Rating based credit risk models

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- Bond market

$T > 0$ fixed horizon date for all market activities, probability space with filtration.

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• risk-free saving account with short term interest rate r_t. If at moment 0 one puts into the bank account 1 unit of money then at moment t one has :

$$B_t = \exp\left(\int_0^t r_u du\right)$$

What are connection between saving account process and bond price processes?

If short term interest rate is deterministic, then

$$B(t,\theta) = e^{-\int_t^\theta r_s ds}.$$

Definition 1. *A family* $B(t, \theta)$*,* $0 \le t \le \theta \le T$ *of adapted processes is called an arbitrage-free family of bond prices relatively to* r *if*

i) $B(\theta, \theta) = 1$ *for every* $\theta \le T$*,*

ii) there exist a probability measure P^* *such that* $B^*(t, \theta) = B(t, \theta)/B_t$ *is a* P^**-martingale for any* $\theta \le T$*.*

Hence

$$B(t, \theta) = E_{P^*}(e^{-\int_t^\theta r_s ds} | \mathcal{F}_t)$$

Conversely, given r, P^*, the family $B(t, \theta)$ defined above is an arbitrage-free family of bond prices relatively to r.

• Model short term interest rate by Ito process

$$dr(t) = b(t)dt + \sigma(t)dW(t).$$

• Standard models:

Vasicek :

$$dr(t) = (b + \beta r(t))dt + \sigma(t)dW(t).$$

Cox, Ingersoll and Ross

$$dr(t) = (a - br(t))dt + \sigma\sqrt{r(t)}dW(t),$$

where a, b, σ are constants with $a \geq 0$, $b > 0$, $\sigma > 0$

• Most of models provide an affine term structure

$$B(t,\theta) = \text{exp}(-A(t,\theta) - B(t,\theta)r_t).$$

• Initial term structure:

$$B(0,\theta) = E_{P^*}(e^{-\int_0^\theta r_s ds}), \quad 0 \leq \theta \leq T$$

Bond price models based on a specific short term interest rate process makes the problem of matching the initial term structure.

• HJM model

Heath, Jarrow and Morton proposed to use the forward rate curve i.e. a function $f(t,\theta)$ defined for $\theta \geq t$ and such that

$$B(t,\theta) = e^{-\int_t^\theta f(t,s)ds}.$$

$f(t,\theta),\ t \leq \theta$, describe our expectation at the moment t of the value of short term interest rate at the moment θ i.e. it is usually interpreted as the anticipated short rate at time θ as seen by the market at time t.

$$f(t,\theta) = -\frac{\partial \ln B(t,u)}{\partial u}\Big|_{u=\theta}$$

Heath, Jarrow and Morton proposed to model the forward curves as Itô processes

$$df(t,\theta) = \alpha(t,\theta)dt + < \sigma(t,\theta),\ dZ(t) >, \qquad 0 \le t \le \theta, \qquad (1)$$

with Z d-dimensional standard Wiener process, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \boldsymbol{P})$.

Equivalently, for $t \le \theta$,

$$f(t,\theta) = f(0,\theta) + \int_0^t \alpha(s,\theta)\,ds + \int_0^t < \sigma(s,\theta),\ dZ(s) > . \qquad (2)$$

For each θ the processes $\alpha(t,\theta)$, $\sigma(t,\theta)$, $t \le \theta$, are assumed to be predictable with respect to a given filtration $(\mathcal{F}_t)$ and such that integrals in (2) are well defined.

It is convenient to assume that once a bond has matured its cash equivalent goes to the bank account. Thus $B(t,\theta)$, the market price at time t of a bond paying 1 at the maturity time θ, is defined also for $t \geq \theta$ by the formula

$$B(t,\theta) = e^{\int_\theta^t r(\sigma)\, d\sigma}. \tag{3}$$

For $\theta < t$ we put

$$\alpha(t,\theta) = \sigma(t,\theta) = 0, \tag{4}$$

so the forward rate f is defined for $t, \theta \in [0, T]$. By (4) we deduce from (2) that for $t > \theta$,

$$f(t,\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)\, ds + \int_0^\theta < \sigma(s,\theta),\, dZ(s) > .$$

Consequently, for each $\theta > 0$ the process $f(t,\theta)$, $t > \theta$, is constant in t and could be identified with the short rate:

$$r(\theta) = f(0,\theta) + \int_0^\theta \alpha(s,\theta)\, ds + \int_0^\theta < \sigma(s,\theta),\, dZ(s) > . \tag{5}$$

So $r(\theta) = f(\theta,\theta)$.

From now on we assume (1) and (4) and that the short rate is given by (5).

HJM condition (d=1):

$$\int_t^\theta \alpha(t,v)\,dv = \frac{1}{2}\left(\int_t^\theta \sigma(t,v)\,dv\right)^2 \tag{6}$$

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We consider a generalization of this model by taking, instead of the Wiener process W, a Lévy process Z with values in a separable Hilbert space U with the scalar product denoted by $< \cdot, \cdot >_U$.

• Lévy processes - main properties

We assume that the basic probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ is complete. By μ we denote the measure associated to jumps of Z i.e. for any $A \in \mathcal{B}(U)$ such that $\overline{A} \subset U \backslash \{0\}$ we have:

$$\mu([0,t], A) = \sum_{0<s\leq t} \mathbf{1}_A(\Delta Z(s)).$$

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The measure ν defined by:

$$\nu(A) = E(\mu([0,1], A)),$$

is called Levy measure of process Z, stationarity of increments implies that we have also:

$$E(\mu([0,t], A)) = t\nu(A).$$

The Lévy-Khintchine formula shows that characteristic function of Lévy process has a form:

$$Ee^{i<\lambda,Z(t)>_U} = e^{t\psi(\lambda)},$$

where

$$\psi(\lambda) = i < a, \lambda >_U - \frac{1}{2} < Q\lambda, \lambda >_U + \int_U (e^{i\langle\lambda,x\rangle_U} - 1 - i < x, \lambda >_U \mathbf{1}_{[-1,1]}(|x|_U))\nu(dx),$$

and $a \in U$, Q is symmetric non negative nuclear operator on U, ν is a measure on U with $\nu(\{0\}) = 0$ and

$$\int_{\mathcal{U}} (|x|^2 \wedge 1)\nu(dx) < \infty. \tag{7}$$

Example. Let X be a compound Poisson, i.e.

$$X_t = \sum_{k=1}^{N_t} Y_k,$$

where Y_i are i.i.d., $F = F_Y$. Then

$$E(e^{iuX_t}) = \exp\left(\lambda t \int_R (e^{iuy} - 1) F(dy)\right).$$

If $E(e^{\alpha Y_1}) < \infty$, then

$$E(e^{\alpha X_t}) = \exp\left(\lambda t \int_R (e^{\alpha y} - 1) F(dy)\right).$$

Moreover Z has a well known Lévy-Itô decomposition:

$$Z(t) = at + W(t) + \int_0^t \int_{|y|_U \leq 1} y(\mu(ds, dy) - dt\nu(dy)) + \int_0^t \int_{|y|_U > 1} y\mu(ds, dy),$$

where W is a Wiener process with values in U and covariance operator Q.

Under additional conditions

$$\boldsymbol{E}\, e^{-<u,Z(t)>} = e^{tJ(u)},$$

where

$$J(u) = - < a, u > + \frac{1}{2} < Qu, u > + J_0(u), \tag{8}$$

$$J_0(u) = \int_U \Big[e^{-<u,y>} - 1 + < u, y > 1_{\{|y| \le 1\}} \Big] \nu(dy), \quad u \in U. \tag{9}$$

In other words

$$J_0(u) = \int_{\{|y| \le 1\}} \Big(e^{-<u,y>} - 1 + < u, y > \Big) \nu(dy) \tag{10}$$
$$+ \int_{\{|y| > 1\}} (e^{-<u,y>} - 1) \nu(dy).$$

It is convenient to express the HJM condition in terms of the logarithm of moment generating function of Lévy process Z, i.e. in terms of the functional J.

We will regard coefficients α and σ, in the equation (1) i.e. in :

$$df(t,\theta) = \alpha(t,\theta)dt + < \sigma(t,\theta),\ dZ(t) >, \qquad 0 \le t \le \theta,$$

as, respectively, $H = L^2([0,T]),\ \ L(U,H)$ valued, predictable processes:

$$\alpha(t)(\theta) = \alpha(t,\theta), \theta \in [0,T],$$

$$\sigma(t)u(\theta) =< \sigma(t,\theta), u > ,\ u \in U,\ \theta \in [0,T].$$

Then (1) can be written as

$$df(t) = \alpha(t)dt + \sigma(t)dZ(t). \qquad (11)$$

Let us recall that *HJM postulate* is the requirement that the discounted bond price processes $B^*(\cdot,\theta)$, $\theta \in [0,T]$:

$$B^*(t,\theta) = \frac{B(t,\theta)}{B_t} = e^{-\int_t^\theta f(t,s)ds} e^{-\int_0^t f(t,s)ds} = e^{-\int_0^\theta f(t,s)ds}$$

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Let b be the Laplace transform of the measure ν restricted to the complement of the ball $\{y : |y| \le 1\}$,

$$b(u) = \int_{|y|>1} e^{-<u,y>} \nu(dy), \tag{12}$$

and B the set of those $u \in U$ for which the Laplace transform is finite:

$$B = \{u \in U : b(u) < \infty\}.$$

Assumption:

$(H1)$ Processes α and σ are predictable and, with probability one, have bounded trajectories.

