Optimal funding of a defined benefit pension plan.

Donatien Hainaut, Griselda Deelstra.

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Abstract

In this paper, we address the issue of determining the optimal contribution rate of a stochastic defined benefit pension fund. The affiliate's mortality is modelled by a jump process and the benefits paid at retirement are function of the evolution of stochastic salaries. Assets of the fund are invested in cash, stocks and a rolling bond. Interest rates are driven by a Vasicek model. The objective is to minimize both the quadratic spread between the contribution rate and the normal cost, and the quadratic spread between the terminal wealth and the mathematical reserve required to cover benefits. The optimization is done under a budget constraint that guarantees the actuarial equilibrium between the current asset and future contributions and benefits. The method of resolution is based on the Cox and Huang's approach and on dynamic programming.

KEYWORDS : defined benefit, pension fund, asset allocation, optimal rate of contribution.

1 Introduction.

There mainly exist two categories of pension funds: the defined contribution pension plan and the defined benefit pension plan. In the first one, the financial risk is beared by the affiliate: in case of poor performance of assets, his savings may be insufficient to maintain his standard of living at retirement. Whereas in a defined benefit pension plan, the risk is beared by the pension fund: whatsoever the return of assets, benefits paid to pensioners are proportional to his salary. In this context, the choice of the investment policy and of the contribution pattern is hence crucial for the agent financing the fund.

Defined benefit pension plans have been extensively studied in the literature. Some authors like Sundaresan and Zapatero (1997) argue that the investor should maximize the expected utility of the surplus of assets over the liabilities of the fund. However, especially from the employer's point of view who pays for the defined benefit pension plan of his employees, the important issue is to find a contribution process which has small fluctuations and which leads as exactly as possible to the value of the mathematical reserve necessary for covering the liabilities promised in the pension plan. Therefore a whole branch of papers has studied the minimization of a loss function of contributions and the wealth to be obtained. In the papers of e.g. Haberman and Sung (1994, 2005), Boulier et al. (1995), Josa Fombellida and Rincon-Zapatero (2004, 2006), the fund manager keeps the value of the assets as close as possible to liabilities by controlling the level of contributions. Cairns (1995, 2000) has discussed the role of objectives in selecting an asset allocation strategy and has analysed some current problems faced by defined benefit pension funds. Huang and Cairns (2006) have studied the optimal contribution rate for defined benefit pension plans when interest rates are stochastic.

The most novel features of our work are the modelling of the affiliates' mortality by a jump process, the use of stochastic interest rates and salaries. Furthermore, we minimize both contribution adjustments and a terminal surplus. By contribution adjustment, we mean the spread between the sponsor's contribution and the normal cost. Whereas the terminal surplus is here defined as the difference between the terminal wealth and the fair value of liabilities at retirement. The optimization is done under a budget constraint that ensures the actuarial equilibrium between the current assets and future deflated cash flows and with initial negative unfunded liabilities. We will see that the last condition is important for obtaining a target wealth higher than the mathematical reserve for the promised liabilities. The objective function in the optimization problem further contains some weighting expressing the importance given to the minimization of contribution adjustments and of the terminal surplus. Numerical results will show that the optimal contribution process depends on the weights corresponding to the minimization of the surplus variation in comparison with the weight corresponding to the minimization of the contribution fluctuations.

In this paper we deal with the difficulty that the presence of random salary and mortality entails that the market is incomplete. The set of equivalent martingale measures counts therefore more than one element and we need to fix the deflator used by the insurer to value liabilities in order to apply the Cox & Huang (1989) martingale method. This approach was used in a similar setting by Brennan and Xia (2002) and translates in fact common practice of actuaries who traditionally already used security adjustments, which means that they already chose a certain probability measure to work under. A more annoying implication of the incompleteness implied by the salary and mortality risk, is that the optimal target wealth process found by the martingale method is not fully replicable. However, it is possible to determine the investment strategy replicating at best this solution by using the dynamic programming principle as in Hainaut and Devolder (2006a,b) who are both studies of asset allocation of deterministic insurance liabilities with a stochastic mortality risk.

The outline of this paper is as follows. Sections 2 and 3 respectively present the financial market and the defined benefit pension plans. In section 4, the form of the deflator is discussed. Section 5 introduces the optimization problem and in section 6, we propose a solution. Section 7 contains a numerical illustration and the last section concludes.

2 The financial market.

In this section, we introduce the market structure of our model and define the dynamics of interest rates and asset values. The uncertainty involved by the financial market is described by a 2-dimensional standard Brownian motion $W_t^{P^f} = \left(W_t^{r,P^f}, W_t^{S,P^f}\right)$ defined on a complete probability space $(\Omega^f, \mathcal{F}^f, P^f)$. $\mathcal{F}^f$ is the filtration generated by $W_t^{P^f}$:

$$\mathcal{F}^f = \left(\mathcal{F}_t^f\right)_t = \sigma\left\{\left(W_t^{r,P^f}, W_t^{S,P^f}\right) : u \leq t\right\}.$$

P^f represents the historical financial probability measure. The two Wiener processes W_t^{r,P^f} and W_t^{S,P^f} are independent. The financial market is complete and there exists therefore a unique equivalent measure under which the discounted prices of assets are martingales. This risk neutral measure is denoted by Q^f. The assets of the defined benefit pension fund are invested in cash, rolling bonds and stocks. The return of cash is the risk free rate r_t and is modelled by an Ornstein-Uhlenbeck process (Vasicek model):

$$dr_t = a.(b - r_t).dt + \sigma_r.dW_t^{r,P^f}. \tag{2.1}$$

The constant parameters a, b, σ_r are respectively the speed of mean reversion, the level of mean reversion and the volatility of r_t. Let λ_r be a negative constant being the market price of risk and so implying the dynamics of r_t under the risk neutral measure Q^f. Indeed under Q^f, the risk free rate is the solution of the following SDE:

$$dr_t = a.(\underbrace{b - \sigma_r.\frac{\lambda_r}{a}}_{b^Q} - r_t).dt + \sigma_r.\underbrace{\left(dW_t^{r,P^f} + \lambda_r.dt\right)}_{dW_t^{r,Q^f}}, \tag{2.2}$$

where W_t^{r,Q^f} is a Wiener process under Q^f.
The second category of assets is a rolling bond of maturity K whose price is denoted R_t^K. This bond is a zero coupon bond continuously rebalanced in order to keep a constant maturity and its price obeys to the dynamics:

$$\begin{aligned}\frac{dR_t^K}{R_t^K} &= r_t.dt - \sigma_r.n(K).\left(dW_t^{r,P^f} + \lambda_r.dt\right)\\ &= r_t.dt - \sigma_r.n(K).dW_t^{r,Q^f}\end{aligned}$$

where $n(K)$ is a function of the maturity K :

$$n(K) = \frac{1}{a}.\left(1 - e^{-a.K}\right).$$

Remark that the risk premium of the rolling bond is denoted by $\nu_R = -\sigma_r.n(K).\lambda_r$.
The last kind of assets available on the financial market is a stock. Its price process S_t is modelled by a geometric Brownian motion and is correlated with the interest rates fluctuations:

$$\begin{aligned}\frac{dS_t}{S_t} &= r_t.dt + \sigma_{Sr}.\left(dW_t^{r,P^f} + \lambda_r.dt\right) + \sigma_S.\left(dW_t^{S,P^f} + \lambda_S.dt\right)\\ &= r_t.dt + \sigma_{Sr}.dW_t^{r,Q^f} + \sigma_S.dW_t^{S,Q^f}.\end{aligned}$$

The constant parameters σ_{Sr}, σ_S and λ_S denote respectively the correlation between stocks and the risk free interest rates, the embedded volatility of the stocks and the market price of risk parameter. The stock risk premium is defined by $\nu_S = \sigma_{Sr}.\lambda_r + \sigma_S.\lambda_s$.

