MIDTERM 5000

Problem 1 Let R be a ring and G be a finite group. For any $g \in G$, let δ_g be an element of the group ring $R[G]$ defined by

$$\delta_g(h) := \begin{cases} 1 & \text{for} \quad h = g, \\ 0 & \text{for} \quad h \neq g. \end{cases}$$

Prove that $\sum_{g \in G} \delta_k * \delta_g = \sum_{g \in G} \delta_g$ for any $k \in G$.

Solution: First, we show that $\delta_k * \delta_g = \delta_{kg}$:

$$(\delta_k * \delta_g)(h) = \sum_{\substack{m, n \in G \\ mn = h}} \delta_k(m)\delta_g(n) = \begin{cases} 1 & kg = h \\ 0 & kg \neq h \end{cases} = \delta_{kg}(h).$$

Next, we compute:

$$\sum_{g \in G} (\delta_k * \delta_g)(h) = \sum_{g \in G} \delta_{kg}(h) = \sum_{g \in G} \delta_g(h).$$

Here we used the fact that, for any $k \in G$, the map:

$$G \ni g \longmapsto kg \in G$$

is a bijection.

Problem 2 Prove that $\mathbb{Q} + \mathbb{Q}\sqrt{2}$ is a subfield of $\mathbb{R}$.

Solution: Since $\mathbb{Q}+\mathbb{Q}\sqrt{2}$ is a subset of the field $\mathbb{R}$ containing its neutral elements 0 and 1, we only need to show that $\mathbb{Q} + \mathbb{Q}\sqrt{2}$ is closed under addition, multiplication and taking inverses of non-zero elements:

$$\forall\, a, b, c, d \in \mathbb{Q} \,:\, (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in \mathbb{Q} + \mathbb{Q}\sqrt{2},$$

$$\forall\, a, b, c, d \in \mathbb{Q} \,:\, (a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2} \in \mathbb{Q} + \mathbb{Q}\sqrt{2},$$

$\forall\, a, b \in \mathbb{Q}$ such that $a + b\sqrt{2} \neq 0$:

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{(a + b\sqrt{2})(a - b\sqrt{2})} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2} \in \mathbb{Q} + \mathbb{Q}\sqrt{2}.$$

Here we used the fact that $a+b\sqrt{2} \neq 0$ implies $a-b\sqrt{2} \neq 0$. Indeed, if $a-b\sqrt{2} = 0$, then $b = 0$ as $\sqrt{2} \notin \mathbb{Q}$, which implies $a = 0$.

Problem 3 Prove that $A := \{f \in \mathrm{Map}(\mathbb{R},\mathbb{R}) \mid f([0,1]) = \{0\}\}$ is an algebra with respect to the pointwise operations. Show that its minimal unitization is the algebra $B := \{f \in \mathrm{Map}(\mathbb{R},\mathbb{R}) \mid \forall\, t,s \in [0,1] : f(t) = f(s)\}$ with the pointwise operations.

Solution: Since A is a subset of the algebra $\mathrm{Map}(\mathbb{R},\mathbb{R})$ containing 0, we only need to check that it is closed under the pointwise addition, scalar multiplication and multiplication:

$$\begin{aligned}
&\forall\, f,g \in A,\ t \in [0,1] :\ (f+g)(t) = f(t)+g(t) = 0+0 = 0,\\
&\forall\, \alpha \in \mathbb{R},\ f \in A,\ t \in [0,1] :\ (\alpha f)(t) = \alpha(f(t)) = \alpha\, 0 = 0,\\
&\forall\, f,g \in A,\ t \in [0,1] :\ (fg)(t) = f(t)g(t) = 0\, 0 = 0.
\end{aligned}$$

For the second part, we prove that the following map

$$A^+ \ni (f,\alpha) \overset{\varphi}{\longmapsto} f + \alpha\, 1 \in B$$

is an isomorphism of unital algebras. The above map is well defined because, for any $\alpha \in \mathbb{R}$, $f \in A$, $t,s \in [0,1]$, we have:

$$(f+\alpha\, 1)(t) = f(t)+\alpha = \alpha = f(s)+\alpha = (f+\alpha\, 1)(s).$$

Furthermore, it is clearly linear and unital. It is also bijective as any $t_0 \in [0,1]$ yields the inverse map:

$$B \ni \widetilde{f} \longmapsto (\widetilde{f} - \widetilde{f}(t_0)1, \widetilde{f}(t_0)) \in A^+.$$

Indeed, the above map is well defined because, for any $\widetilde{f} \in B$ and any $t \in [0,1]$, we have $(\widetilde{f} - \widetilde{f}(t_0)1)(t) = \widetilde{f}(t) - \widetilde{f}(t_0) = 0$, an the verification that it is a two-sided inverse of φ is immediate. Finally, we prove that φ intertwines the multiplications:

$$\begin{aligned}
\varphi((f,\alpha)(g,\beta))(x) &= \varphi((fg+\alpha g+\beta f, \alpha\beta))(x)\\
&= f(x)g(x) + \alpha g(x) + \beta f(x) + \alpha\beta\\
&= (f(x)+\alpha)(g(x)+\beta)\\
&= \varphi((f,\alpha))(x)\varphi((g,\beta))(x).
\end{aligned}$$

Since the above equality is true for any $x \in \mathbb{R}$, we conclude that

$$\varphi((f,\alpha)(g,\beta)) = \varphi((f,\alpha))\varphi((g,\beta)).$$

Problem 4 An element x of a ring R is called nilpotent iff $\exists\, n \in \mathbb{N} \setminus \{0\} : x^n = 0$. Show that the set I of all nilpotent elements in a commutative ring R is an ideal of R.

Solution: Clearly, $0 \in I$. Next, let $x, y \in I$. Then $\exists\ m, n \in \mathbb{N} \setminus \{0\}$: $x^m = 0$ and $y^n = 0$. With the help of the commutativity of R, it follows that

$$(x+y)^{m+n-1} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} x^{m+n-1-k} y^k = 0.$$

Indeed, if $k \le n-1$, then $x^{m+n-1-k} = 0$, and, if $k \ge n$, then $y^k = 0$. Hence all the elements of the above sum are zero. This shows that I is an abelian subgroup of R. Finally, we verify the ideal property. If $x^m = 0$, then for any $r \in R$, taking again an advantage of the commutativity of R, we obtain:

$$(xr)^m = x^m r^m = 0r^m = 0, \quad (rx)^m = r^m x^m = r^m 0 = 0.$$

Problem 5 Let $H \xrightarrow{\phi} G \xrightarrow{\psi} K$ be an exact sequence of abelian groups. Let $m, n \in \mathbb{N} \setminus \{0\}$ be such that $mH = 0$ and $nK = 0$ (i.e., any element of H taken m times and added to itself is zero and any element of K taken n times and added to itself is zero). Prove that $mn\,G = 0$.

Solution: For any $g \in G$, we have $0 = n\psi(g) = \psi(ng)$, so $ng \in \ker\psi$. By the exactness of the sequence, there is $h \in H$ such that $\phi(h) = ng$. Consequently,

$$0 = \phi(0) = \phi(mh) = m\phi(h) = mng.$$