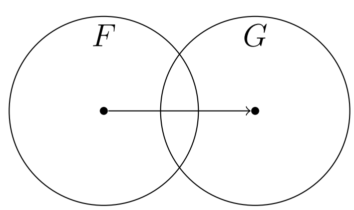
   (a) 
   

   $\{w\}$ is hereditary in $F$ but not in $G$.

   (b) 
   

   $\{v\}$ is saturated in $F$ but not in $G$.

   However, both properties are preserved by special inclusions $F \subseteq F \cup G$ for the special set $F^0 \setminus (F^0 \cap G^0)$ because there are no edges like this:

2. Restriction of graphs to subgraphs does not preserve the saturation property even in the special case of $F^0 \setminus (F^0 \cap G^0)$ in $F \subseteq F \cup G$. However, it always preserves the property of being hereditary: if $H \subseteq F^0$, $F \subseteq G$, is not hereditary in $F$, it is not hereditary in $G$. Indeed, if there is a path starting at $v \in H$ and ending at $w \in F^0 \setminus H$, then it is also a path starting at $v \in H$ and ending at $w \in G^0 \setminus H$.

Extended graph:

Let $E = (E^0, E^1, s_E, t_E)$ be a graph. The extended graph $\bar{E} := (\bar{E}^0, \bar{E}^1, s_{\bar{E}}, t_{\bar{E}})$ of the graph $E$ is defined as follows

$$\bar{E}^0 := E^0, \quad \bar{E}^1 := E^1 \sqcup (E^1)^*, \quad (E^1)^* := \{e^* \mid e \in E^1\},$$

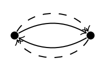
$$\forall\, e \in E^1 : \quad s_{\bar{E}}(e) := s_E(e), \quad t_{\bar{E}}(e) := t_E(e),$$
$$\forall\, e^* \in (E^1)^* : \quad s_{\bar{E}}(e^*) := t_E(e), \quad t_{\bar{E}}(e^*) := s_E(e).$$

Thus $E$ is a subgraph of $\bar{E}$.

Examples:

1. $E =$  $\bar{E} =$ 

2. $E =$  $\bar{E} =$ 

3. $E =$  $\bar{E} =$ 

# 7 Graph algebras

## 7.1 Path algebras

Let $V$ be any vector space over a field $k$. To endow $V$ with an algebra structure, we have to define the multiplication map $V \times V \overset{m}{\mapsto} V$, which is a bilinear map satisfying some conditions. Any such a map is uniquely determined by its value on pairs of basis elements $(e_i, e_j)$, and any assignment $(e_i, e_j) \mapsto v_{ij} \in V$ defines a bilinear map from $V \times V$ to $V$. Now, let $E$ be any graph, and $FP(E)$ the set of all its finite paths. Consider the vector space

$$kE := \{f \in \mathrm{Map}(FP(E), k) \mid f(p) \neq 0 \text{ for finitely many } p \in FP(E)\},$$

where the addition and scalar multiplication are pointwise. Then the set of functions $\{\chi_p\}_{p \in FP(E)}$ given by

$$\chi_p(q) = \begin{cases} 1 & \text{for } p = q \\ 0 & \text{otherwise} \end{cases}$$

is a linear basis of $kE$. Indeed, let $\{q_1, \ldots, q_n\}$ be the support of $f \in kE$. Then

$$f = \sum_{i=1}^{n} f(q_i)\chi_{q_i}$$

because, $\forall\, p \in FP(E)$:

$$\left(\sum_{i=1}^{n} f(q_i)\chi_{q_i}\right)(p) = \sum_{i=1}^{n} f(q_i)\chi_{q_i}(p)$$
$$= \begin{cases} 0 & \text{if } p \notin \{q_1,\ldots,q_n\} \\ f(q_i) & \text{if } p = q_i \text{ for } i \in \{1,\ldots,n\} \end{cases}$$
$$= f(p).$$

Hence $\{\chi_p\}_{p\in FP(E)}$ spans $kE$.