3 The pension fund.

The pension plan considered in this work provides benefits to affiliates which are defined in terms of a member's final salary. For the sake of simplicity, one assumes that the pension fund counts initially n_x members of the same age x and earning the same salary, denoted $(A_t)_t$. All members retire at the age $x+T$ and in case of death, no benefits are paid. The evolution of the individual salary is stochastic and correlated to the financial market. More precisely, one supposes that the dynamics of an affiliate's salary are defined by the following SDE:

$$\frac{dA_t}{A_t} = \mu_A(t).dt + \sigma_{Ar}dW_t^{r,P^f} + \sigma_{AS}dW_t^{S,P^f} + \sigma_A.dW_t^{A,P^a} \tag{3.1}$$

where $\mu_A(t)$ is the average growth of the salary and W_t^{A,P^a} is a Wiener process that represents the intrinsic randomness of the salary and is independent from W_t^{r,P^f} and W_t^{S,P^f}. As this salary risk is not traded, W_t^{A,P^a} is a source of incompleteness. We will come back to this point in the next section. The constants σ_{Ar} and σ_{AS} model the correlation of the salary with resp. interest rates and stocks; and σ_A denotes the embedded wage volatility. W_t^{A,P^a} is defined on a probability space $(\Omega^a, \mathcal{F}^a, P^a)$ where $\mathcal{F}^a$ is the filtration generated by W_t^{A,P^a}.

Benefits are defined in terms of the salary at retirement date. Each pensioner will receive a continuous annuity whose rate B is a fraction, α of the last wage:

$$B = A_T.\alpha.$$

These benefits are financed during the accumulation phase. c_t is the contribution rate made by the sponsor to the funding process at time t.

The fair value of liabilities will be discussed in the next section. We now detail the jump process modelling the mortality of the covered employees. The mortality process is defined as in Møller (1998) on a probability space $(\Omega^m, \mathcal{F}^m, P^m)$ and is assumed to be independent from the filtration generated by W_t^{r,P^f} , W_t^{S,P^f}, W_t^{S,P^a}. The remaining lifetimes of the affiliates are exponential random variables, denoted $T_1, T_2, \ldots, T_{n_x}$ and their hazard rate (namely the mortality rate), at time t, is given by $\mu(x+t)$. N_t points out the total number of deaths observed till time t:

$$N_t = \sum_{i=1}^{n_x} I(T_i \leq t)$$

where $I(.)$ is an indicator function. The filtration $\mathcal{F}^m$ is generated by N_t and the expectation of the infinitesimal variation of N_t verifies:

$$\mathbb{E}\left(dN_t | \mathcal{F}^m_{t-}\right) = (n_x - N_{t-}).\mu(x+t).dt.$$

As the mortality is not traded in our model, this is a second source of incompleteness. The compensated process M_t of the mortality process is defined as follows:

$$M_t = N_t - \int_0^t (n_x - N_{u-}).\mu(x+u).du$$

and M_t is a martingale under the historical measure P^m. The expected number of survivors under P^m is equal to the current number of survivors times a survival probability:

$$\begin{aligned}
\mathbb{E}\left((n_x - N_s) | \mathcal{F}^m_t\right) &= \mathbb{E}\left(\sum_{i=1}^{n_x} I(T_i > s) | \mathcal{F}^m_t\right) \\
&= \sum_{T_i > s} \mathbb{E}\left(I(T_i > s) | \mathcal{F}^m_t\right) \\
&= (n_x - N_t).\underbrace{\exp\left(-\int_t^s \mu(x+u).du\right)}_{{}_{s-t}p_{x+t}}.
\end{aligned}$$

${}_{s-t}p_{x+t}$ is the actuarial notation for the probability that an individual of age $x+t$ survives till age $x+s$.

4 The deflator and the fair value of liabilities.

Let $(\Omega, \mathcal{F}, P)$ be the probability space resulting from the product of the financial, wage and mortality probability spaces:

$$\Omega = \Omega^f \times \Omega^a \times \Omega^m \qquad \mathcal{F} = \mathcal{F}^f \otimes \mathcal{F}^a \otimes \mathcal{F}^m \vee \mathcal{N} \qquad P = P^f \times P^a \times P^m$$

where the sigma algebra $\mathcal{N}$ is generated by all subsets of null sets from $\mathcal{F}^f \otimes \mathcal{F}^a \otimes \mathcal{F}^m$. The prices of pension fund liabilities are defined on $(\Omega, \mathcal{F}, P)$. In this setting, the market of pension fund liabilities is incomplete owing to the presence of two unhedgeable risks: the salary risk and the mortality risk. It entails that prices may differ from one insurance company to another. The next subsections describe the insurer's deflator that is here composed of three elements called abusively the financial, wage and actuarial deflators, and is an extension of the deflators used in Hainaut and Devolder (2006 b).

4.1 Financial deflator.

The completeness of the financial market entails that there exists one unique equivalent measure, namely the risk neutral measure, under which the discounted prices of assets are martingales. This measure is denoted Q^f and is defined by the following change of measure:

$$\left(\frac{dQ^f}{dP^f}\right)_t = \exp\left(-\frac{1}{2}.\int_0^t ||\Lambda^f||^2.du - \int_0^t \Lambda^f.dW_u^{P^f}\right)$$

where $\Lambda^f = (\lambda_r, \lambda_S)'$. The dynamics of the assets under Q^f have been discussed in section 2. The financial deflator $H^f(t,s)$ at time t for a cash flow paid at time $t \leq s$ is equal to the product of the discount factor and of the change of measure:

$$\begin{aligned} H^f(t,s) &= \frac{\exp\left(-\int_0^s r_u.du\right).\left(\frac{dQ^f}{dP^f}\right)_s}{\exp\left(-\int_0^t r_u.du\right).\left(\frac{dQ^f}{dP^f}\right)_t} \\ &= \exp\left(-\int_t^s r_u.du - \frac{1}{2}.\int_t^s ||\Lambda^f||^2.du - \int_t^s \Lambda^f.dW_u^{P^f}\right). \end{aligned}$$

4.2 The wage deflator.

As the intrinsic salary risk is not traded, the market of pension fund liabilities is incomplete and for any $\mathcal{F}^a$ adapted process $\lambda_{a,t}$, an equivalent probability measure Q^{a,λ_a} can be defined by the following Radon-Nikodym derivative:

$$\left(\frac{dQ^{a,\lambda_a}}{dP^a}\right)_t = \exp\left(-\frac{1}{2}.\int_0^t |\lambda_{a,u}|^2.du - \int_0^t \lambda_{a,u}.dW_u^{A,P^a}\right)$$

and under Q^{a,λ_a} , $dW_u^{A,Q^{a,\lambda_a}} = dW_u^{A,P^a} + \lambda_{a,u}.du$ is a Brownian motion. For the sake of simplicity, $\lambda_{a,u}$ is assumed to be constant and denoted λ_a in the sequel of this paper. The dynamics of the salary process under $Q^f \times Q^{a,\lambda_a}$ are:

$$\begin{aligned} \frac{dA_t}{A_t} &= \underbrace{(\mu_A(t) - \sigma_{Ar}.\lambda_r - \sigma_{AS}.\lambda_S - \sigma_A.\lambda_a)}_{\mu_A^Q(t)}.dt \\ &\quad +\sigma_{Ar}dW_t^{r,Q^f} + \sigma_{AS}dW_t^{S,Q^f} + \sigma_A.dW_t^{A,Q^{a,\lambda_a}} \end{aligned} \tag{4.1}$$

and $H^a(t,s)$ denotes the wage deflator at instant t, for a payment occurring at time $s \geq t$:

$$H^a(t,s) = \frac{\left(\frac{dQ^{a,\lambda_a}}{dP^a}\right)_s}{\left(\frac{dQ^{a,\lambda_a}}{dP^a}\right)_t} = \exp\left(-\frac{1}{2}.\int_t^s |\lambda_{a,u}|^2.du - \int_t^s \lambda_{a,u}.dW_u^{A,P^a}\right).$$

