MIDTERM 4000

Problem 1 Let R be a ring and $(G, \cdot, e)$ be a monoid. For any $g \in G$, let δ_g be an element of the monoidal ring $R[G]$ defined by

$$\delta_g(h) := \begin{cases} 1 & \text{for} \quad h = g, \\ 0 & \text{for} \quad h \neq g. \end{cases}$$

Prove that δ_e is the neutral element for the convolution multiplication in $R[G]$.

Solution: Let $\alpha \in R[G]$. Then

$$(\alpha * \delta_e)(g) = \sum_{\substack{h,\, k \in G \\ hk = g}} \alpha(h)\delta_e(k) = \sum_{\substack{h \in G \\ h = g}} \alpha(h) = \alpha(g) \quad \text{for all } g \in G,$$

$$(\delta_e * \alpha)(g) = \sum_{\substack{h,\, k \in G \\ hk = g}} \delta_e(h)\alpha(k) = \sum_{\substack{k \in G \\ k = g}} \alpha(k) = \alpha(g) \quad \text{for all } g \in G.$$

Problem 2 Show that a field K has characteristic zero if and only if it contains a subfield isomorphic with the field $\mathbb{Q}$ of rational numbers.

Solution: If K has characteristic 0, then $\forall\ q \in \mathbb{N} \setminus \{0\} : q \cdot 1_K \neq 0$, so there is $(q \cdot 1_K)^{-1} \in K$. Consequently, $\forall\ p \in \mathbb{Z} :\ p \cdot (q \cdot 1_K)^{-1} \in K$. Denote the set of all such elements by K'. Since

$$p \cdot (q \cdot 1_K)^{-1} = pr \cdot (qr \cdot 1_K)^{-1} \iff (p \cdot 1_K)(qr \cdot 1_K) = (pr \cdot 1_K)(q \cdot 1_K),$$

we have the map $f : \mathbb{Q} \ni \frac{p}{q} \mapsto p \cdot (q \cdot 1_K)^{-1}$ that has K' as its image. Furthermore, as

$$p \cdot (q \cdot 1_K)^{-1} + p' \cdot (q' \cdot 1_K)^{-1} = (pq' + p'q) \cdot (qq' \cdot 1_K)^{-1},$$

because

$$(p \cdot 1_K)(q \cdot 1_K)^{-1}(qq' \cdot 1_K) + (p' \cdot 1_K)(q' \cdot 1_K)^{-1}(qq' \cdot 1_K) = (pq' + p'q) \cdot 1_K\ ,$$

f preserves the addition. It also preserves the multiplication, because

$$(p \cdot (q \cdot 1_K)^{-1})(p' \cdot (q' \cdot 1_K)^{-1}) = pp' \cdot (qq' \cdot 1_K)^{-1},$$

and maps 1 to 1_K. Hence f is a field homomorphism, so it must be injective and its image must be a field, as for $f(x) \neq 0$ we have $x \neq 0$ and $f(x)^{-1} = f(x^{-1})$. Consequently, $\mathbb{Q}$ is isomorphic with the subfield K'.

Vice versa, if $f : \mathbb{Q} \to K$ is a field homomorphism, then

$$\forall\ n \in \mathbb{N} \setminus \{0\} :\ f(n) = n \cdot 1_K \neq 0,$$

so K has characteristic 0.

Problem 3 Prove that $A := \{f \in \mathrm{Map}(\mathbb{R},\mathbb{R}) \mid f(\sqrt{\pi}) = 0\}$ is an algebra with respect to the pointwise operations. Show that its minimal unitization is the algebra $\mathrm{Map}(\mathbb{R},\mathbb{R})$ with the pointwise operations.

Solution: Since A is a subset of the algebra $\mathrm{Map}(\mathbb{R},\mathbb{R})$ containing 0, we only need to check that it is closed under the pointwise addition, scalar multiplication and multiplication:

$$\begin{aligned}
&\forall\, f,g \in A :\ (f+g)(\sqrt{\pi}) = f(\sqrt{\pi}) + g(\sqrt{\pi}) = 0+0 = 0,\\
&\forall\, \alpha \in \mathbb{R},\ f \in A :\ (\alpha f)(\sqrt{\pi}) = \alpha(f(\sqrt{\pi})) = \alpha\, 0 = 0,\\
&\forall\, f,g \in A :\ (fg)(\sqrt{\pi}) = f(\sqrt{\pi})g(\sqrt{\pi}) = 0\,0 = 0.
\end{aligned}$$

