Torbjörn Lundh
Mathematical Sciences, Chalmers and University of Gothenburg
e-mail: torbjorn.lundh@chalmers.se
Jun Udagawa
Department of Developmental Biology, Shimane University
e-mail: jun@med.shimane-u.ac.jp
Sven-Erik Hänel
IFP Research AB, 431 22 Mölndal, Sweden
e-mail: haenel@gotanet.com
Hiroki Otani
Department of Developmental Biology, Shimane University
e-mail: hotani@med.shimane-u.ac.jp

Invariances of cross- and trippel-ratios of human limbs?

Recall that in the complex plain, four points, p, q, r, s, can be mapped to four other points, $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$, by a Möbius transformation, $z \mapsto \frac{az+b}{cz+d}$, if and only if the cross-ratio, $\frac{(p-r)(q-s)}{(p-s)(q-r)}$, equals the cross-ratio of $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$. In [1], a bold and highly inspiring statement was given that the cross-ratio of consecutive joints of human limbs, are invariant, not only over time, but also between different limbs, and even different persons! In order to investigate this intriguing statement, but also to develop new morphometric tools for development studies, we geometrically analyze the morphological development of the human body, and we examined the cross-ratio of three consecutive body parts that are segmented by four landmarks in their configuration. Moreover, we introduce an generalization of the cross-ratio: the triple-ratio of five landmarks that segments four consecutive parts (e.g. the shoulder, upper arm, forearm, and hand) and examined their growth patterns. The triple-ratio was defined for five arbitrary points, p, q, r, s, and t as:

$$\kappa(p, q, r, s, t) = \frac{|p-r||q-s||r-t|}{|q-r||r-s||p-t|}.$$

It is easy to show that also the trippel-ratio is invariant under Möbius transformations. The cross- and triple-ratios of the upper limb and shoulder girdle in fetuses were constant when biomechanical landmarks were used although the cross-ratio of the upper limb varied when the anatomical landmarks were used. The cross-ratios of the lower limbs, trunk, and pelvic girdles of fetuses differed from their corresponding cross-ratios in adults. These results suggest the Möbius growth in the fetal upper limb and shoulder girdle, but not in the other body parts we examined. However, the growth balance of the three contiguous body parts was represented by the developmental change in the cross-ratio. Therefore, the cross- and triple-ratios may be applicable for the assessment of growth balance or proportion of the body parts.

References

[1] S.V. Petukhov *Non-Euclidean geometries and algorithms of living bodies* Comput. Math. Appl. **17**:505–534.