4.3 The actuarial deflator.

The second source of incompleteness is the mortality risk. For any $\mathcal{F}^m$-predictable process h_s, such that $h_s > -1$, an equivalent actuarial measure $Q^{m,h}$ is defined by the random variable solution of the SDE:

$$\begin{aligned} d\left(\frac{dQ^{m,h}}{dP^m}\right)_t &= \left(\frac{dQ^{m,h}}{dP^m}\right)_t.h_t.d\left(N_t - \int_0^t (n_x - N_{u-}).\mu(x+u).du\right) \\ &= \left(\frac{dQ^{m,h}}{dP^m}\right)_t.h_t.dM_t \end{aligned} \tag{4.2}$$

and we have the property that the process $M_t^{m,h}$ defined by

$$M_t^{m,h} = N_t - \int_0^t (n_x - N_{u-}) . \mu(x+u).(1+h_u).du$$

is a martingale under $Q^{m,h}$. We adopt the notation $\lambda_{N,u} = (n_x - N_{u-}) . \mu(x+u)$ for the intensity of jumps. The solution of the SDE (4.2) is (for details, see Duffie 2001, appendix I on counting processes):

$$\begin{aligned} \left(\frac{dQ^{m,h}}{dP^m}\right)_t &= \prod_{T_i \le t} (1+h_{T_i}) . \exp\left(-\int_0^t h_u.\lambda_{N,u}.du\right) \\ &= \exp\left(\int_0^t \ln(1+h_u).dN_u - \int_0^t h_u.\lambda_{N,u}.du\right) \end{aligned}$$

and $H^m(t,s)$ denotes the actuarial deflator at instant t, for a payment occurring at time $s \ge t$, defined by:

$$H^m(t,s) = \frac{\left(\frac{dQ^{m,h}}{dP^m}\right)_s}{\left(\frac{dQ^{m,h}}{dP^m}\right)_t} = \exp\left(\int_t^s \ln(1+h_u).dN_u - \int_t^s h_u.\lambda_{N,u}.du\right). \tag{4.3}$$

Under $Q^{m,h}$, the expected number of survivors at time s is equal to the number of survivors at time t multiplied by a modified probability of survival ${}_{s-t}p^h_{x+t}$:

$$\mathbb{E}^{Q^{m,h}}\left((n_x - N_s)|\mathcal{F}_t^m\right) = (n_x - N_t).\underbrace{\exp\left(-\int_t^s \mu(x+u).(1+h_u).du\right)}_{{}_{s-t}p^h_{x+t}}.$$

In the sequel of this work, we restrict our field of research to a constant process $h_u = h$. The reason motivating this choice is that, in this particular case, some interesting analytic results can be presented. Remark that if $h > 0$, h can be seen as a security margin against an adverse evolution of the mortality.

4.4 The deflator and the price of liabilities.

The deflator used to price liabilities, written $H(t,s)$ is in our settings the product of the financial, wage and actuarial deflators:

$$H(t,s) = \frac{\exp\left(-\int_0^s r_u.du\right)}{\exp\left(-\int_0^t r_u.du\right)}.\frac{\left(\frac{dQ^f}{dP^f}\right)_s}{\left(\frac{dQ^f}{dP^f}\right)_t}.\frac{\left(\frac{dQ^{a,\lambda_a}}{dP^a}\right)_s}{\left(\frac{dQ^{a,\lambda_a}}{dP^a}\right)_t}.\frac{\left(\frac{dQ^{m,h}}{dP^m}\right)_s}{\left(\frac{dQ^{m,h}}{dP^m}\right)_t}. \tag{4.4}$$

The pricing of pension fund liabilities is hence done under a probability measure Q which is equal to the product of Q^f, Q^a and Q^m. Q is thus defined by the deflator $H(t,s)$ which depends on the particular choice of h and λ_a, which are decided by the insurer and depend on the way he evaluates the mortality risk and salary risk.

Remark that the expectation of the deflator $H(t,s)$ is equal to the price of a zero coupon bond, denoted $B(t,s)$:

$$\begin{aligned} B(t,s) &= \mathbb{E}\left(H(t,s)|\mathcal{F}_t\right) \\ &= \mathbb{E}^Q\left(e^{-\int_t^s r_u.du}|\mathcal{F}_t\right) \end{aligned}$$

and the analytic expression of $B(t,s)$ is reminded in appendix A. The fair value at time t of the liabilities at the date of retirement, denoted L_t, is defined as the expectation of the deflated value of future contributions and benefits. L_t will be used in the sequel to state the optimization problem. In particular, if T^m is the maximum time horizon of the insurer's commitments, L_t is equal to:

$$L_t = \mathbb{E}\left(-\int_t^T H(t,s).c_s.ds + \int_T^{T^m} H(t,s).\left(n_x - N_s\right).B.ds|\mathcal{F}_t\right).$$

Generally, the minimum asset value that the fund must hold to ensure his solvency is set larger than or equal to L_t (this minimum depends evidently on the local regulation).

5 The optimization problem.

As motivated in the introduction, the insurer's objective is to minimize the quadratic spread between the contribution rate and a constant target one (namely the normal cost) and to minimize the deviation of the terminal target asset from the mathematical reserve required to cover benefits at the date of retirement. The normal cost, denoted NC, is the contribution rate allowing equality between expected assets and liabilities:

$$NC = \frac{\mathbb{E}\left(H(0,T).L_T|\mathcal{F}_0\right)}{\mathbb{E}\left(\int_0^T H(0,s).ds|\mathcal{F}_0\right)}.$$

The target total asset is denoted $\tilde{X}_T$. Following Brennan and Xia (2002), we will use the Cox-Huang method and minimize first with respect to the contributions and the associated terminal target wealth. The value function is defined as follows:

$$V(t,x,n,a) = \min_{c_t,\tilde{X}_T\in\mathcal{A}_t(x)} \mathbb{E}\left[\int_t^T u_1.\left(c_s - NC\right)^2.ds + u_2.(\tilde{X}_T - L_T)^2 \mid \underbrace{\tilde{X}_t = x,\ N_t = n,\ A_t = a}_{\mathcal{F}_t}\right] \quad (5.1)$$

where u_1 and u_2 are constant weights. The contribution rate and the target wealth are chosen in a set $\mathcal{A}_t(x)$ which is delimited by a constraint ensuring the actuarial equilibrium between future deflated cash flows and the current asset x.

$$\mathcal{A}_t(x) = \left\{\left(\left(c_s\right)_{s\in[t,T]}, \tilde{X}_T\right) \quad \textit{such that} \quad \mathbb{E}\left(-\int_t^T H(t,s).c_s.ds + H(t,T).\tilde{X}_T|\mathcal{F}_t\right) \le x\right\}. \quad (5.2)$$

In the sequel, this constraint is called the budget constraint. However, as the market is incomplete, the fact that $\tilde{X}_T$ belongs to $\mathcal{A}_t(x)$ doesn't guarantee that this process is replicable by an adapted investment policy. This is the reason why we use the terminology of "target" terminal wealth, denoted by $\tilde{X}_T$. This point is detailed in section 6.2, in which we introduce also a replicable wealth X_T at time of retirement.