$(H2)$ For arbitary $r > 0$ the function b is bounded on

$$\{u : |u| \leq r, b(u) < \infty\}.$$

Theorem 1. *Assume that (H1) holds.*

i) *If HJM postulate holds then, for arbitrary $\theta \leq T$, $\boldsymbol{P}-$ almost surely,*

$$\int_t^\theta \sigma(t,v)\,dv \in B, \tag{13}$$

for almost all $t \in [0,\theta]$.

ii) *Assume (H2) and that for all $\theta \leq T$, $\boldsymbol{P}-$ almost surely* (13) *holds for almost all $t \in [0,\theta]$. Then HJM postulate holds if and only if, the following HJM condition is satisfied,*

$$\int_t^\theta \alpha(t,v)\,dv = J\left(\int_t^\theta \sigma(t,v)\,dv\right), \tag{14}$$

for almost all $t \in [0,\theta]$

Remark 1. *Explicit formulation* (14)*, in terms of the function* J*, indicates that the drift term is completely determined by the diffusion term. In the particular case when* $a = 0$*,* $\mu = 0$*, one arrives at the classical HJM condition.*

Remark 2. *Part (i) of Theorem 1 is in the spirit of Theorem 25.3 in Sato book which implies that for a finite dimensional Lévy process the conditions*

$$\boldsymbol{E} e^{-\langle u, Z_t \rangle} < \infty \tag{15}$$

and

$$\int_{|y|>1} e^{-\langle u, y \rangle} \nu(dy) < \infty \tag{16}$$

are equivalent. In Theorem 1 we generalized the implication (15) $\Rightarrow$ (16) *taking a stochastic integral with respect to a Lévy process* Z *instead of* Z*. For a fixed* T*, we have proved that if for a bounded process* Σ*, the process*

$$Y_t = \text{exp}\left(- \int_0^t \langle \Sigma(s), dZ_s \rangle - \int_0^t J(\Sigma(s))\, ds \right) \tag{17}$$

is a local martingale, then

$$\Sigma(t) \in B \qquad dt \otimes dP \ \text{ almost surely.} \tag{18}$$

Remark 3. *This condition, even in the finite dimensional case, is more general than that given in the paper by Eberlein and Özkan who assume there exists a constant* $M > 0$ *such that*

$$\int_{|y|>1} e^{-\langle c,y\rangle}\nu(dy) < \infty \quad \text{for all } c \in [-M, M]^d. \tag{19}$$

and the volatility function takes values in that interval. However, as follows from Theorem 1, if σ *is a positive process and* Z *has only positive jumps, no a priori requirements on* ν *are necessary. Indeed, if a* 1*-dimensional Lévy process* Z *has positive jumps and the volatility is non-negative, then condition* (13) *is always satisfied (condition* (19) *might not be satisfied).*

Corollary 1. *If ν is a Lévy measure of the $\alpha-$ stable symmetric process Z in $\mathbb{R}$, then*

$$\nu(dy) = c|y|^{-1-\alpha}dy \quad \textit{and} \quad \forall u \neq 0 \int_{\{|y|>1\}} e^{<u,y>}|y|^{-1-\alpha}dy = \infty.$$

Therefore, as a consequence of Theorem 1, we obtain that HJM postulate could not be satisfied for the $\alpha-$stable symmetric process Z, so Z can not be used for modelling forward rate.

Corollary 2. *Under mild assumption the HJM-type condition can be written as*

$$\alpha(t,\theta) = \Big\langle DJ\Big(\int_0^\theta \sigma(t,v)dv\Big), \sigma(t,\theta)\Big\rangle_U,$$

where

$$DJ(x) = -a + Qx - \int_U \Big(e^{-\langle x,y\rangle_U} - \mathbf{1}_{|y|_U\le 1}(y)\Big)y\,\nu(dy),$$

so the HJM-type condition has the following form:

$$\begin{aligned}\alpha(t,\theta) = \quad & -\Big\langle a, \sigma(t,\theta)\Big\rangle_U + \Big\langle Q\int_0^\theta \sigma(t,v)dv, \sigma(t,\theta)\Big\rangle_U \\ & - \int_U \Big(e^{-\langle \int_0^\theta \sigma(t,v)dv, y\rangle_U} - \mathbf{1}_{|y|_U\le 1}(y)\Big)\Big\langle y, \sigma(t,\theta)\Big\rangle_U \nu(dy).\end{aligned}$$

The dynamics of the forward rate f under the HJM condition.

Theorem 2. *Assume that*

$$\int_{|y|\geq 1} e^{-\langle u,y\rangle}\nu(dy) < \infty \tag{20}$$

for all u *from some neighborhood of the set in which* $\int_t^\theta \sigma(t,v)dv$ *takes values. Then the HJM condition* (14) *implies that the dynamics of* f *has the form*

$$df(t,\theta) = \Big\langle DJ\Big(\int_t^\theta \sigma(t,v)dv\Big), \sigma(t,\theta)\Big\rangle\, dt + \langle \sigma(t,\theta), dZ(t)\rangle, \tag{21}$$

where DJ *is the gradient of* J.

Thus, under very mild assumptions, the HJM postulate holds if and only if

$$df(t,\theta) = \Big\langle DJ\Big(\int_t^\theta \sigma(t,v)\,dv\Big), \sigma(t,\theta)\Big\rangle\, dt + \langle \sigma(t,\theta), dZ(t)\rangle.$$

Now I indicate the difference between finite and infinite dimensional case.

For $U = \mathbb{R}^d$ under a natural condition on the volatility σ the function b is finite on a certain set.

Remark 4. *Let $U = \mathbb{R}^d$. If b is finite on a dense subset of K and $IntK \neq \emptyset$, then*

$$b(u) < \infty \tag{22}$$

for all $u \in IntK$ i.e. $IntK \subset B$.

Proof. We first prove that if $U = \mathbb{R}^d$ and (22) is satisfied on a dense subset D of the open ball $B(x,r)$, $x \in \mathbb{R}^d$, $r > 0$, then it holds for all $c \in B(x,r)$. Indeed, for every $c \in B(x,r)$ there exist $c_1, \ldots, c_{d+1} \in D$ such that c belongs to the simplex with vertices $c_1, \ldots, c_{d+1}$, i.e. $c = \sum_{i=1}^{d+1} \lambda_i c_i$, $\lambda_i \in [0,1]$, $\sum_{i=1}^{d+1} \lambda_i = 1$. Hence, by convexity of the exponential function,

$$\int_{|y|>1} e^{-\langle c,y\rangle} \nu(dy) \leq \sum_{i=1}^{d+1} \lambda_i \int_{\{|y|>1\}} e^{-\langle c_i,y\rangle} \nu(dy) < \infty,$$

since $c_1, \ldots, c_{d+1} \in D$.

Next, since $G = \mathtt{Int} K$ is open, for every $x \in G$ there exists $r > 0$ such that $B(x,r) \subset G$. By assumption (22) holds for a dense subset of $B(x,r)$, so by the previous considerations (22) holds for all $y \in B(x,r)$, in particular for $y = x$. $\square$

It turns out that many properties of B which hold in finite dimensions are not true in infinite dimensions. In particular, in infinite dimensions the set B could be the difference of an open set and a dense subset.

Theorem 3. *There exists a model of the form* (1) *for which (H1) holds, HJM postulate is satisfied,*

$$\int_{|y|>1} e^{-<c,y>}\nu(dy) < \infty, \tag{23}$$

for c *in a dense subset of* $B(0,r)$*,*

$$\int_{|y|>1} e^{-<c_n,y>}\nu(dy) = \infty \tag{24}$$

for a sequence $c_n \to 0$ *as* $n \to \infty$*.*

HJM conditions for defaultable bonds

- Our aim is to derive the HJM-type conditions for the market containing a risk free bond and defaultable bonds.

In a defaultable case we have several variants describing amount and timing of so called *recovery payment* which is paid to bond holders if default has occurred before bond's maturity. If by τ we denote the moment of default, then, generally speaking, the payoff of the defaultable bond is as follows:

$$D(\theta, \theta) = 1_{\{\tau > \theta\}} + 1_{\{\tau \leq \theta\}} \cdot \textit{ recovery payment}.$$

If δ is a recovery rate process, then *recovery payment* can take different forms:

- $\delta_\tau D(\tau-,\theta)\frac{B_\theta}{B_\tau}$ - *fractional recovery of market value* - at time of default bondholders receive a fraction of pre-default market value of defaultable bond (i.e. $D(\tau-,\theta)$):

$$D(\theta,\theta) = 1_{\{\tau>\theta\}} + 1_{\{\tau\leq\theta\}} \cdot \delta_\tau D(\tau-,\theta)\frac{B_\theta}{B_\tau},$$

where δ_t is an $\mathbb{F}$ predictable and takes values in $[0,1]$.

• δ - *fractional recovery of Treasury value* - a fixed fraction δ of bond's face value is paid to bondholders at maturity θ:

$$D^{\delta}(\theta,\theta) = 1_{\{\tau>\theta\}} + 1_{\{\tau\leq\theta\}} \cdot \delta.$$

• $\frac{\delta B_\theta}{B_\tau}$ - *fractional recovery of par value*- a fixed fraction δ of bond's face value is paid to bondholders at default time τ:

$$D^{\triangle}(\theta,\theta) = 1_{\{\tau>\theta\}} + 1_{\{\tau\leq\theta\}} \cdot \delta\frac{B_\theta}{B_\tau}.$$

• *fractional recovery with multiple defaults* - defaultable bonds such that their face values, at each default time τ_i, is reduced by fraction L_{τ_i}, where L_s is $\mathbb{F}$ - predictable process taking values in $[0,1]$:

$$D^m(\theta,\theta) = \prod_{\tau_i \le \theta} (1 - L_{\tau_i})$$

τ_i are moments of jumps of Cox process N_t with stochastic intensity process $(\lambda_t)_{t\ge 0}$.
Note that $1 - L_t$ can be interpreted as a recovery process and therefore we will denote it by δ_t. Thus $\delta_t = 1 - L_t$.