For the second part, we prove that the following map

$$A^{+} \ni (f,\alpha) \overset{\varphi}{\longmapsto} f + \alpha\, 1 \in \mathrm{Map}(\mathbb{R},\mathbb{R})$$

is an isomorphism of unital algebras. The above map is clearly linear and unital. It is also bijective as it has the inverse map:

$$\mathrm{Map}(\mathbb{R},\mathbb{R}) \ni \widetilde{f} \longmapsto (\widetilde{f} - \widetilde{f}(\sqrt{\pi})1, \widetilde{f}(\sqrt{\pi})) \in A^{+}.$$

Indeed, the above map is well defined because, for any $\widetilde{f} \in \mathrm{Map}(\mathbb{R},\mathbb{R})$, we have

$$(\widetilde{f} - \widetilde{f}(\sqrt{\pi})1)(\sqrt{\pi}) = \widetilde{f}(\sqrt{\pi}) - \widetilde{f}(\sqrt{\pi}) = 0,$$

and the verification that it is a two-sided inverse of φ is immediate. Finally, we prove that φ intertwines the multiplications:

$$\begin{aligned}
\varphi((f,\alpha)(g,\beta))(x) &= \varphi((fg + \alpha g + \beta f, \alpha\beta))(x)\\
&= f(x)g(x) + \alpha g(x) + \beta f(x) + \alpha\beta\\
&= (f(x)+\alpha)(g(x)+\beta)\\
&= \varphi((f,\alpha))(x)\varphi((g,\beta))(x).
\end{aligned}$$

Since the above equality is true for any $x \in \mathbb{R}$, we conclude that

$$\varphi((f,\alpha)(g,\beta)) = \varphi((f,\alpha))\varphi((g,\beta)).$$

Problem 4 Let s be a central element in a ring R: $sr = rs$ for any $r \in R$. Show that $I := \{r \in R \mid rs = 0\}$ is an ideal of R.

Solution: As $0 \in I$ and $\forall\, r, r' \in I$:

$$(r + r')s = rs + r's = 0 + 0 = 0,$$

the set I is an additive subgroup of R. Due to the centrality of s in R, it also satisfies the ideal property:

$$\begin{aligned} \forall\, r \in I,\, r' \in R : (r'r)s &= r'(rs) = r'0 = 0, \\ (rr')s &= r(r's) = r(sr') = (rs)r' = 0r' = 0. \end{aligned}$$

Problem 5 Consider the homomorphisms of abelian groups:

$$\mathbb{Z} \ni n \longmapsto 2n \in \mathbb{Z} \quad \text{and} \quad \mathbb{Z} \ni n \longmapsto 3n \in \mathbb{Z}.$$

Show that they induce the homomorphisms of quotient groups

$$\mathbb{Z}/3\mathbb{Z} \ni [n]_3 \xmapsto{f_2} [2n]_6 \in \mathbb{Z}/6\mathbb{Z} \quad \text{and} \quad \mathbb{Z}/6\mathbb{Z} \ni [n]_6 \xmapsto{f_3} [3n]_2 \in \mathbb{Z}/2\mathbb{Z},$$

respectively. Then prove that the sequence of group homomorphisms

$$0 \longrightarrow \mathbb{Z}/3\mathbb{Z} \xrightarrow{f_2} \mathbb{Z}/6\mathbb{Z} \xrightarrow{f_3} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact.

Solution: The induced maps are well defined because $2 \cdot 3\mathbb{Z} = 6\mathbb{Z}$ and $3 \cdot 6\mathbb{Z} = 18\mathbb{Z} \subset 2\mathbb{Z}$. Furthermore, f_3 is surjective because $f_3([1]_6) := [3]_2 = [1]_2$. Next, f_2 is injective because

$$f_2([n]_3) := [2n]_6 = [0]_6 \iff 2n \in 6\mathbb{Z} \iff n \in 3\mathbb{Z} \iff [n]_3 = 0.$$

To see that $f_2(\mathbb{Z}/3\mathbb{Z}) \subseteq \ker f_3$, note that, for any $n \in \mathbb{Z}$,

$$f_3(f_2([n]_3)) = f_3([2n]_6) = [6n]_2 = [0]_2.$$

Vice versa, $\ker f_3 \subseteq f_2(\mathbb{Z}/3\mathbb{Z})$ because

$$f_3([n]_6) := [3n]_2 = [0]_2 \iff 3n \in 2\mathbb{Z} \iff n \in 2\mathbb{Z} \iff [n]_6 \in f_2(\mathbb{Z}/3\mathbb{Z}).$$