6 The martingale solution.

6.1 Optimal contribution rate and wealth.

In this section, we solve the optimization problem (5.1)-(5.2). Let $y_t \in \mathbb{R}^+$ be the Lagrange multiplier associated to the the budget constraint at instant t. The Lagrangian is defined by:

$$\begin{aligned} &\mathcal{L}\left(t, x, n, a, (c_s)_s\,,\ \tilde{X}_T,\ y_t\right) = \\ &\qquad \mathbb{E}\left(\int_t^T u_1.\left(c_s - NC\right)^2.ds + u_2.(\tilde{X}_T - L_T)^2)|\mathcal{F}_t\right) - \\ &\qquad y_t.\left(x - \mathbb{E}\left(-\int_t^T H(t,s).c_s.ds + H(t,T).\tilde{X}_T|\mathcal{F}_t\right)\right). \end{aligned} \tag{6.1}$$

A sufficient condition to obtain an optimal contribution rate $(c_s^*)_{s\in[t,T]}$ and an optimal target wealth $\tilde{X}_T^*$, is the existence of an optimal Lagrange multiplier $y_t^* > 0$ such that the couple $\left((c_s^*)_{s\in[t,T]},\ \tilde{X}_T^*\right)$ is a saddle point of the Lagrangian. The value function may therefore be reformulated as:

$$\begin{aligned} V(t,x,n,a) &= \sup_{y_t\in\mathbb{R}^+}\left(\inf_{(c_s)_s,\,\tilde{X}_T}\mathcal{L}\left(t,x,n,a,(c_s)_s\,,\ \tilde{X}_T,\ y_t\right)\right) \\ &= \sup_{y_t\in\mathbb{R}^+}\ \tilde{V}(t,x,n,a,y_t) \end{aligned} \tag{6.2}$$

and

$$V(t,x,n,a) = \tilde{V}(t,x,n,a,y_t^*).$$

It can be proved under technical conditions (see Karatzas and Shreve 1998, for details) that the optimal contribution rate and target wealth are:

$$c_s^* = y_t^*.H(t,s).\frac{1}{2.u_1} + NC \tag{6.3}$$

$$\tilde{X}_T^* = -y_t^*.H(t,T).\frac{1}{2.u_2} + L_T. \tag{6.4}$$

Formally, c_s^* and $\tilde{X}_T^*$ are obtained by offsetting the derivatives of equation (6.1) with respect to c_s and X_T. The optimal Lagrange multiplier, y_t^*, is such that the budget constraint (5.2) is binding:

$$y_t^* = \frac{\mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right) - x - NC.\int_t^T \mathbb{E}\left(H(t,s)|\mathcal{F}_t\right)ds}{\frac{1}{2.u_1}.\int_t^T \mathbb{E}\left(H(t,s)^2|\mathcal{F}_t\right)ds + \frac{1}{2.u_2}.\mathbb{E}\left(H(t,T)^2|\mathcal{F}_t\right)}. \tag{6.5}$$

The numerator of (6.5) is precisely the part of the benefits that are not yet financed: the expected fair value of reserves less the current asset and less the normal cost times a financial annuity. This quantity is called unfunded liabilities in the sequel of this paper and noted as follows:

$$UL_t = \mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right) - x - NC.\underbrace{\int_t^T \mathbb{E}\left(H(t,s)|\mathcal{F}_t\right)ds}_{\bar{a}_{t,T}} \tag{6.6}$$

where $\bar{a}_{t,T}$ is a financial annuity of maturity $T-t$. It is important to note that if $UL_t \leq 0$ then ${}^*_T - L_T \geq 0$, meaning that the optimal target wealth at the date of retirement is larger than the mathematical reserve required to cover the liabilities at that date. Remark that if one is in a situation that $UL_t > 0$, then the optimal target value would not be large enough.

If we insert (6.3) and (6.4) in the objective (5.1), the value function is rewritten in terms of unfunded liabilities:

$$V(t,x,n,a) = \frac{UL_t^2}{\frac{1}{u_1}.\int_t^T \mathbb{E}\left(H(t,s)^2|\mathcal{F}_t\right)ds + \frac{1}{u_2}.\mathbb{E}\left(H(t,T)^2|\mathcal{F}_t\right)} \tag{6.7}$$

The following propositions detail the expectations intervening in the calculation of the Lagrange multiplier (6.5) and of the value function (6.7).

Proposition 6.1. *Under the assumptions that interest rates follow (2.1), that the deflator is defined by (4.4), and that the process defining the actuarial measure $Q^{a,h}$ is constant, $h_t = h$ with $h > -\frac{1}{2}$, the conditional expectation of the square of the deflator is equal to:*

$$\begin{aligned}
&\mathbb{E}\left(H(t,s)^2|\mathcal{F}_t\right) = \\
&\quad \exp\left(\int_t^s \left(\lambda_r^2+\lambda_S^2+\lambda_a^2\right).du\right). \\
&\quad \exp\left(-\beta^{\tilde{P}}.(s-t)+n(s-t).(\beta^{\tilde{P}}-2.r_t)-\frac{\sigma_r^2}{a}.n(s-t)^2\right). \\
&\quad \sum_{n=1}^{n_x-N_t}\frac{(n_x-N_t)!}{(n_x-N_t-n)!\,n!}\left(k^n.\left({}_{s-t}p_{x+t}^{2h}\right)^{n_x-N_t-n}.\left(1-{}_{s-t}p_{x+t}^{2h}\right)^n\right)
\end{aligned}$$

where $\beta^{\tilde{P}}$ and k are constant and defined by:

$$\beta^{\tilde{P}} = 2.b - 4.\frac{\sigma_r.\lambda_r}{a} - 2.\frac{\sigma_r^2}{a^2}$$

$$k = \frac{(1+h)^2}{(1+2.h)}$$

${}_{s-t}p_{x+t}^{2h}$ is a probability of survival under a modified measure of probability:

$${}_{s-t}p_{x+t}^{2h} = \exp\left(-\int_t^s \mu(x+u).(1+2.h).du\right)$$

and $n(s-t)$ is a positive decreasing function, null when s=t,

$$n(s-t) = \frac{1-e^{-a(s-t)}}{a} \tag{6.8}$$

The proof is provided in appendix B.

Proposition 6.2. *The expectation of the deflated value of liabilities, at time $t \leq T$, is:*

$$\mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right) = (n_x-N_t).\alpha.\int_t^T {}_{s-t}p_{x+t}^h.\mathbb{E}^Q\left(e^{-\int_t^T r_u.du}.A_T.B(T,s)|\mathcal{F}_t\right).ds$$

where

$$\begin{aligned}
\mathbb{E}^Q\left(e^{-\int_t^T r_u.du}.A_T.B(T,s)|\mathcal{F}_t\right) &= A_t\,.\,e^{\int_t^T \mu_A^Q(u).du}\,.\,B(t,s) \\
&\quad .e^{\left(\frac{\sigma_{A_r}.\sigma_r}{a}.(-(T-t)+n(s-t)-n(s-T))\right)}
\end{aligned}$$

and $n(s-t)$ is defined by equation (6.8).

The proof is detailed in appendix C. Note that, in the example detailed in section 7, the integrals $\int_t^T \mathbb{E}\left(H(t,s)^2|\mathcal{F}_t\right)ds$ and $\int_t^T {}_{s-t}p_{x+t}^h.\mathbb{E}^Q\left(e^{-\int_t^T r_u.du}.A_T.B(T,s)|\mathcal{F}_t\right).ds$ are computed numerically.

6.2 The best replicating strategy.

We now turn to the issue that the optimal target wealth $\tilde{X}_T^*$ is in general not hedgeable due to the incompleteness of the market caused by mortality and salary risk. From the previous section, we recall that $\tilde{X}_T^*$ depends on L_T which has the following expression:

$$\begin{aligned} L_T &= \mathbb{E}\left(\int_T^{Tm} H(T,s).\left(n_x - N_s\right).B.ds|\mathcal{F}_T\right) \\ &= \left(n_x - N_T\right).\alpha.A_T.\int_T^{Tm} {}_{s-T}p^h_{x+T}.B(T,s).ds. \end{aligned}$$

As L_T is a function both of the mortality and of the salary which are not replicable, it is easily seen that $\tilde{X}_T^*$ is not hedgeable. However, it is possible to find the investment strategy replicating at best this process. We refer the interested reader to Hainaut and Devolder (2006a), in which two conceivable ways to establish the best investment policy are studied in order to determine the optimal asset allocation in case of pure endowment insurance contracts. Our reasoning in this paper is based on dynamic programming (see e.g. Fleming and Rishel 1975 for details) and is also applied in Hainaut and Devolder (2006b), which is a study of the dividend policy and the asset allocation of a portfolio of life insurance policies with predetermined contributions and benefits.