This model describes situation, where company has had to declare default is not liquidated but is restructured. After that firm may default again in the future.

We denote by $g_1(t,u)$ the pre-default forward rate corresponding to pre-default term structure observed on the market. We postulate here that

$$dg_1(t,\theta) = \alpha_1(t,\theta)dt + < \sigma_1(t,\theta), dZ_1(t) >_U,$$

where Z_1 is Lévy process with values in U which has the following Lévy-Itô decomposition:

$$Z_1(t) = \alpha_1 t + W_1(t) + \int_0^t \int_{|y|_U \leq 1} y(\mu_1(ds,dy) - ds\nu_1(dy)) + \int_0^t \int_{|y|_U > 1} y\mu_1(ds,dy).$$

$$g_1(t,u) > f(t,u)$$

So if $D_1(t,\theta) = e^{-\int_t^\theta g_1(t,u)du}$, then

$$D_1(t,\theta) = e^{-\int_t^\theta g_1(t,u)du} < B(t,\theta) = e^{-\int_t^\theta f(t,u)du}.$$

By applying Itô lemma

Theorem 4. *Dynamics of the process $D_1(t,\theta)$ is given by*

$$dD_1(t,\theta) = D_1(t-,\theta)\Big(\Big(g_1(t,t) + \bar{a}_1(t,\theta)\Big)dt + \int_U \Big[e^{-<\tilde{\sigma}_1^*(t)1_{[0,\theta]},y>_U} - 1\Big](\mu_1(dt,dy) - dt\nu_1(dy)) - <\tilde{\sigma}_1^*(t)1_{[0,\theta]}>_{U,dW_1(t)}\Big),$$

where $\bar{a}_1(t,\theta)$ satisfies

$$\bar{a}_1(t,\theta) = -<1_{[0,\theta]},\alpha_1(t)> + J_1(\tilde{\sigma}_1^*(t)1_{[0,\theta]}).$$

We assume that the moment of default τ is a $\mathbb{G}$ stopping time, and that our filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ are filtrations generated by Levy processes and observing default time i.e. $\mathcal{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$, respectively.

We assume that τ admits an $\mathbb{F}$ intensity $(\lambda_t)_{t\geq 0}$ which is an $\mathbb{F}$ adapted process such that for $H_t = \mathbf{1}_{\{\tau\leq t\}}$, process M_t given by the formula

$$M_t = H_t - \int_0^{t\wedge\tau} \lambda_u du = H_t - \int_0^t (1 - H_u)\lambda_u du$$

follows a $\mathbb{G}$–martingale.
All result are proved under
Hypothesis (H): We say that *hypothesis (H)* holds for filtrations $\mathbb{F}$ and $\mathbb{G}$, with $\mathbb{F} \subseteq \mathbb{G}$, if every an $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale.

Assume

Hypothesis (J): For the default time τ and for all θ we have:

$$\Delta D_1(\tau, \theta) = 0 \quad a.s.$$

We make a standard assumption on the short term rate for defaultable bonds (see e.g. Jarrow et al. 1997) including information on intensity.

Hypothesis (H1):

$$g_1(t,t) = f(t,t) + \lambda_t(1 - \delta(t)), \tag{25}$$

so the short term rate for a defaultable bond with rating class i is equal to the risk-free short term rate plus spread (premium) for bearing credit risk. Spread for credit risk depends on recovery payment and the intensity of probability of the credit event (different for different rating classes).

Hypothesis (H1) is natural, which one can see from the following fact.

Remark 5. *If the price of a defaultable bond with fractional recovery of market value is given in traditional way, it means that it is given by the intensity proces* λ *and the risk-free short term rate* r *in the following way:*

$$\mathbf{1}_{\{\tau>t\}}\widehat{D}(t,\theta) = \mathbf{1}_{\{\tau>t\}}E(e^{-\int_t^\theta [r_u+(1-\delta_u)\lambda_u]du}|\mathcal{F}_t);$$

then, for bounded λ *and* r*, we have*

$$
\begin{aligned}
g_1(t,t) &\triangleq -\lim_{\theta\downarrow t}\frac{\partial}{\partial\theta}\ln E(e^{-\int_t^\theta[r_u+(1-\delta_u)\lambda_u]du}|\mathcal{F}_t)\\
&= -\lim_{\theta\downarrow t}\frac{\frac{\partial}{\partial\theta}E(e^{-\int_t^\theta[r_u+(1-\delta_u)\lambda_u]du}|\mathcal{F}_t)}{E(e^{-\int_t^\theta[r_u+(1-\delta_u)\lambda_u]du}|\mathcal{F}_t)}\\
&= -\lim_{\theta\downarrow t}\frac{E(\frac{\partial}{\partial\theta}e^{-\int_t^\theta[r_u+(1-\delta_u)\lambda_u]du}|\mathcal{F}_t)}{E(e^{-\int_t^\theta[r_u+(1-\delta_u)\lambda_u]du}|\mathcal{F}_t)}\\
&= \lim_{\theta\downarrow t}\frac{E([r_\theta+(1-\delta_\theta)\lambda_\theta]e^{-\int_t^\theta[r_u+(1-\delta_u)\lambda_u]du}|\mathcal{F}_t)}{E(e^{-\int_t^\theta[r_u+(1-\delta_u)\lambda_u]du}|\mathcal{F}_t)}\\
&= r_t+(1-\delta(t))\lambda_t,
\end{aligned}
$$

so (25) holds. The same conclusions can be drawn for other kinds of recovery.

Fractional recovery of market value

$$D(\theta,\theta) = \mathbf{1}_{\{\tau>\theta\}} + \mathbf{1}_{\{\tau\le\theta\}}\delta_\tau D(\tau-,\theta)\frac{B_\theta}{B_\tau}$$

and for $t \le \theta$ we model a value of defaultable bond by

$$D(t,\theta) = \mathbf{1}_{\{\tau>t\}} e^{-\int_t^\theta g_1(t,u)du} + \mathbf{1}_{\{\tau\le t\}}\delta_\tau D(\tau-,\theta)\frac{B_t}{B_\tau},$$

where $g_1(t,u)$ is the pre-default forward rate corresponding to pre-default term structure. Our first objective is to derive the HJM drift condition.

Using the process $H_t = \mathbf{1}_{\{\tau\le t\}}$ we can represent D as

$$D(t,\theta) = (1-H_t)D_1(t,\theta) + H_t\delta_\tau D_1(\tau-,\theta)\frac{B_t}{B_\tau}.$$

Theorem 5. *(HJM drift condition for $D(t,\theta)$)*
Discounted prices of defaultable bonds with fractional recovery of market value are local martingales if and only if the following condition holds: for all $\theta \in [0, T^*]$ and for almost all $t \le \theta$ on the set $\{\tau > t\}$

$$\int_t^\theta \alpha_1(t,v)dv = J_1\left(\int_t^\theta \sigma_1(t,v)dv\right). \tag{26}$$

Lemma 6. *Let τ and H_t be as above and D_t be a process of the form:*

$$D_t = (1 - H_t)X_t + H_t Y_t + H_t Z_\tau,$$

where processes X_t, Y_t have local martingale parts M_t^X and M_t^Y and absolutely continuous drifts α_t^X, α_t^Y, which means that processes X_t, Y_t have decompositions:

$$dX_t = \alpha_t^X dt + dM_t^X, \quad dY_t = \alpha_t^Y dt + dM_t^Y.$$

Moreover we assume that X and Y have no jumps at τ i.e. $\Delta X_\tau = \Delta Y_\tau = 0$. Then D_t is local martingale if and only if for each $t \in [0, T]$ the following conditions hold:

$$\alpha_t^X = \lambda_t(X_{t-} - Y_{t-} - Z_t) \quad \text{on the set } \{\tau > t\} \tag{27}$$

$$\alpha_t^Y = 0 \quad \text{on the set } \{\tau \leq t\} \tag{28}$$

• Fractional recovery of treasury

$$D^{\delta}(\theta,\theta) = \mathbf{1}_{\{\tau>\theta\}} + \mathbf{1}_{\{\tau\le\theta\}}\cdot\delta.$$

So $$D^{\delta}(t,\theta) = \mathbf{1}_{\{\tau>t\}} e^{-\int_t^{\theta} g_1(t,u)du} + \mathbf{1}_{\{\tau\le t\}}\cdot\delta\cdot B(t,\theta).$$

Therefore

$$D^{\delta}(t,\theta) = (1-H_t)D_1(t,\theta) + H_t\delta B(t,\theta). \tag{29}$$

Theorem 7. *(HJM drift condition for $D^{\delta}(t,\theta)$)*
The processes of discounted defaultable bond prices with fractional recovery of treasury are local martingales if and only if the following condition holds:
for all $\theta\in[0,T^*]$ and for almost all $t\le\theta$ on the set $\{\tau>t\}$

$$\int_t^{\theta}\alpha_1(t,v)dv = J_1\left(\int_t^{\theta}\sigma_1(t,v)dv\right) + \delta\left(\frac{B(t-,\theta)}{D_1(t-,\theta)} - 1\right)\lambda_t. \tag{30}$$