To see the linear independence, take any finite subset $\{\chi_{p_1},\ldots,\chi_{p_m}\} \subseteq \{\chi_p\}_{p\in FP(E)}$, and suppose that $\sum_{i=1}^{m} \alpha_i \chi_{p_i} = 0$. Then $\forall\, j \in \{1,\ldots,m\}$

$$0 = \left(\sum_{i=1}^{m} \alpha_i\chi_{p_i}\right)(p_j) = \sum_{i=1}^{m} \alpha_i\chi_{p_i}(p_j) = \alpha_j\,.$$

Thus we have shown that $\{\chi_p\}_{p\in FP(E)}$ is a basis of $kE$. Now we will use $\{\chi_p\}_{p\in FP(E)}$ to define a bilinear map:

$$m : kE \times kE \longrightarrow kE, \qquad m(\chi_p,\chi_q) := \begin{cases} \chi_{pq} & \text{if } t(p) = s(q) \\ 0 & \text{otherwise} \end{cases}.$$

**Proposition 7.1.** *The bilinear map $m : kE \times kE \to kE$ defines an algebra structure on $kE$.*

*Proof.* To check the associativity of $m$ it suffices to verify it on basis elements:

$$m(m(\chi_p,\chi_q),\chi_x) = \begin{cases} m(\chi_{pq},\chi_x) & \text{if } t(p) = s(q) \\ 0 & \text{if } t(p) \neq s(q) \end{cases}$$
$$= \begin{cases} \chi_{pqx} & \text{if } t(q) = s(x) \text{ and } t(p) = s(q) \\ 0 & \text{if } t(q) \neq s(x) \text{ and } t(p) = s(q) \\ 0 & \text{if } t(p) \neq s(q) \end{cases},$$

$$m(\chi_p, m(\chi_q,\chi_x)) = \begin{cases} m(\chi_p,\chi_{qx}) & \text{if } t(q) = s(x) \\ 0 & \text{if } t(q) \neq s(x) \end{cases}$$
$$= \begin{cases} \chi_{pqx} & \text{if } t(p) = s(q) \text{ and } t(q) = s(x) \\ 0 & \text{if } t(p) \neq s(q) \text{ and } t(q) = s(x) \\ 0 & \text{if } t(q) \neq s(x) \end{cases}.$$

Hence $m(m(\chi_p,\chi_q),\chi_x) = m(\chi_p, m(\chi_q,\chi_x))$ for any $p,q,x \in FP(E)$. (The distributivity follows from the bilinearity of $m$.) □

**Definition 7.2.** *Let $E$ be a graph. The above constructed algebra $(kE,+,0,m)$ is called the path algebra of $E$.*

Elementary facts:

The path algebra $kE$ of a graph $E$ is

1. finite dimensional $\iff$ $E$ is finite and acyclic (no loops),
2. unital $\iff$ $E^0$ is finite,
3. commutative $\iff$ $E^1 = \emptyset$ or each edge is a loop starting/ending at a different vertex.

## 7.2 Leavitt path algebras

**Definition 7.3.** *Let $A$ be a $k$-algebra and $S$ a subset of $A$. The ideal generated by $S$ is the set of all finite sums*

$$\sum_{i \in F} x_i s_i y_i \,,$$

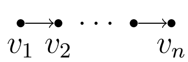
*where $s_i \in S$ and $x_i\,, y_i \in A$ for all $i \in F$.*

**Definition 7.4.** *Let $E$ be a graph and $k$ be a field. The Leavitt path algebra $L_k(E)$ of $E$ is the path algebra $k\bar{E}$ of the extended graph $\bar{E}$ divided by the ideal generated by the following elements:*

1. $\{\chi_{e^*}\chi_f - \delta_{ef}\chi_{t(e)} \mid e, f \in E^1\}$,
2. $\{\sum_{e \in s^{-1}(v)} \chi_e \chi_{e^*} - \chi_v \mid v \in E^0,\ 0 < |s^{-1}(v)| < \infty\}$.

Examples:

1. Matrix algebras:

$E =$  $\qquad L_k(E) = M_n(k)$

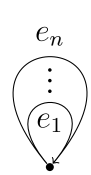


$E =$  $\qquad L_k(E) = M_\infty(k) = \bigcup_{n \in \mathbb{N}\setminus\{0\}} M_n(k)$

$$M_n(k) \ni M \longmapsto \begin{bmatrix} M & & 0 \\ & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \in M_{n+1}(k)$$

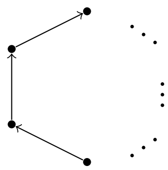
(arbitrary size finite matrices).