Let $\mathcal{A}_t^\pi(x)$ be the set of replicable wealth processes. If (π_t^S, π_t^R) denote respectively the fraction of the wealth invested in stocks and rolling bonds, $\mathcal{A}_t^\pi(x)$ is defined as follows:

$$\begin{aligned} \mathcal{A}_t^\pi(x) &= \Big\{\left(\left(c_s\right)_{s\in[t,T]}, X_T\right) \quad | \quad \exists\, (\pi_t^S)_t\, (\pi_t^R)_t \quad F_t - adapted \quad : \\ & e^{-\int_t^T r_s.ds}.X_T = x + \int_t^T e^{-\int_t^s r_u.du}.c_s.ds \\ & + \int_t^T e^{-\int_t^s r_u.du}.\pi_s^S.X_s.dS_s + \int_t^T e^{-\int_t^s r_u.du}.\pi_s^R.X_s.dR_s^K\Big\}. \end{aligned}$$

By definition, the set $\mathcal{A}_t^\pi(x)$ is included in $\mathcal{A}_t(x)$ and the dynamics of the replicable wealth process are such that:

$$\begin{aligned} dX_t &= \left(\left(r_t + \pi_t^S.\nu_S + \pi_t^R.\nu_R\right).X_t + c_t\right).dt + \pi_t^S.\sigma_S.X_t.dW_t^{S,Pf} \\ & + \left(\pi_t^S.\sigma_{Sr} - \pi_t^R.\sigma_r.n(K)\right).X_t.dW_t^{r,Pf}. \end{aligned}$$

For a small step of time Δt, the dynamic programming principle states that:

$$V(t,x,n,a) = \mathbb{E}\left[\int_t^{t+\Delta t} u_1.\left(c_s^* - NC\right)^2.ds + V\left(t+\Delta t, \tilde{X}^*_{t+\Delta t}, N_{t+\Delta t}, A_{t+\Delta t}\right) | \mathcal{F}_t\right] \quad (6.9)$$

Given that $(\tilde{X}_t^*)_t$ is the process minimizing the value function, any other process $(X_t)_t \in \mathcal{A}_t^\pi(x) \subset \mathcal{A}_t(x)$ verifies the inequality:

$$V(t,x,n,a) \leq \mathbb{E}\left[\int_t^{t+\Delta t} u_1.\left(c_s^* - NC\right)^2.ds + V\left(t+\Delta t, X_{t+\Delta t}, N_{t+\Delta t}, A_{t+\Delta t}\right) | \mathcal{F}_t\right] \quad (6.10)$$

The Ito's lemma for jump processes (see for e.g. Øksendal and Sulem 2004, chapter one), leads to the following expression for the expectation of the value function at time $t+\Delta t$:

$$\begin{aligned} &\mathbb{E}\left(V(t+\Delta t, X_{t+\Delta t}, N_{t+\Delta t}, A_{t+\Delta t})|\mathcal{F}_t\right) = \\ &\qquad V(t,x,n,a) + \mathbb{E}\left(\int_t^{t+\Delta t} G^\pi(s, X_s, N_s, A_s).ds|\mathcal{F}_t\right) + \\ &\qquad \mathbb{E}\left(\int_t^{t+\Delta t} \left(V(s,X_s,N_s,A_s) - V(s,X_s,N_{s-},A_s)\right).dN_s|\mathcal{F}_t\right) \end{aligned}$$

where $G^{\pi}(s, X_s, N_s, A_s)$ is the generator of the value function:

$$\begin{aligned}
&G^{\pi}(s, X_s, N_s, A_s) = \\
&\quad V_s + a.(b - r_s).V_r + \mu_A(s).A_s.V_A + \frac{1}{2}.\sigma_r^2.V_{rr} + \frac{1}{2}.A_s^2.(\sigma_A^2 + \sigma_{Ar}^2 + \sigma_{AS}^2).V_{AA} \\
&\quad +\sigma_{Ar}.A_s.\sigma_r.V_{Ar} + X_s.A_s.\left(\sigma_{AS}.\pi_s^S.\sigma_S + \sigma_{Ar}.\left(\pi_s^S.\sigma_{Sr} - \pi_s^R.\sigma_r.n(K)\right)\right).V_{XA} \\
&\quad + \left(\left(r_s + \pi_s^S.\nu_S + \pi_s^R.\nu_R\right).X_s + c_s^*\right).V_X + X_s.\sigma_r.\left(\pi_s^S.\sigma_{Sr} - \pi_s^R.\sigma_r.n(K)\right).V_{Xr} \\
&\quad +\frac{1}{2}.X_s^2.\left(\left(\pi_s^S.\sigma_S\right)^2 + \left(\pi_s^S.\sigma_{Sr} - \pi_s^R.\sigma_r.n(K)\right)^2\right).V_{XX}.
\end{aligned}$$

V_s, V_X, V_r, V_A, V_{XX}, V_{Xr}, V_{XA}, V_{rr}, V_{AA} are partial derivatives of first and second orders with respect to time, fund, wage and interest rate. When Δt tends to zero, minimizing the right hand term of the inequality (6.10) is equivalent to minimizing the generator $G^{\pi}(s, X_s, N_s, A_s)$. The investment strategy replicating at best the process $\tilde{X}_t^*$ is then obtained by deriving $G^{\pi}(t, X_t, N_t, A_t)$ with respect to π_t^S and π_t^R:

$$\pi_t^{S*} = \left(-\frac{\nu_R.\sigma_{Sr}}{\sigma_S^2.\sigma_r.n(K)} - \frac{\nu_S}{\sigma_S^2}\right).\frac{V_X}{V_{XX}}.\frac{1}{X_t} - \frac{\sigma_{AS}}{\sigma_S}.\frac{V_{XA}}{V_{XX}}.\frac{A_t}{X_t} \tag{6.11}$$

$$\begin{aligned}
\pi_t^{R*} = &\left(-\frac{\nu_S.\sigma_{Sr}}{\sigma_S^2.\sigma_r.n(K)} - \frac{\nu_R}{\sigma_r^2.n(K)^2}.\left(1 + \frac{\sigma_{Sr}^2}{\sigma_S^2}\right)\right).\frac{V_X}{V_{XX}}.\frac{1}{X_t} + \\
&\left(\frac{\sigma_{Ar}}{\sigma_r.n(K)} - \frac{\sigma_{AS}.\sigma_{Sr}}{\sigma_S.\sigma_r.n(K)}\right)\frac{V_{XA}}{V_{XX}}.\frac{A_t}{X_t} + \frac{1}{n(K)}.\frac{V_{Xr}}{V_{XX}}.\frac{1}{X_t}.
\end{aligned} \tag{6.12}$$

As the value function is known (see expression (6.7)), it suffices to derive it with respect to X_t, r_t and A_t to obtain the optimal part of the funds invested in stocks and bonds:

$$\pi_t^{S*} = \left(\frac{\nu_R.\sigma_{Sr}}{\sigma_S^2.\sigma_r.n(K)} + \frac{\nu_S}{\sigma_S^2}\right).\frac{UL_t}{X_t} + \frac{\sigma_{AS}}{\sigma_S}.\frac{\mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right)}{X_t} \tag{6.13}$$

$$\begin{aligned}
\pi_t^{R*} = &\left(\frac{\nu_S.\sigma_{Sr}}{\sigma_S^2.\sigma_r.n(K)} + \frac{\nu_R}{\sigma_r^2.n(K)^2}.\left(1 + \frac{\sigma_{Sr}^2}{\sigma_S^2}\right)\right).\frac{UL_t}{X_t} \\
&-\left(\frac{\sigma_{Ar}}{\sigma_r.n(K)} - \frac{\sigma_{AS}.\sigma_{Sr}}{\sigma_S.\sigma_r.n(K)}\right).\frac{\mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right)}{X_t} \\
&+\underbrace{\frac{1}{n(K)}.\frac{V_{Xr}}{V_{XX}}.\frac{1}{X_t}}_{correction\ term}.
\end{aligned} \tag{6.14}$$