- Fractional recovery of par

The payoff at maturity has form

$$D^{\triangle}(\theta,\theta) = \mathbf{1}_{\{\tau>\theta\}} + \mathbf{1}_{\{\tau\le\theta\}} \cdot \delta \frac{B_\theta}{B_\tau},$$

and before maturity has form

$$D^{\triangle}(t,\theta) = \mathbf{1}_{\{\tau>t\}} D_1(t,\theta) + \mathbf{1}_{\{\tau\le t\}} \cdot \delta \frac{B_t}{B_\tau}.$$

Theorem 8. *(HJM drift condition for $D^{\triangle}(t,\theta)$)*
Discounted prices of defaultable bond with fractional recovery of par are local martingales if and only if the following condition holds:
for all $\theta \in [0, T^*]$ and for almost all $t \leq \theta$ on the set $\{\tau > t\}$

$$\int_t^\theta \alpha_1(t,v)dv = J_1\left(\int_t^\theta \sigma_1(t,v)dv\right) + \delta\left(\frac{1}{D_1(t-,\theta)} - 1\right)\lambda_t. \qquad (31)$$

• Fractional recovery with multiple defaults

A holder of such defaultable bond receives, at maturity θ,

$$D^m(\theta,\theta) = \prod_{\tau_i \le \theta} (1 - L_{\tau_i}).$$

If we introduce process V_t by the formula:

$$V_t = \prod_{\tau_i \le t} (1 - L_{\tau_i}),$$

then $D^m(\theta,\theta) = V_\theta$ and for $t \le \theta$:

$$D^m(t,\theta) = V_t e^{-\int_t^\theta g_1(t,u)du} = V_t D_1(t,\theta).$$

Moreover, we assume that τ_i are moments of jumps of Cox process N_t (doubly stochastic Poisson process) with stochastic intensity process $(\lambda_t)_{t \ge 0}$. It can be shown that V_t solves the following SDE:

$$dV_t = -V_{t-} L_t dN_t, \tag{32}$$

and the process

$$M_t = N_t - \int_0^t \lambda_u du \tag{33}$$

follows $\mathbb{G}$ - martingale.

Theorem 9. *Discounted prices of defaultable bonds with multiple defaults and fractional recovery are local martingales if and only if the following condition holds:*

for all $\theta \in [0, T^*]$ *and for almost all* $t \leq \theta$ *on the set* $\{V_{t-} > 0\}$

$$\int_t^\theta \alpha_1(t,v)dv = J_1\bigg(\int_t^\theta \sigma_1(t,v)dv\bigg). \tag{34}$$

These results can be generalize to the rating migration case.

The set of rating classes $\mathcal{K}$ is identical with $= \{1, \ldots, K\}$, where the state $i = 1$ represents the highest rank and the state $i = K$ the default event. The credit rating migration process will be denoted by C^1 and assumed to be a conditional Markov chain relative to $\mathbb{F}$ with the unique absorption state K. We follow approach from Bielecki and Rutkowski book.

The conditional infinitesimal generator of the process C^1 describing credit rating migration, at time t given $\mathbb{G}_t$ has the form

$$\Lambda(t) = \begin{pmatrix} \lambda_{1,1}(t) & \lambda_{1,2}(t) & \cdots & \lambda_{1,K-1}(t) & \lambda_{1,K}(t) \\ \lambda_{2,1}(t) & \lambda_{2,2}(t) & \cdots & \lambda_{2,K-1}(t) & \lambda_{2,K}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{K-1,1}(t) & \lambda_{K-1,2}(t) & \cdots & \lambda_{K-1,K-1}(t) & \lambda_{K-1,K}(t) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where off-diagonal processes $\lambda_{i,j}(t)$, $i \neq j$ are non-negative processes adapted to $\mathbb{G}$ and diagonal elements are negative and are determined by off-diagonals by the formula $\lambda_{i,i}(t) = -\sum_{j \in \mathcal{K} \setminus \{i\}} \lambda_{i,j}(t)$. We can regard $p_{i,j}(t) = -\frac{\lambda_{i,j}(t)}{\lambda_{i,i}(t)}$ as a probability of jumping from the state i to the state j given that we jump-off the state i.

By f we denote the forward process associated with risk free bond and by $g_1, g_2, ..., g_{K-1}$ the pre-default term structures associated with ratings $1, 2, ..., K-1$. The pre-default term structure g is thus given by the formula

$$g(t,u) = g_{C^1(t)}(t,u) = 1_{\{C^1(t)=1\}} g_1(t,u) + \ldots + \\ + 1_{\{C^1(t)=K-1\}} g_{K-1}(t,u).$$

To avoid arbitrage it is reasonable to assume that

$$g_{K-1}(t,\theta) > g_{K-2}(t,\theta) > \ldots > g_1(t,\theta) > f(t,\theta)$$

for all $t \in [0,\theta]$ and all $\theta \in [0,T^*]$.

Hypothesis (H4):

$$g_i(t,t) - f(t,t) = \lambda_{i,K}(t)(1-\delta_i(t)), \qquad i = 1,\ldots,K-1, \tag{35}$$

so the intensity of migration from rating i into default state K is equal to the short term spread for rating i divided by one minus recovery from rating i. Of course, (35) implies

$$g_{C^1(t)}(t,t) = f(t,t) + (1-\delta_{C^1(t)}(t))\lambda_{C^1(t),K}(t), \tag{36}$$

Recovery payment depends on credit rating before default i.e.

$$\delta_t = \delta_{C^2(t)}(t) = 1_{\{C^2(t)=1\}}\delta_1(t) + 1_{\{C^2(t)=2\}}\delta_2(t) + \ldots + 1_{\{C^2(t)=K-1\}}\delta_{K-1}(t),$$

where δ_i is a recovery payment connected with $i-$th rating class and C^2 is so called process of the previous ratings:

$$C^2(t) = C^1(\tau_{k-1}), \qquad t \in [\tau_k, \tau_{k+1}),$$

where $\tau_1, \tau_2, \tau_3, \ldots$ denote the consecutive moments of jumps of credit migration process C^1 (of course $C^1(t) = C^1(\tau_k)$ for $t \in [\tau_k, \tau_{k+1})$) .

Hypothesis (H5): We assume that for $(\tau_k)_{k\geq 0}$ the consecutive times of jumps of credit migration process and for all $\theta \in [0, T^*]$ we have

$$\boldsymbol{P}\left(\Delta B(\tau_k, \theta) \neq 0\right) = 0, \qquad \boldsymbol{P}\left(\Delta D_i(\tau_k, \theta) \neq 0\right) = 0 \ \forall i = 1, \ldots K-1$$

Auxiliary processes. Define process $H_i(t) = 1_{\{i\}}(C^1(t))$ and for $i \neq j$

$$H_{i,j}(t) = \sum_{0<u\leq t} H_i(u-)H_j(u), \qquad \forall t \in \mathbb{R}_+.$$

Then the the processes

$$M_i(t) = H_i(t) - \int_0^t \lambda_{C^1(u),i}(u)du,$$

$$M_{i,j}(t) = H_{i,j}(t) - \int_0^t \lambda_{i,j}(u)H_i(u)du = H_{i,j}(t) - \int_0^t \lambda_{C^1(u),j}(u)H_i(u)du,$$

and

$$M_K(t) = H_K(t) - \int_0^t \sum_{i=1}^{K-1} \lambda_{i,K} H_i(u)du = H_K(t) - \int_0^t \lambda_{C^1(u),K}(1 - H_K(u))du,$$

are $\mathbb{G} = \mathbb{F} \vee \mathbb{F}^{C^1} \vee \mathbb{F}^{\Lambda}$-martingales.

Fractional recovery of market value with rating migrations.

Let us focus on defaultable bonds with fractional recovery of market value $D(t, \theta)$. This kind of bond pays 1 -unit of cash if default has not occurred before maturity θ, i.e., if the default time satisfies $\tau > \theta$, and if the bond defaults before θ we have recovery payment at the default time which is a fraction $\delta(t)$ of its market value just before the default time, so the recovery payment is equal to $\delta(\tau)D(\tau-, \theta)$. Therefore, in the case of rating migration, the price process of the defaultable bond with credit migrations and fractional recovery of market value should satisfy

$$D(\theta, \theta) = \mathbf{1}_{\{\tau > \theta\}} + \mathbf{1}_{\{\tau \le \theta\}} \delta_{C^2(\tau)}(\tau) D_{C^2(\tau)}(\tau-, \theta) \frac{B_\theta}{B_\tau},$$

where $\tau = \inf\{t > 0 : C^1(t) = K\}$.

