MIDTERM 6000

Problem 1 (2 points) Let R be a ring and $R[\mathbb{N}]$ be the polynomial ring with coeffcients in R. Furthermore, let x^l, $l \in \mathbb{N}$, be the elements of $R[\mathbb{N}]$ defined by

$$x^l(n) := \begin{cases} 1 & \text{for} \quad n = l, \\ 0 & \text{for} \quad n \neq l. \end{cases}$$

Prove that, for all natural numbers k, m, n,

$$\left(\left(\sum_{i=0}^{k} x^{2i} \right) * \left(\sum_{j=0}^{m} x^{2j+1} \right) \right) (2n) = 0.$$

Solution: Note first that $x^{m_1} * x^{m_2} = x^{m_1+m_2}$ for all natural numbers m_1 and m_2:

$$\forall\, n \in \mathbb{N}\colon (x^{m_1} * x^{m_2})(n) = \sum_{n_1+n_2=n} x^{m_1}(n_1) x^{m_2}(n_2) = \begin{cases} 1 & \text{for} \quad n = m_1 + m_2, \\ 0 & \text{for} \quad n \neq m_1 + m_2. \end{cases}$$

Now we can directly compute

$$\left(\left(\sum_{i=0}^{k} x^{2i} \right) * \left(\sum_{j=0}^{m} x^{2j+1} \right) \right) (2n) = \sum_{i=0}^{k} \sum_{j=0}^{m} x^{2(i+j)+1}(2n) = 0.$$

Indeed, $2(i+j)+1$ is always odd and $2n$ is always even, so they can never be equal.

Problem 2 (3 points) Let R be a ring and let $r \in R$ be a non-zero element such that $r^n = 0$ for some $n \in \mathbb{N} \setminus \{0\}$. Show that R cannot be a field.

Solution: We prove it by contradiction. Suppose R is a field. Then it is commutative and all its non-zero elements are invertible. Hence there exists the multiplicative inverse r^{-1}, and using the commutativity of R, we obtain

$$1 = 1^n = (r^{-1}r)^n = r^{-n}r^n = r^{-n} \cdot 0 = 0.$$

This yields the desired contradiction because $1 \neq 0$ in any field.

Problem 3 (4 points) Let $f : A \to B$ be a homomorphism of k-algebras. Assume that B is unital, and let $\iota : A \to A^+$ denote the inclusion of A into its minimal unitization A^+. Construct a unital algebra homomorphism $\tilde{f} : A^+ \to B$ such that $f = \tilde{f} \circ \iota$.

Solution: We define $\tilde{f} : A^+ \to B$ by

$$\tilde{f}(a, \alpha) := f(a) + \alpha \cdot 1_B \,,$$

where 1_B is the unit of B. This map is evidently k-linear and it satisfies the condition

$$\forall\, a \in A\colon\ (\tilde{f} \circ \iota)(a) = \tilde{f}(a, 0) = f(a) + 0 \cdot 1_B = f(a),$$

so it only remains to show that it preserves the algebra multiplication:

$$\begin{aligned}
\forall\, a, b \in A,\ \alpha, \beta \in k\colon\ & \tilde{f}\big((a, \alpha)(b, \beta)\big) \\
&= \tilde{f}(ab + \alpha b + \beta a, \alpha\beta) \\
&= f(ab + \alpha b + \beta a) + \alpha\beta \cdot 1_B \\
&= f(a)f(b) + \alpha f(b) + \beta f(a) + (\alpha \cdot 1_B)(\beta \cdot 1_B) \\
&= \big(f(a) + \alpha \cdot 1_B\big)\big(f(b) + \beta \cdot 1_B\big) \\
&= \tilde{f}((a, \alpha))\tilde{f}((b, \beta)).
\end{aligned}$$

Problem 4 (3 points) Let R be a commutative ring and let I be an ideal of R. Prove that the set

$$\sqrt{I} := \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N} \setminus \{0\}\}$$

is an ideal of R.

Solution: We check first that $\sqrt{I}$ is a subgroup of R. Take any $r, s \in \sqrt{I}$. Then there exist positive integers m and n such that $r^m \in I$ and $s^n \in I$. Consequently,

$$(r+s)^{m+n-1} = \sum_{k=0}^{m+n-1} \binom{m+n-1}{k} r^{m+n-1-k} s^k \in I.$$

Indeed, for $k \geq n$ elements of the sum are in I because $s^k \in I$. For $k < n$, we have $m+n-1-k \geq m$, so elements of the sum are in I because $r^{m+n-1-k} \in I$. Finally, $\sqrt{I}$ is closed under the multiplication by any elements of R because

$$\forall\, r \in \sqrt{I} \text{ such that } r^n \in I,\ n \in \mathbb{N} \setminus \{0\},\ x \in R : (rx)^n = r^n x^n \in I.$$

Problem 5 (4 points) Consider the homomorphisms of abelian groups:

$$\mathbb{Z} \ni n \longmapsto 6n \in \mathbb{Z} \quad \text{and} \quad \mathbb{Z} \ni n \longmapsto 18n \in \mathbb{Z}.$$

Show that they induce the homomorphisms of quotient groups

$$\mathbb{Z}/18\mathbb{Z} \ni [n]_{18} \overset{f_6}{\longmapsto} [6n]_{54} \in \mathbb{Z}/54\mathbb{Z} \quad \text{and} \quad \mathbb{Z}/54\mathbb{Z} \ni [n]_{54} \overset{f_{18}}{\longmapsto} [18n]_{108} \in \mathbb{Z}/108\mathbb{Z},$$

respectively. Then prove that the sequence of group homomorphisms

$$\mathbb{Z}/18\mathbb{Z} \xrightarrow{f_6} \mathbb{Z}/54\mathbb{Z} \xrightarrow{f_{18}} \mathbb{Z}/108\mathbb{Z}$$

is exact.

Solution: The induced maps are well defined because $6 \cdot 18\mathbb{Z} = 108\mathbb{Z} \subset 54\mathbb{Z}$ and $18 \cdot 54\mathbb{Z} = 972\mathbb{Z} \subset 108\mathbb{Z}$. To see that $f_6(\mathbb{Z}/18\mathbb{Z}) \subseteq \ker f_{18}$, note that, for any $n \in \mathbb{Z}$,

$$f_{18}(f_6([n]_{18})) = f_{18}([6n]_{54}) = [108n]_{108} = [0]_{108}.$$

Vice versa, $\ker f_{18} \subseteq f_6(\mathbb{Z}/18\mathbb{Z})$ because

$$f_{18}([n]_{54}) := [18n]_{108} = [0]_{108} \iff 18n \in 108\mathbb{Z} \iff n \in 6\mathbb{Z} \iff [n]_{54} \in f_6(\mathbb{Z}/18\mathbb{Z}).$$