The correction term has no simple analytic expression:

$$\begin{aligned}
\frac{1}{n(K)}.\frac{V_{Xr}}{V_{XX}}.\frac{1}{X_t} = &\frac{1}{n(K).X_t}.\left(NC.\int_t^T \frac{\partial B(t,s)}{\partial r_t}.ds - \frac{\partial \mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right)}{\partial r_t}\right) + \\
&\frac{UL_t}{n(K).X_t}.\frac{\left(\frac{1}{u_1}.\int_t^T \frac{\partial \mathbb{E}\left(H(t,s)^2|\mathcal{F}_t\right)}{\partial r_t}ds + \frac{1}{u_2}.\frac{\partial \mathbb{E}\left(H(t,T)^2|\mathcal{F}_t\right)}{\partial r_t}\right)}{\left(\frac{1}{u_1}.\int_t^T \mathbb{E}\left(H(t,s)^2|\mathcal{F}_t\right)ds + \frac{1}{u_2}.\mathbb{E}\left(H(t,T)^2|\mathcal{F}_t\right)\right)^2}
\end{aligned}$$

where

$$\frac{\partial B(t,s)}{\partial r_t} = -n(s-t).B(t,s) \qquad \frac{\partial \mathbb{E}\left(H(t,s)^2|\mathcal{F}_t\right)}{\partial r_t} = -2.n(s-t).\mathbb{E}\left(H(t,s)^2|\mathcal{F}_t\right)$$

$$\frac{\partial \mathbb{E}\left(H(t,T).L_T|\mathcal{F}_t\right)}{\partial r_t} = -(n_x - N_t).\alpha.\int_t^T {}_{s-t}p^h_{x+s}.n(T-s).\mathbb{E}^Q\left(e^{-\int_t^T r_u.du}.A_T.B(T,s)|\mathcal{F}_t\right).ds$$

An interesting characteristic of this correction term is that it tends to zero when $t \to T$. Indeed, all terms intervening in the numerator of the correction term are integrals or function of $n(T-t)$ which tend to zero when $t \to T$.

7 Example.

We consider a male population; age 50, of n_{50} =10000 affiliates, and who earns a wage $A_{t=0}$ of 2500 Eur. We assume that all individuals go on retirement at 65 years and receive till their death, a continuous annuity equal to $\alpha = 20\%$ of the last salary A_T. Market parameters are presented in the table 7.1.

Table 7.1: Parameters.

a	12.72%	σ_{SR}	-0.10%
b	3.88%	ν_S	5.35%
σ_r	1.75%	μ_A	2.00%
λ_r	-2.36%	σ_{Ar}	2.00%
$r_{t=0}$	2.00%	σ_{AS}	2.00%
K	8 years	μ_A^Q	2.00%
ν_R	2.77%	σ_A	5.00%
λ_S	34.94%	λ_a	-4.54%
σ_S	15.24%	h	0.0

The normal cost is set to

$$NC = \frac{\mathbb{E}\left(H(0,T).L_T|\mathcal{F}_0\right)}{\bar{a}_{0,T}} = 26.763\,Eur$$

According to equation (6.7), this is the normal cost minimizing the value function at time $t = 0$ (indeed, it implies that $UL_{t=0} = 0$ since the initial wealth is null at $t = 0$). Three choices of weights u_1, u_2 are tested. In the first test, the asset manager seeks mainly to limit the volatility of the contribution rate : $u_1 = 1$, $u_2 = 0.1$. In the second case studied, u_1 and u_2 are set equal to one. In the last test, the aim is mainly to limit the volatility of the terminal surplus : $u_1 = 1$, $u_2 = 10$. We have opted for Monte Carlo simulations. 5000 sample paths are generated for each test and the discretization step of time Δt is set to one year (Contributions and asset allocation are both changed once a year). In the following figures, we compare resp. the average contribution rates and the average negative unfunded liabilities ($-UL_t$ which is equal at time T to the terminal surplus, see eq. (6.6)).

Figure 7.1: Contribution rates.

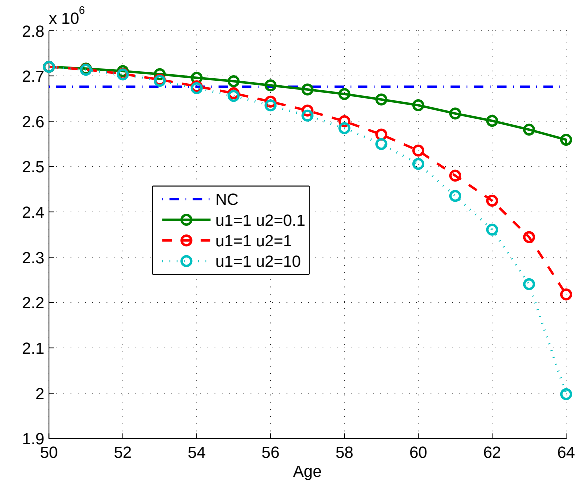
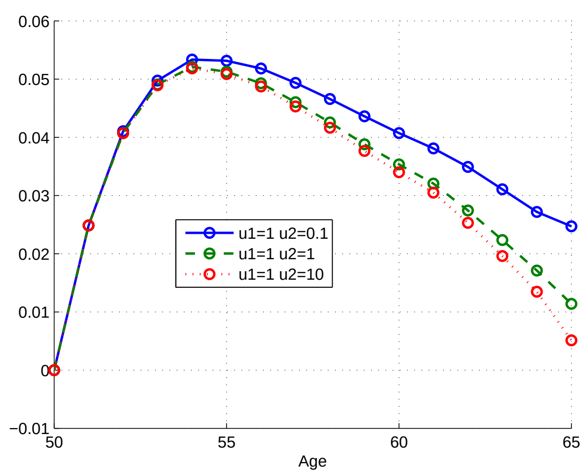
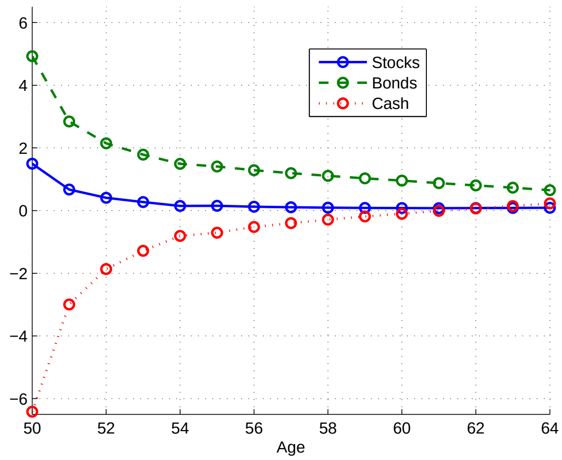


Figure 7.2: $-UL_t$ (% of the fund).



For each set of weights, the contribution rate decreases on average. The higher is the weight u_2 granted to minimize the terminal surplus variation, the higher is the decrease of the contribution rate and the lower is the average negative unfunded liabilities.
The next figure depicts the evolution of the average asset allocation for $u_1 = 1$ and $u_2 = 0.1$ as obtained in equations (6.13) and (6.14). Over the first nine years, huge amounts of cash are borrowed and invested in stocks and bonds. This short position in cash is reduced with time. One year before T, the asset allocation is as follows: 68.3% in bonds, 21.3% in cash and 10.4% in stocks. We also observe that weights mainly influences the contribution rate and the terminal surplus: the average asset allocation for the two other sets of weights are nearly identical to the one displayed in figure (7.3).

Figure 7.3: Asset mix for $u_1 = 1$ and $u_2 = 10$.



8 Conclusions.

In this paper, we have investigated a model for defined benefits pension plans which incorporates stochastic interest rates, mortality and salary. In particular, we have studied the problem of pension funding from the perspective of an asset manager who wishes to minimize the deviation of contributions and terminal surplus from target ones, under a budget constraint and using a quadratic criterion.