Hence we postulate that for $t \le \theta$

$$
\begin{aligned}
D(t,\theta) = \mathbf{1}_{\{C^1(t)\neq K\}} D_{C^1(t)}(t,\theta) + \\
\mathbf{1}_{\{C^1(t)=K\}} \delta_{C^2(\tau)}(\tau) D_{C^2(\tau)}(\tau-,\theta)\frac{B_t}{B_\tau} \\
= \sum_{i=1}^{K-1} \mathbf{1}_{\{C^1(t)\neq K\}} \mathbf{1}_{\{C^1(t)=i\}} D_i(t,\theta) + \\
\sum_{i=1}^{K-1} \mathbf{1}_{\{C^1(t)=K\}} \mathbf{1}_{\{C^2(t)=i\}} \delta_i(\tau) D_i(\tau-,\theta)\frac{B_t}{B_\tau}
\end{aligned}
$$

or, equivalently,

$$
D(t,\theta) = \sum_{i=1}^{K-1} \left(H_i(t) D_i(t,\theta) + H_{i,K}(t)\delta_i(\tau) D_i(\tau-,\theta)\frac{B_t}{B_\tau}\right). \quad (37)
$$

Theorem 10. *The processes of discounted prices of a defaultable bond with credit migration and fractional recovery of market value are local martingales if and only if the following condition holds:*
for all $\theta \in [0, T^*]$ *and for almost all* $t \leq \theta$ *on the set* $\{\tau > t\}$

$$\int_t^\theta \alpha_{C^1(t)}(t,v)dv = J_{C^1(t)}\left(\int_t^\theta \sigma_{C^1(t)}(t,v)dv\right) + \sum_{i=1, i\neq C^1(t)}^{K-1}\left[\frac{D_i(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),i}(t). \tag{38}$$

Fractional recovery of par value

In the case of fractional recovery of par value the holder of defaultable bond receives 1 unit cash if there is no default prior to maturity and if bond has defaulted a fixed fraction δ of par value is paid at default time. Therefore the payoff at maturity has form

$$D(\theta,\theta) = 1_{\{\tau>\theta\}} + 1_{\{\tau\le\theta\}}\delta_{C^2(\tau)}\frac{B_\theta}{B_\tau},$$

hence

$$D(t,\theta) = \sum_{i=1}^{K-1}\left(H_i(t)D_i(t,\theta) + H_{i,K}(t)\delta_i\frac{B_t}{B_\tau}\right).$$

Theorem 11. *The processes of discounted prices of defaultable bond with fractional recovery of par value are local martingales if and only if the following condition holds*
for all $\theta \in [0, T^*]$ *and for almost all* $t \leq \theta$ *we have on the set* $\{C^1(t) \neq K\}$*:*

$$\begin{aligned}\int_t^\theta \alpha_{C^1(t)}(t,u)du =& J_{C^1(t)}\left(\int_t^\theta \sigma_{C^1(t)}(t,v)dv\right) \\ &+ \delta_{C^1(t)}\left[\frac{1}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),K}(t) \\ &+ \sum_{j=1,j\neq C^1(t)}^{K-1}\left[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),j}(t).\end{aligned} \tag{39}$$

Fractional recovery of Treasury value with rating migrations

The holder of a defaultable bond with fractional recovery of Treasury value receives 1 if there is no default by θ, and if default has occurred before maturity θ, then the a fixed amount $\delta \in [0, 1]$ is paid to the bondholder at maturity. Therefore, since paying δ at maturity θ is equivalent to paying $\delta B(\tau, \theta)$ at the default time τ, in the case of fractional recovery of Treasury value with rating migrations we have

$$D(\theta, \theta) = \mathbf{1}_{\{\tau > \theta\}} + \mathbf{1}_{\{\tau \le \theta\}} \delta_{C^2(t)},$$

hence

$$
\begin{aligned}
D(t,\theta) &= \mathbf{1}_{\{C^1(t)\neq K\}} D_{C^1(t)}(t,\theta) + \mathbf{1}_{\{C^1(t)=K\}} \delta_{C^2(t)} B(t,\theta) \\
&= \sum_{i=1}^{K-1} \mathbf{1}_{\{C^1(t)\neq K\}} \mathbf{1}_{\{C^1(t)=i\}} D_i(t,\theta) + \\
&\quad \sum_{i=1}^{K-1} \mathbf{1}_{\{C^1(t)=K\}} \mathbf{1}_{\{C^2(t)=i\}} \delta_i B(t,\theta)
\end{aligned}
$$

or, equivalently,

$$
D(t,\theta) = \sum_{i=1}^{K-1} \Big(H_i(t) D_i(t,\theta) + H_{i,K}(t) \delta_i B(t,\theta) \Big). \tag{40}
$$

Theorem 12. *The processes of discounted prices of a defaultable bond with fractional recovery of Treasury value are local martingales if and only if the following condition holds:*
for all $\theta \in [0, T^*]$ *and for almost all* $t \leq \theta$ *on the set* $\{\tau > t\}$

$$\int_t^\theta \alpha_{C^1(t)}(t,u)du = J_{C^1(t)}\left(\int_t^\theta \sigma_{C^1(t)}(t,v)dv\right) + \delta_{C^1(t)}\left[\frac{B(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),K}(t) + \sum_{j=1, j\neq C^1(t)}^{K-1}\left[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right]\lambda_{C^1(t),j}(t). \tag{41}$$

Fractional recovery with multiple defaults and rating migrations

This model describes situation, where company has had to declare default is not liquidated but is restructured. After that firm may default again in the future. Schönbucher investigated defaultable bonds whose face value is reduced by an fraction L_{τ_i} at each default time τ_i, where L_s is an $\mathbb{F}$-predictable process taking values in $[0,1]$. Therefore, a holder of such a defaultable bond receives, at maturity θ,

$$D^m(\theta,\theta) = \prod_{\tau_i \le \theta} (1 - L_{\tau_i}),$$

so

$$D^m(t,\theta) = V_t e^{-\int_t^\theta g_1(t,u)du} = V_t D_1(t,\theta). \tag{42}$$

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so

$$D^m(t, \theta) = V_t e^{-\int_t^\theta g_1(t,u)du} = V_t D_1(t, \theta). \tag{42}$$

As before, we assume that τ_i are jump times of a Cox process N_t with stochastic intensity process $(\gamma_t)_{t \geq 0}$.

We add a rating migration process to the model. Since a company after default is restructured, the rating migration process has no absorbing state and for the rating migration process $C(t)$ we take a conditional Markov chain with values in the set $\{1, \ldots, K-1\}$. Moreover, we assume that the process describing fractional losses does not depend on the credit migration process.

Note that $1 - L_t$ can be interpreted as a recovery process and therefore we will denote it by $\delta(t)$, so $\delta(t) = 1 - L_t$. Thus the bond price process should satisfy the following terminal condition:

$$D(\theta, \theta) = V_\theta = \prod_{\tau_i \leq \theta} (1 - L_{\tau_i}) = \prod_{\tau_i \leq \theta} \delta_{\tau_i},$$

and before maturity it should be given by the formula

$$D(t, \theta) = V_t D_{C^1(t)}(t, \theta) = V_t \sum_{i=1}^{K-1} H_i(t) D_i(t, \theta).$$

In this case the filtration $\mathbb{G}$ is specified as $\mathbb{G} = \mathbb{F} \vee \mathbb{F}^N \vee \mathbb{F}^C$, i.e. $\mathbb{G}_t = \mathcal{F}_t \vee \mathcal{F}_t^N \vee \mathcal{F}_t^C$, and hypothesis (H1), i.e. formula (35) takes the form

$$g_{C^1(t)}(t, t) = f(t, t) + (1 - \delta(t))\gamma_t \ .$$

Theorem 13. *The discounted prices of a bonds with fractional recovery with multiple defaults and rating migrations are local martingales if and only if the following condition holds:*
for all $\theta \in [0, T^*]$ *and for almost all* $t \leq \theta$ *on the set* $\{V_{t-} > 0\}$

$$\int_t^\theta \alpha_{C^1(t)}(t,v)dv = J_{C^1(t)}\left(\int_t^\theta \sigma_{C^1(t)}(t,v)dv\right) + \sum_{j=1, j\neq C^1(t)}^{K-1} \left[\frac{D_j(t-,\theta)}{D_{C^1(t)}(t-,\theta)} - 1\right] \lambda_{C^1(t),j}(t) \ . \tag{43}$$

Consistency conditions

Bielecki and Rutkowski generalized HJM model to defaultable bonds with ratings under consistency conditions. We investigate relationship between the the HJM type conditions and consistency conditions analogous to that in Bielecki and Rutkowski.

Consistency conditions

Bielecki and Rutkowski generalized HJM model to defaultable bonds with ratings under consistency conditions. We investigate relationship between the the HJM type conditions and consistency conditions analogous to that in Bielecki and Rutkowski.

In the case of fractional recovery of par value with rating migrations the consistency condition has the form

$$\sum_{i=1, i\neq C^1(t)}^{K-1} \Big[\Big(D_i(t-,\theta) - D_{C^1(t)}(t-,\theta)\Big)\lambda_{C^1(t),i}(t) + \\ (\delta_{C^1(t)}(t) - D_{C^1(t)}(t-,\theta))\lambda_{C^1(t),K}(t) + \\ \Big(g_{C^1(t)}(t,t) - f(t,t) + \bar{a}_{C^1(t)}(t,\theta)\Big) D_{C^1(t)}(t-,\theta)\Big] = 0. \tag{44}$$

Theorem 14. *For defaultable bonds with fractional recovery of par value with rating migration consistency condition* (44) *holds if and only if HJM type condition* (39) *holds.*

Theorem 14. *For defaultable bonds with fractional recovery of par value with rating migration consistency condition* (44) *holds if and only if HJM type condition* (39) *holds.*

In the case of other kind of recoveries we have very similar situation altho-ugh consistency conditions have slightly different form. So for all kinds of recovery

Consistency condition holds if and only if HJM type condition holds.