2. Laurent polynomial algebra: $E =$  $\qquad L_k(E) = k[\mathbb{Z}]$.

3. Leavitt algebras: $E =$  $\qquad L_k(E) = L_k(1, n)$



($\sum_{i=1}^n \chi_{e_i}\chi_{e_i^*} \Rightarrow L_k(1,n)^n \cong L_k(1,n)$ as modules.)

4. $E =$  $L_k(E) \cong M_n(k)[\mathbb{Z}] \cong M_n(k[\mathbb{Z}])$

(Laurent polynomials with matrix coefficients or matrices over Laurent polynomials.)

**Lemma 7.5.** *Let $E \hookrightarrow F$ be an admissible inclusion of row-finite (no infinite emitters) graphs and $k$ be a field. Then the formulas*

$$\chi_v \longmapsto \begin{cases} \chi_v & \text{if } v \in E^0 \\ 0 & \text{if } v \in F^0 \setminus E^0 \end{cases},$$

$$\chi_e \longmapsto \begin{cases} \chi_e & \text{if } e \in E^1 \\ 0 & \text{if } e \in F^1 \setminus E^1 \end{cases},$$

$$\chi_{e^*} \longmapsto \begin{cases} \chi_{e^*} & \text{if } e^* \in (E^1)^* \\ 0 & \text{if } e^* \in (F^1)^* \setminus (E^1)^* \end{cases},$$

*define a homomorphism $L_k(F) \stackrel{\pi}{\to} L_k(E)$ of algebras yielding the short exact sequence*

$$0 \longrightarrow I(F^0 \setminus E^0) \stackrel{\text{inclusion map}}{\longrightarrow} L_k(F) \stackrel{\pi}{\longrightarrow} L_k(E) \longrightarrow 0,$$

*where $I(F^0 \setminus E^0)$ is the ideal of $L_k(E)$ generated by $F^0 \setminus E^0$.*

**Corollary 7.6.**

$$L_k(E) \cong L_k(F)/I(F^0 \setminus E^0)$$

**Remark 7.7.** *If $A \stackrel{f}{\to} B$ is a surjective homomorphism of algebras and $A$ is unital, then $B$ is also unital and $f(1_A) = 1_B$. Indeed, $\forall\, b \in B$:*

$$f(1_A)b = f(1_A)f(a) = f(a) = b, \qquad bf(1_A) = f(a)f(1_A) = f(a) = b.$$

**Definition 7.8.** *Let $A_1 \stackrel{f_1}{\longrightarrow} B \stackrel{f_2}{\longleftarrow} A_2$ be homomorphisms of algebras. The pullback algebra $P(f_1, f_2)$ of $f_1$ and $f_2$ is*

$$\boxed{P(f_1, f_2) := \{(x, y) \in A_1 \oplus A_2 \mid f_1(x) = f_2(y)\}\,.}$$

*Here $A_1 \oplus A_2$ is viewed as an algebra with componentwise multiplication. $P(f_1, f_2)$ is a subalgebra of $A_1 \oplus A_2$ because $f_1$ and $f_2$ are algebra homomorphisms.*

**Theorem 7.9.** *Let $F_1$ and $F_2$ be row-finite graphs whose intersection $F_1 \cap F_2$ is admissible, and let $k$ be a field. Furthermore, let*

$$L_k(F_1) \stackrel{\pi_1}{\longrightarrow} L_k(F_1 \cap F_2) \stackrel{\pi_2}{\longleftarrow} L_k(F_2)$$

*and*

$$L_k(F_1) \stackrel{p_1}{\longleftarrow} L_k(F_1 \cup F_2) \stackrel{p_2}{\longrightarrow} L_k(F_2)$$

*be the canonical surjections of the preceding lemma. Then the map*

$$L_k(F_1 \cup F_2) \ni x \longmapsto (p_1(x), p_2(x)) \in L_k(F_1) \oplus L_k(F_2)$$

*corestricts to an isomorphism $L_k(F_1 \cup F_2) \to P(\pi_1, \pi_2)$ of algebras.*