The presence of stochastic mortality and salary entails that the market of pension fund liabilities is incomplete and the set of deflators used to valuate liabilities counts more than one element. In order to apply the Cox & Huang martingale method, it is then necessary to choose a deflator that reflects the pricing preferences of the fund manager. This assumption is not really impeding and corresponds to the actuarial practice. Another drawback of the market incompleteness is that the optimal wealth process found by the martingale approach is not perfectly replicable. However, we can find the optimal investment hedging this process at best by a reasoning based on the dynamic programming principle.

We have seen that the optimal contribution rate is the sum of the normal cost and of the unfunded liabilities amortized by a factor, function of the market conditions. The optimal investment strategy also depends on the unfunded liabilities; in particular: for initial negative unfunded liabilities, the optimal target wealth will be larger than the mathematical reserve at retirement date necessary to cover the promised liabilities. An illustrative example has been given which shows the dependence between the contribution rate and the weights respectively given to the minimization of the contribution risk and of the surplus risk.

Appendix A.

As mentioned early in section 4.4, the expected value of the deflator, $\mathbb{E}\left(H(t,s)\,|\,\mathcal{F}_t\right)$, is the price of a zero coupon bond $B(t,s)$, because of independency of W_u^{r,P^f}, W_t^{S,P^f} and W_t^{A,P^a}. If interest rates are driven by a Vasicek model (for details on this model, we refer to Cairns 2004), the price of a zero coupon bond is given by

$$B(t,s) \quad = \quad \exp\left(-\beta.(s-t) + n(s-t).(\beta - r_t) - \frac{\sigma_r^2}{4.a}.n(s-t)^2\right) \qquad (8.1)$$

where

$$\beta = b^Q - \frac{\sigma_r^2}{2.a^2} = b - \sigma_r.\frac{\lambda_r}{a} - \frac{\sigma_r^2}{2.a^2} \tag{8.2}$$

and $n(s-t)$ is a positive decreasing function, null when $s = t$:

$$n(s-t) = \frac{1}{a}.\left(1 - e^{-a.(s-t)}\right).$$

The derivative of the bond price with respect to r_t , used in paragraph 6.2 to calculate the correction term of the optimal bonds strategy (6.14), is:

$$\frac{\partial B(t,s)}{\partial r_t} = -n(s-t).B(t,s).$$

Appendix B.

This appendix presents the proof of the proposition 6.1. The deflator (4.4) can be rewritten as follows:

$$\begin{aligned} H(t,s) &= \exp\left(-\int_t^s r_u.du - \frac{1}{2}.\int_t^s ||\Lambda||^2.du - \int_t^s \Lambda.dW_u^P\right). \\ &\quad \exp\left(\int_t^s \ln(1+h).dN_u - \int_t^s h.\lambda_{N,u}.du\right) \end{aligned}$$

where $\Lambda = (\lambda_r, \lambda_S, \lambda_a)'$ and $W_u^P = \left(W_u^{r,P^f}, W_u^{S,P^f}, W_u^{A,P^a}\right)'$. $\mathbb{E}^P\left(H(t,s)^2|\mathcal{F}_t\right)$ can therefore be decomposed in two independent terms abusively called in the sequel financial and actuarial components which are next calculated separately:

$$\begin{aligned} \mathbb{E}^P\left(H(t,s)^2|\mathcal{F}_t\right) &= \underbrace{\mathbb{E}^P\left(\exp\left(-2.\int_t^s r_u.du - \int_t^s ||\Lambda||^2.du - 2.\int_t^s \Lambda.dW_u^P\right)|\mathcal{F}_t\right)}_{Financial\ component} \\ &\quad \underbrace{\mathbb{E}^P\left(\exp\left(\int_t^s \ln\left((1+h)^2\right).dN_u - \int_t^s 2.h.\lambda_{N,u}.du\right)|\mathcal{F}_t\right)}_{Actuarial\ component} \end{aligned} \tag{8.3}$$

Calculation of the financial component.

The following random variable defines a change of measure from P to $\tilde{P}$:

$$\left(\frac{d\tilde{P}}{dP}\right)_t = \exp\left(-\int_0^t 2.\Lambda.dW_u^P - \frac{1}{2}.\int_0^t ||2.\Lambda||^2.du\right)$$

and under $\tilde{P}$, the following elements are Brownian motions:

$$\begin{aligned} d\tilde{W}_u^{r,\tilde{P}} &= dW_u^{r,P^f} + 2.\lambda_r.du \\ d\tilde{W}_u^{S,\tilde{P}} &= dW_u^{S,P^f} + 2.\lambda_S.du \\ d\tilde{W}_u^{A,\tilde{P}} &= dW_u^{A,P^a} + 2.\lambda_a.du. \end{aligned}$$

The financial component of (8.3) becomes:

$$\begin{aligned} &\mathbb{E}^P\left(\exp\left(-2.\int_t^s r_u.du - \int_t^s ||\Lambda||^2.du - 2.\int_t^s \Lambda.dW_u^P\right)|\mathcal{F}_t\right) \\ &\qquad = \exp\left(\int_t^s ||\Lambda||^2.du\right).\mathbb{E}^{\tilde{P}}\left(e^{-\int_t^s 2.r_u.du}|\mathcal{F}_t\right) \end{aligned}$$

and as $2.r_u$ has mean reverting dynamics under $\tilde{P}$,

$$d(2.r_u) = a.\left(2.b - 4.\frac{\sigma_r.\lambda_r}{a} - 2.r_u\right).dt + 2.\sigma_r.d\tilde{W}_u^{r,\tilde{P}},$$

it suffices to apply the Vasicek's formula to obtain that:

$$\mathbb{E}^{\tilde{P}}\left(e^{-\int_t^s 2.r_u.du}|\mathcal{F}_t\right) = \exp\left(-\beta^{\tilde{P}}.(s-t) + n(s-t).(\beta^{\tilde{P}} - 2.r_t) - \frac{\sigma_r^2}{a}.n(s-t)^2\right)$$

where

$$\beta^{\tilde{P}} = 2.b - 4.\frac{\sigma_r.\lambda_r}{a} - 2.\frac{\sigma_r^2}{a^2}$$

and

$$n(s-t) = \frac{1}{a}.\left(1 - e^{-a.(s-t)}\right).$$

Calculation of the actuarial component.

By the assumption that $h > -\frac{1}{2}$, it is possible to define a positive constant k:

$$k = \frac{(1+h)^2}{(1+2.h)}$$

such that the actuarial component of equation (8.3) can be rewritten as:

$$\mathbb{E}^P\left(\exp\left(\int_t^s \ln\left((1+h)^2\right).dN_u - \int_t^s 2.h.\lambda_{N,u}.du\right)|\mathcal{F}_t\right) =$$
$$\mathbb{E}^{P^a}\left(\exp\left(\int_t^s \ln(k).dN_u\right).\underbrace{\exp\left(\int_t^s \ln(1+2.h).dN_u - \int_t^s 2.h.\lambda_{N,u}.du\right)}_{\frac{dQ^{a,2.h}}{dP^a}}|\mathcal{F}_t\right) \quad (8.4)$$

The term $\frac{dQ^{a,2.h}}{dP^a}$ defines a new actuarial measure $Q^{a,2.h}$, under which the following centered process

$$M_t^{a,2.h} = N_t - \int_0^t (n_x - N_{u-}).\mu(x+u).(1+2.h).du$$

is a martingale. The expected number of survivors at time s, conditionally to instant t is given by:

$$\mathbb{E}^{Q^{a,2.h}}((n_x - N_s)|\mathcal{F}_t) = (n_x - N_t).\underbrace{\exp\left(-\int_t^s \mu(x+u).(1+2.h).du\right)}_{{}_{s-t}p_{x+t}^{2.h}}$$

Equation (8.4) is finally rewritten as the expectation under $Q^{a,2.h}$ of a constant k to the power $N_s - N_t$, the number of deaths.

$$\mathbb{E}^P\left(\exp\left(\int_t^s \ln\left((1+h)^2\right).dN_u - \int_t^s 2.h.\lambda_{N,u}.du\right)|\mathcal{F}_t\right) = \mathbb{E}^{Q^{a,2.h}}\left(k^{N_s-N_t}|\mathcal{F}_t\right).$$

Under $Q^{a,2.h}$, the probability of observing n deceases in the interval of time (t,s) is a binomial variable of parameters $(n_x - N_t, 1 - {}_{s-t}p^{2.h}_{x+t})$. The expected value of $k^{N_s - N_t}$ is then computable by the following formula:

$$\begin{aligned}
&\mathbb{E}^P \left(\exp \left(\int_t^s \ln \left((1+h)^2 \right) .dN_u - \int_t^s 2.h.\lambda_{N,u}.du \right) |\mathcal{F}_t \right) \\
&\quad = \mathbb{E}^{Q^{a,2.h}} \left(k^{N_s - N_t} | \mathcal{F}_t \right) \\
&\quad = \sum_{n=1}^{n_x - N_t} \frac{(n_x - N_t)!}{(n_x - N_t - n)!\, n!} \left(k^n . \left({}_{s-t}p^{2.h}_{x+t} \right)^{n_x - N_t - n} . \left(1 - {}_{s-t}p^{2.h}_{x+t} \right)^n \right).
\end{aligned}$$

Appendix C.