HJM conditions in terms of derivatives of Laplace exponents J_i

Under mild assumption HJM conditions have the form

i) Condition (39) for fractional recovery of market value and condition (43) for fractional recovery with multiple defaults have the form

$$\alpha_{C^1(t)}(t,\theta) = \left\langle DJ_{C^1(t)}\left(\int_0^\theta \sigma_{C^1(t)}(t,v)dv\right), \sigma_{C^1(t)}(t,\theta)\right\rangle_U + \sum_{i=1, i\neq C^1(t)}^{K-1} \lambda_{C^1(t),i}(t)\left(g_{C^1(t)}(t-,\theta) - g_i(t-,\theta)\right)e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))du}$$

ii) Condition (41) for fractional recovery of Treasury value has the form

$$\alpha_{C^1(t)}(t,\theta) = \left\langle DJ_{C^1(t)}\left(\int_0^\theta \sigma_{C^1(t)}(t,v)dv\right), \sigma_{C^1(t)}(t,\theta)\right\rangle_U +$$

$$\sum_{i=1, i\neq C^1(t)}^{K-1} \lambda_{C^1(t),i}(t)\Big(g_{C^1(t)}(t-,\theta) - g_i(t-,\theta)\Big)e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))du}$$

$$+\ \delta_{C^1(t)}\lambda_{C^1(t),K}\Big(g_{C^1(t)}(t-,\theta) - f(t-,\theta)\Big)e^{\int_t^\theta (g_{C^1(t)}(t-,u)-f(t-,u))du}.$$

iii) Condition (39) for fractional recovery of par value has the form

$$
\begin{aligned}
\alpha_{C^1(t)}(t,\theta) &= \left\langle DJ_{C^1(t)}\left(\int_0^\theta \sigma_{C^1(t)}(t,v)dv\right), \sigma_{C^1(t)}(t,\theta)\right\rangle_U + \\
&\sum_{i=1, i\neq C^1(t)}^{K-1} \lambda_{C^1(t),i}(t)\Big(g_{C^1(t)}(t-,\theta) - g_i(t-,\theta)\Big)e^{\int_t^\theta (g_{C^1(t)}(t-,u)-g_i(t-,u))du} \\
&+ \delta_{C^1(t)}\lambda_{C^1(t),K}\, g_{C^1(t)}(t-,\theta)e^{\int_t^\theta g_{C^1(t)}(t-,u)du}.
\end{aligned}
$$

II. Modeling of credit migration processes

We consider an arbitrage-free market on which defaultable contingent claims are also traded. Credit rating migration process C of a defaultable instrument (eg. bond) is a càdlàg process which takes values in the set of rating classes $\mathcal{K} = \{1, \ldots, K\}$, where the state $i = 1$ represents the highest rank, $i = K - 1$ the lowest rank and the state $i = K$ the default event. So

$$\tau := \inf \{t > 0 : C_t = K\}.$$

is a default time.

II. Modeling of credit migration processes

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$$\tau := \inf\{t > 0 : C_t = K\}.$$

is a default time.

Goals:

1. Describing a good model of a credit migrations process.

2. Pricing of defaultable rating-sensitive claims

Doubly Stochastic Markov Chains We assume that all processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We also fix a filtration $\mathbb{F}$, which plays a role of reference filtration (corresponding to observation of market without credit rating i.e. filtration corresponding to interest rate risk and other market factors that drives credit risk).

Doubly Stochastic Markov Chains We assume that all processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We also fix a filtration $\mathbb{F}$, which plays a role of reference filtration (corresponding to observation of market without credit rating i.e. filtration corresponding to interest rate risk and other market factors that drives credit risk).

Definition 2. *A càdlàg process C is called the $\mathbb{F}$–doubly stochastic Markov chain with state space $\mathcal{K} \subset \{\ldots, -1, 0, 1, 2, \ldots\}$ if there exists a family of stochastic matrices $P(s,t) = (p_{i,j}(s,t))_{i,j\in\mathcal{K}}$ for $0 \leq s \leq t$ such that*

1. *a matrix $P(s,t)$ is $\mathcal{F}_t$ measurable, $P(s,\cdot)$ is progressively measurable*

2. *for all $i, j \in \mathcal{K}$, $s \leq t$*

$$\mathbf{P}(C_t = j | \mathcal{F}_\infty \vee \mathcal{F}_s^C) \mathbf{1}_{\{C_s = i\}} = \mathbf{1}_{\{C_s = i\}} p_{i,j}(s,t). \tag{45}$$

Example 1 (Compound Poisson process). *A compound Poisson process X with jumps in $\mathbb{Z}$ is an $\mathbb{F}$–DS Markov chain. We know that $X_t = \sum_{i=1}^{N_t} Y_i$, where N is a Poisson process with intensity λ, Y_i is a sequence of independent identically distributed random variables with values in $\mathbb{Z}$ and distribution ν. Moreover $(Y_i)_i$ and N are independent. Hence for $\mathcal{F}_\infty = \sigma(N)$, $j \geq i$, $s \leq t$:*

$$\mathbf{P}(X_t = j|\mathcal{F}_\infty \vee \mathcal{F}_s^X)\mathbf{1}_{\{X_s=i\}} = \mathbf{P}(X_t = j, X_s = i|\mathcal{F}_\infty \vee \mathcal{F}_s^X)\mathbf{1}_{\{X_s=i\}} =$$

$$\mathbf{P}(X_t - X_s = j - i|\mathcal{F}_\infty)\mathbf{1}_{\{X_s=i\}} = \mathbf{P}(\sum_{m=N_s+1}^{N_t} Y_m = j - i|\mathcal{F}_\infty)\mathbf{1}_{\{X_s=i\}} =$$

$$\nu^{\otimes(N_t-N_s)}(j - i)\mathbf{1}_{\{X_s=i\}}.$$

Thus $p_{i,j}(s,t) = \nu^{\otimes(N_t-N_s)}(j-i)$ satisfy conditions 1) and 2) of Definition 2.

Example 2. *The Cox process C is a DS–Markov chain with $\mathcal{K} = \mathbb{N}$. Indeed, the definition of Cox process implies that*

$$\mathbf{P}(C_t - C_s = k | \mathcal{F}_\infty \vee \mathcal{F}_s^C) = e^{-\int_s^t \lambda_u du} \frac{\left(\int_s^t \lambda_u du\right)^k}{k!} \tag{46}$$

for some $\mathbb{F}$–adapted intensity process λ such that $\lambda \geq 0$, $\int_0^t \lambda_s ds < \infty$ for all $t \geq 0$ and $\int_0^\infty \lambda_s ds = \infty$. Hence follows that

$$\mathbf{P}(C_t - C_s = k | \mathcal{F}_\infty \vee \mathcal{F}_s^C) = \mathbf{P}(C_t - C_s = k | \mathcal{F}_\infty),$$

so increments are conditionally independent from the past and $\mathcal{F}_\infty$.

Therefore for $j \geq i$

$$\mathbf{P}(C_t = j|\mathcal{F}_\infty \vee \mathcal{F}_s^C)\mathbf{1}_{\{C_s=i\}} = \mathbf{P}(C_t = j, C_s = i|\mathcal{F}_\infty \vee \mathcal{F}_s^C)\mathbf{1}_{\{C_s=i\}} =$$

$$\mathbf{1}_{\{C_s=i\}}\mathbf{P}(C_t - C_s = j - i|\mathcal{F}_\infty \vee \mathcal{F}_s^C) = \mathbf{1}_{\{C_s=i\}} e^{-\int_s^t \lambda_u du} \frac{\left(\int_s^t \lambda_u du\right)^{j-i}}{(j-i)!}.$$

Thus

$$p_{i,j}(s,t) = \begin{cases} e^{-\int_s^t \lambda_u du} \frac{\left(\int_s^t \lambda_u du\right)^{j-i}}{(j-i)!}, & \text{for } j \geq i \\ 0, & \text{for } j < i \end{cases}$$

satisfy conditions 1) and 2) of definition 2, so a Cox process is a DS–Markov chain with $\mathcal{K} = \mathbb{N}$.

Example 3 (Time changed discrete Markov chain). *Assume that $\bar{C}$ is a discrete time Markov chain with values in $\mathcal{K} = \{1, \ldots K\}$, N is a Cox process and processes $(\bar{C}_k)_{k \geq 0}$ and $(N_t)_{t \geq 0}$ are independent and conditionally independent given $\mathcal{F}_\infty$. Then the process*

$$C_t := \bar{C}_{N_t},$$

is a DS–Markov chain.

Simple calculations gives another elementary example:

Example 4 (Truncated Cox process). *The process $C_t := \min\{N_t, K\}$, where N is the Cox process and $K \in \mathbb{N}$, is a DS–Markov chain with state space $\mathcal{K} = \{0, \ldots K\}$.*

Since we are interested in using the class of DS–Markov chains to model financial markets with rating migrations, we restrict ourselves to a finite set $\mathcal{K}$, i.e. $\mathcal{K} = \{1, \dots, K\}$, with $K < \infty$. Moreover we make assumption that $C_0 = l$ for some $l \in \mathcal{K}$. This assumption does not restrict generality of the results.

Proposition 15. *If C is a DS–Markov chain, then*

$$\mathbf{P}(C_{u_1} = i_1, \dots C_{u_n} = i_n | \mathcal{F}_\infty \vee \mathcal{F}^C_{u_0}) \mathbf{1}_{\{C_{u_0} = i_0\}} \\ = \mathbf{1}_{\{C_{u_0} = i_0\}} p_{i_0, i_1}(u_0, u_1) \prod_{k=1}^{n-1} p_{i_k, i_{k+1}}(u_k, u_{k+1}) \tag{47}$$

for arbitrary $0 \le u_0 \le \dots \le u_n$ and $(i_0, \dots, i_n) \in \mathcal{K}^{n+1}$.

The following hypothesis is standard in credit risk theory

HYPOTHESIS H : For every bounded $\mathcal{F}_\infty$ measurable random variable Y we have for each $t \geq 0$

$$\mathbf{E}(Y|\mathcal{F}_t \vee \mathcal{F}_t^C) = \mathbf{E}(Y|\mathcal{F}_t).$$

It is well known that Hypothesis H is equivalent to martingale invariance property of filtration $\mathbb{F}$ w.r.t $\mathbb{F} \vee \mathbb{F}^C$ i.e any $\mathbb{F}$ martingale is $\mathbb{F} \vee \mathbb{F}^C$ martingale. This hypothesis is satisfied for the DS–Markov chains.

Proposition 16. *If C is an DS–Markov chain then Hypothesis H holds.*

Proposition 17. *Assume that C is an $\mathbb{F}$ DS–Markov chain. Then C is an $\mathbb{F}$ conditional $\mathbb{F} \vee \mathbb{F}^C$ Markov chain.*

Proof. We have to check that for $s \leq t$

$$\mathbf{P}(C_t = i | \mathcal{F}_s \vee \mathcal{F}_s^C) = \mathbf{P}(C_t = i | \mathcal{F}_s \vee \sigma(C_s))$$

By definition of a DS–Markov chain

$$\mathbf{P}(C_t = i | \mathcal{F}_s \vee \mathcal{F}_s^C) = \mathbf{E}(\mathbf{E}(\mathbf{1}_{\{C_t = i\}} | \mathcal{F}_\infty \vee \mathcal{F}_s^C) | \mathcal{F}_s \vee \mathcal{F}_s^C) =$$

$$\mathbf{E}\left(\sum_{j=1}^K \mathbf{1}_{\{C_s = j\}} p_{j,i}(s,t) | \mathcal{F}_s \vee \mathcal{F}_s^C\right) = \sum_{j=1}^K \mathbf{1}_{\{C_s = j\}} \mathbf{E}\left(p_{j,i}(s,t) | \mathcal{F}_s \vee \mathcal{F}_s^C\right) = I.$$

$$I = \sum_{j=1}^K \mathbf{1}_{\{C_s = j\}} \mathbf{E}\left(p_{j,i}(s,t) | \mathcal{F}_s\right)$$

since hypothesis H holds and this ends the proof.

Theorem 18. *Let C be a DS–Markov chain with transition matrices $P(s,t)$. Then for any $u \geq t \geq s$ we have*

$$P(s,u) = P(s,t)P(t,u) \;\; a.s. \;, \tag{48}$$

so on the set $\{C_s = i\}$ holds

$$p_{i,j}(s,u) = \sum_{k=1}^{K} p_{i,k}(s,t) p_{k,j}(t,u).$$

Definition 3. *We say that a DS–Markov chain C has intensity if there exists a matrix valued process $\Lambda = (\Lambda_s)_{s\geq 0} = (\lambda_{i,j}(s))_{s\geq 0}$ which is $\mathbb{F}$–progressively measurable and satisfies the following conditions:*

1) Λ *is locally integrable i.e. for any $T > 0$*

$$\int_{]0,T]} \sum_{i\in\mathcal{K}} |\lambda_{ii}(s)| ds < \infty \qquad a.s. \tag{49}$$

2) Λ *satisfies conditions:*

$$a)\ \lambda_{i,j}(s) \geq 0 \quad \forall i,j \in \mathcal{K}, i \neq j \quad , \quad \lambda_{i,i}(s) = -\sum_{j\neq i} \lambda_{i,j}(s) \quad \forall i \in \mathcal{K}, \tag{50}$$

b) the Kolmogorov backward equation: for all v, $v \leq t$,

$$P(v,t) - \mathbb{I} = \int_v^t \Lambda(u) P(u,t) du, \tag{51}$$

c) the Kolmogorov forward equation: for all v, $v \leq t$,

$$P(v,t) - \mathbb{I} = \int_v^t P(v,u) \Lambda(u) du. \tag{52}$$

This process Λ is called the intensity of DS–Markov chain C.

Theorem 19 (Existence of Intensity). *Let C be a DS–Markov chain. Assume that*

1. *P as a matrix value mapping is measurable i.e.* $P : (\mathbb{R}_+ \times \mathbb{R}_+ \times \Omega, \mathcal{B}(R_+ \times R_+) \otimes \mathcal{F}) \to (\mathbb{R}^{K\times K}, \mathcal{B}(\mathbb{R}^{K\times K}))$.

2. *There exists a version of process P which is continuous in (s,t).*

3. *For every $t \geq 0$ the following limit exists in almost sure sense:*
$$\Lambda_t := \lim_{h\downarrow 0} \frac{P(t,t+h) - \mathbb{I}}{h}, \tag{53}$$
and is locally integrable.

Then Λ is the intensity of C.

Example 5. *If* $C_t = \min\{N_t, K\}$*, where* N *is a Cox process with intensity process* $\tilde{\lambda}$*, then* C *has the intensity process of the following form*

$$\lambda_{i,j}(t) = \begin{cases} -\tilde{\lambda}(t), & \textit{for } i = j \in \{0, \ldots K-1\}; \\ \tilde{\lambda}(t), & \textit{for } j = i+1 \ \textit{ with } i \in \{0, \ldots K-1\}; \\ 0, & \textit{otherwise.} \end{cases}$$

Example 6. *If* $C_t = \bar{C}_{N_t}$*, then* Λ *given by*

$$\lambda_{i,j}(t) = (P - I)_{i,j} \tilde{\lambda}_t$$

is the intensity of C*.*

Let $H_t^i := \mathbf{1}_{\{C_t = i\}}$.

Theorem 20. *Let* $(C_t)_{t\geq 0}$ *be a* $\mathcal{K}$*–valued stochastic process,* $\widehat{\mathcal{G}}_t := \mathcal{F}_\infty \vee \mathcal{F}_t^C$*, and* $(\Lambda_t)_{t\geq 0}$ *be an* $\mathbb{F}$*– progressively measurable matrix valued process satisfying* (49) *and (50). The process* C *is a DS–Markov chain with intensity process* Λ *if and only if the processes*

$$M_t^i := H_t^i - \int_{]0,t]} \lambda_{C_u,i}(u)du,$$

for $i \in \mathcal{K}$*, are* $\widehat{\mathbb{G}}$ *– local martingales.*

Theorem 21. *The processes* M^i *for all* $i \in \mathcal{K}$ *are local martingales if and only if the processes*

$$M_t^{i,j} := H_t^{i,j} - \int_{]0,t]} H_s^i \lambda_{i,j}(s) ds,$$

where

$$H_t^{i,j} := \int_{]0,t]} H_{u-}^i dH_u^j \tag{54}$$

are local martingales for $i \neq j \in \mathcal{K}$.

Processes $H^{i,j}$ defined by (54) counts number of jumps from state i to j up to time t, one can show that

$$H^{i,j} = \sum_{0<u\leq t} H_{u-}^i H_u^j.$$

Since M^i are adapted to $\mathbb{G}$ with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_t^C$, which is sub-filtration of $\widehat{\mathbb{G}}$ we have:

Corollary 3. *If C is Doubly Stochastic Markov Process, then M^i are $\mathbb{G}$ local martingales.*

Remark 6. *Process C obtained by the canonical construction in Bielecki Rutkowski book is a DS–Markov chain. This is a consequence of theorem 20, because calculations analogous to lemma 11.3.2 page 347 shows that M^i are $\widehat{\mathbb{G}}$ martingales and in the canonical construction Λ is bounded.*

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Theorem 22. *Let $(\Lambda_t)_{t\geq 0}$ be an arbitrary locally integrable $\mathbb{F}$– progressively measurable matrix valued process which satisfies condition* (50)*. Then we can construct DS–Markov chain with intensity $(\Lambda_t)_{t\geq 0}$ and given initial state i_0.*

Defaultable Rating-sensitive claims. Description

We consider an arbitrage-free market on which defaultable instruments are also traded.

In what follows, we assume that C is a DS–Markov chain with intensity process Λ which satisfies the following integrability condition:

$$\mathbf{E}\left(\int_{]0,T]}\sum_{i\in\mathcal{K}}|\lambda_{i,i}(s)|ds\right)<\infty \tag{55}$$

Hence, in particularly, follows that local martingales M^i, $M^{i,j}$ are also martingales. By B we denote the value of the saving account (as usual $dB_t = r_t B_t dt, B_0 = 1$, where r is an $\mathcal{F}$-progressively measurable and integrable process).

Definition 4. *By* defaultable rating-sensitive claim *maturing at* T *we mean a quintuple* (X, A, Z, C, τ)*, where* X *is a* $K-1$ *dimensional vector of* $\mathcal{F}_T$ *measurable random variables,* A *is a* $K-1$ *dimensional vector valued* $\mathbb{F}$*-progressively measurable stochastic process of finite variation,* Z *is an* $\mathbb{F}$*-predictable* $K \times K$ *dimensional matrix valued process with zero at diagonal,* C *is a càdlàg process with values in* $\mathcal{K}$ *and* τ *a positive random variable.*

In this definition X describes a promised payoff which is contingent on rating at maturity T i.e. payoff is equal to X_i provided that $\{C_T = i\}$, A models a process of promised dividends which can depend on current credit rating, $Z^{i,j}$ are processes which describe payments that are paid at times when rating changes, in particularly $Z^{j,K}$ specifies recovery payment at default time τ provided that before the default time we are in state j, C is a credit rating process, τ is a default time.

By taking $X_i = 1, \;\; A^i = 0, \;\; Z^{i,K} = \delta_i, \quad$ for $i = 1, \dots, K-1$ we obtain that defaultable bond is a claim in the sense of our definition.