The independence between mortality and the other random variables of our model entails that the fair value of the pension fund liabilities is:

$$\begin{aligned}
L_T &= \mathbb{E} \left(\int_T^{Tm} H(T,s).\left(n_x - N_s \right).B.ds | \mathcal{F}_T \right) \\
&= \left(n_x - N_T \right).\alpha.A_T. \int_T^{Tm} {}_{s-T}p^h_{x+T}.B(T,s).ds
\end{aligned}$$

and that the expectation at time $t \leq T$ of L_T equals:

$$\begin{aligned}
&\mathbb{E} \left(H(t,T).L_T | \mathcal{F}_t \right) \\
&\quad = \alpha.\left(n_x - N_t \right). \int_T^{Tm} {}_{s-t}p^h_{x+t}.\mathbb{E}^Q \left(e^{-\int_t^T r_u.du}.A_T.B(T,s) | \mathcal{F}_t \right).ds.
\end{aligned}$$

The sequel of this paragraph focus then on the calculation of $\mathbb{E}^Q \left(e^{-\int_t^T r_u.du}.A_T.B(T,s) | \mathcal{F}_t \right)$. This step is based on the following four observations. Firstly, A_T is the Dolean-Dade exponential, solution of the SDE (4.1):

$$\begin{aligned}
A_T &= A_t. \exp \left(\int_t^T \left(\mu^Q_A(u) - \frac{\sigma^2_{Ar}}{2} - \frac{\sigma^2_{AS}}{2} - \frac{\sigma^2_A}{2} \right) du \right) \\
&\quad . \exp \left(+ \int_t^T \sigma_A.dW^{A,Q^{a,\lambda_a}}_u + \int_t^T \sigma_{Ar} dW^{r,Q^f}_u + \int_t^T \sigma_{AS} dW^{S,Q^f}_u \right). \qquad (8.5)
\end{aligned}$$

Secondly, as detailed in appendix A, the price of a zero coupon bond is given by:

$$B(T,s) = \exp \left(-\beta.(s-T) + n(s-T).(\beta - r_T) - \frac{\sigma^2_r}{4.a}.n(s-T)^2 \right) \qquad (8.6)$$

where β is defined by equation (8.2). The last useful elements are related to the fact that interest rates are Gaussian in the Vasicek model:

$$r_T = \left(1 - e^{-a.(T-t)} \right).b^Q + e^{-a.(T-t)}.r_t + \int_t^T \sigma_r.e^{-a.(T-u)}.dW^{r,Q^f}_u \qquad (8.7)$$

$$\int_t^T r_u du = b^Q.(T-t) + (r_t - b^Q).n(T-t) + \sigma_r. \int_t^T n(T-u).dW^{r,Q^f}_u \qquad (8.8)$$

The proof of such results can be found in Cairns (2004), appendix B. Combining expressions (8.5), (8.6), (8.7) and (8.8) allows us to rewrite $e^{-\int_t^T r_u.du}.A_T.B(T,s)$ as an exponential of independent normal random variables and the calculation of $\mathbb{E}^Q \left(e^{-\int_t^T r_u.du}.A_T.B(T,s) | \mathcal{F}_t \right)$ directly results from the expectation of lognormal variables.

Appendix D.

In the example presented in this paper, mortality rates obey to a Gompertz-Makeham distribution. The parameters are those defined by the Belgian regulator for the pricing of a life insurance purchased by a man. For an individual of age x, the mortality rate is :

$$\mu(x) = a_\mu + b_\mu.c^x \qquad a_\mu = -\ln(s_\mu) \qquad b_\mu = \ln(g_\mu).\ln(c_\mu)$$

where the parameters s_μ, g_μ, c_μ take the values showed in the table 8.1.

Table 8.1: Belgian legal mortality, for life insurance products and for a male population.

s_μ:	0.999441703848
g_μ:	0.999733441115
c_μ :	1.116792453830

References

[1] Boulier, J.F., Trussant E., Florens D. 1995. "A dynamic model for pension fund management. "Proceedings of the 5th AFIR international symposium Brussels, Vol 1, 361-384.

[2] Brennan M.J., Xia Y. 2002. "Dynamic asset allocation under inflation." The journal of finance, vol LVII (3), 1201-1238.

[3] Cairns A.J.G. 1995 "Pension funding in a stochastic environment: The role of objectives in selecting an asset allocation strategy." Proceedings of the 5th AFIR international symposium Brussels, Vol 1, 429-453.

[4] Cairns A.J.G. 2000 "Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time." ASTIN Bulletin, Vol 30 (1) 19-55.

[5] Cairns A. 2004. Interest rate models, an introduction. Princeton University Press.

[6] Cox J., Huang C.F. 1989. Optimal consumption and portfolio policies when asset prices follow a diffusion process. Journal of economic theory, vol 49, 33-83.

[7] Duffie D. , 2001. Dynamic asset pricing theory. Third edition. Princeton University Press.

[8] Fleming W., Rishel R. , 1975. Deterministic and Stochastic optimal control. Springer.

[9] Haberman S., Sung J.H. 1994. "Dynamic approaches to pension funding." Insurance: Mathematics and Economics 15, 151-162.

[10] Haberman S., Sung J.H. 2005. "Optimal pension funding dynamics over infinite control horizon when stochastic rates of return are stationary." Insurance: Mathematics and Economics, 36, 103-116.

[11] Hainaut D., Devolder P. 2006 (a). "A martingale approach applied to the management of life insurances." Working paper of the institute of actuarial sciences UCL.

[12] Hainaut D., Devolder P. 2006 (b). "Management of a pension funds under stochastic mortality and interest rates." Working paper of the institute of actuarial sciences UCL. Forthcoming in Insurance: Mathematics and economics.

[13] Huang H.-C., Cairns A.J.G. 2006 "On the control of defined-benefit pension plans." Insurance: Mathematics and economics, Vol 38, 113-131.

[14] Josa-Fombellida R., Rincon-Zapatero J.P., 2004, "Optimal risk management in defined benefit stochastic pension funds." Insurance: Mathematics and Economics, Vol 34, 489-503.

[15] Josa-Fombellida R., Rincon-Zapatero J.P., 2006, "Optimal investment decisions with a liability: the case of defined benefits pension plans." Insurance: Mathematics and Economics, Vol 36, 81-98.

[16] Karatzas I., Shreve S., 1998. Methods of mathematical finance. Springer.

[17] Møller T., 1998. Risk minimizing hedging strategies for unit-linked life insurance contracts. ASTIN Bulletin, Vol 28-1, 17-47.

[18] Øksendal B., Sulem A., 2005. Applied stochastic control of jump diffusions. Springer.

[19] Sundaresan S., Zapatero F., 1997. "Valuation, optimal asset allocation and retirement incentives of pension plans." The Review of Financial Studies 10 (3), 631-660.