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Remark 7. *i) If we put $X_i = X$ for each i, then we have promised payment which depends only on default:*

$$\sum_{i=1}^{K-1} X_i \mathbf{1}_{\{C_T=i\}} = X \sum_{i=1}^{K-1} \mathbf{1}_{\{C_T=i\}} = X\mathbf{1}_{\{C_T \neq K\}} = X\mathbf{1}_{\{\tau > T\}}.$$

By taking $X_i = 1,\ \ A^i = 0,\ \ Z^{i,K} = \delta_i, \quad$ for $i = 1, \ldots, K-1$ we obtain that defaultable bond is a claim in the sense of our definition.

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ii)

$$\sum_{i=1}^{K-1} \int_{]0,t\wedge T]} Z_u^{i,K} dH_u^{i,K} = \sum_{i=1}^{K-1} Z_\tau^{i,K} \mathbf{1}_{\{0<\tau\le t\wedge T, C_{\tau-}=i\}} = Z_\tau^{C_{\tau-},K} \mathbf{1}_{\{0<\tau\le t\wedge T\}},$$

so the recovery process allows recovery to depend on the rating before default time τ.

Now, we defined the dividend process which describes cash flows from claim on the interval $[0, T]$.

Definition 5. *Dividend process* $D = (D_t)_{t \geq 0}$ *of claim* (X, A, Z, C, τ) *maturing at* T *equals for* $t \geq 0$

$$D_t = \sum_{i=1}^{K-1} (X_i H_T^i \mathbf{1}_{[T,+\infty[}(t) + \int_{]0,t\wedge T]} H_u^i dA_u^i + \sum_{j \neq i \in \mathcal{K}} \int_{]0,t\wedge T]} Z_u^{i,j} dH_u^{i,j}).$$

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Example. The dividend process for defaultable bond equals

$$D_t = \mathbf{1}_{\{\tau > T\}} \mathbf{1}_{[T,+\infty[}(t) + \delta_{C_{\tau-}} \mathbf{1}_{\{0 < \tau \leq t \wedge T\}}.$$

Example 7 (credit sensitive note). *The coupons of this note are paid at pre-specified coupon dates* $0 < T_1 < T_2 < \ldots < T_n$*, and value of coupon is contingent on rating corporate at coupon date. Recovery payment depends on pre-default rating* $C_{\tau-}$*, and it is assumed that* $\delta_i \in [0, 1)$ *is fixed for each* $i \in \mathcal{K} \setminus K$*. So*

$$X_i = 1, \;\; Z^{i,K} = \delta_i, \;\; F = 0, \;\; A^i_t = \sum_{j=1}^{n} \mathbf{1}_{\{t \geq T_j\}} d_{i,j},$$

where $d_{i,j}$ *are fixed constants chosen in advance and dividend process of this note is given by*

$$D_t = \mathbf{1}_{\{\tau > T\}} \mathbf{1}_{[T,+\infty[}(t) + \sum_{i=1}^{K-1} \int_{]0,t\wedge T]} \delta_i dH^{i,K}_u + \sum_{i=1}^{K-1} \int_{]0,t\wedge T]} H^i_u dA^i_u.$$

Pricing

Definition 56. *The ex-dividend price process S of defaultable rating-sensitive claim (X, A, Z, C, τ) is given by*

$$S_t = B_t \mathbf{E}\left(\int_{]t,T]} B_u^{-1} dD_u \middle| \mathcal{G}_t \right),$$

for $t \geq 0$.

Theorem 23. *Let (X, A, Z, τ, C) be a defaultable rating sensitive claim. Then the ex-dividend price process is given by the formula:*

$$S_t \mathbf{1}_{\{C_t=i\}} = \mathbf{1}_{\{C_t=i\}} \sum_{j=1}^{K-1} B_t \mathbf{E}\Bigg(\frac{X_j p_{i,j}(t,T)}{B_T} + \int_{]t,T]} B_u^{-1} p_{i,j}(t,u) dA_u^j + \int_{]t,T]} \sum_{k=1}^{K} \frac{Z_u^{j,k}}{B_u} p_{i,j}(t,u) \lambda_{j,k}(u) du \Bigg| \mathcal{F}_t \Bigg).$$

Corollary 4. *The ex-dividend price D^δ of defaultable bond with fractional recovery of par value is equal to*

$$\begin{aligned} D^\delta(t,T)\mathbf{1}_{\{C_t=i\}} &= \mathbf{1}_{\{C_t=i\}} \sum_{j=1}^{K-1} B_t \mathbf{E}\Bigg(\frac{p_{i,j}(t,T)}{B_T} \\ &+ \int_{]t,T]} \frac{\delta_j}{B_u} p_{i,j}(t,u) \lambda_{j,K}(u) du \Bigg| \mathcal{F}_t \Bigg) \end{aligned}$$

for $t < T$.

Corollary 5. *The ex-dividend price of the Credit Sensitive Note with coupons with resetting at coupon payment date equals*

$$B_t \mathbf{E}\left(\sum_{k:t<T_k} d_{C_{T_k}} \frac{\mathbf{1}_{\{\tau>T_k\}}}{B_{T_k}} \middle| \mathcal{F}_t \right) \mathbf{1}_{\{C_t=i\}}$$

$$= \mathbf{1}_{\{C_t=i\}} \mathbf{E}\left(\sum_{k:t<T_k} e^{-\int_t^{T_k} r_u du} \left(\sum_{j=1}^{K-1} d_j p_{i,j}(t,T_k) \right) \middle| \mathcal{F}_t \right),$$

for $t < T$.

Examples. Pricing Bonds and CDS in the time changed discrete Markov chain model.

$$C_t := \bar{C}_{N_t},$$

where N is the Cox process with $\mathbb{F}$ adapted intensity process λ, $(\bar{C})$ is a discrete time (homogenous) Markov chain independent of $\mathbb{F}$ with values in set $\mathcal{K} = \{1, \ldots, K\}$ and with one-step transition matrix P of the following form:

$$P = \begin{pmatrix} 0 & p_{1,2} & p_{1,3} & \cdots & p_{1,K} \\ p_{2,1} & 0 & p_{2,3} & \cdots & p_{2,K} \\ p_{3,1} & p_{3,2} & 0 & \cdots & p_{3,K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{K-1,1} & p_{K-1,2} & p_{K-1,2} & \cdots & p_{K-1,K} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover. we assume that processes $(\bar{C}_k)_{k\geq 0}$ and $(N_t)_{t\geq 0}$ are conditionally independent given $\mathcal{F}_\infty$.

Theorem 24.

$$p_{i,j}(s,t) = \left[e^{(P-I)\int_s^t \lambda_u du}\right]_{i,j}. \tag{57}$$

Moreover, for $i, j \in \mathcal{K} \setminus K$

$$p_{i,j}(s,t) = \left[e^{-(I-Q)\int_s^t \lambda_u du}\right]_{i,j} = \sum_{l=1}^{k} \sum_{m=n'_l+1}^{n'_l+n_l} a_{i,m} \left(\sum_{p=m}^{n'_l+n_l} b_{p,j} e^{-(1-d_l)\int_s^t \lambda_u du} \frac{\left(\int_s^t \lambda_u du\right)^{p-m}}{(p-m)!} \right), \tag{58}$$

where Q is the matrix from the canonical form of P, and $A = [a_{i,j}]_{i,j=1}^{K-1}$ is the matrix from the Jordan's decomposition of Q, i.e. a decomposition of the form $Q = AJA^{-1}$ with a nonsingular matrix A, $J = \oplus_{l=1}^{k} J_{n_l}(d_l)$ and $J_{n_l}(d_l)$ is a Jordan's block of dimension n_l associated with eigenvalue d_l and $n'_1 := 0$, $n'_l := n'_{l-1} + n_l$ for $l > 1$, and $b_{p,j} := [A^{-1}]_{p,j}$.

Example 8. *Assume that the matrix Q is diagonalizable i.e. there exist a nonsingular matrix A and a diagonal matrix $D = diag(d_1, \ldots, d_{K-1})$ such that $Q = ADA^{-1}$. Then formula* (58) *simplifies, namely for $i, j \in \mathcal{K} \setminus K$ we have*

$$\left[e^{-(I-Q)\int_s^t \lambda_u du} \right]_{i,j} = \sum_{n=1}^{K-1} a_{i,n} b_{n,j} e^{-(1-d_n)\int_s^t \lambda_u du}, \tag{59}$$

where $b_{n,j} := [A^{-1}]_{n,j}$.

We show how to calculate a joint conditional distribution of a default time τ and a pre-default state of rating migration process C in terms of transition matrix P as well as the matrix from canonical decomposition of P.

Theorem 25. *Let Q be the matrix from canonical decomposition of P and $i, j \in \mathcal{K} \setminus K$. Then for any $v > 0$ and for every $0 \le t \le v$ we have*

$$\begin{aligned} &\mathbf{P}(t < \tau \le v, C_{\tau-} = j | \mathcal{F}_\infty \vee \sigma(C_t)) \mathbf{1}_{\{C_t = i\}} \\ &= \mathbf{1}_{\{C_t = i\}} p_{j,K} \left[(\mathbb{I} - Q)^{-1} \left(\mathbb{I} - e^{-(\mathbb{I} - Q) \int_t^v \lambda_u du} \right) \right]_{i,j}. \end{aligned} \tag{60}$$

Consider bonds with fractional recovery of par value

$$D(t,T) = B_t \mathbf{E}\left(\frac{1}{B_T}\mathbf{1}_{\{\tau>T\}} + \frac{\delta_{C_{\tau-}}}{B_\tau}\mathbf{1}_{\{t<\tau\le T\}}|\mathcal{G}_t\right).$$

Theorem 26. *For $t \le T$, the price of a defaultable bond with fractional recovery of par value is equal to*

$$D(t,T)\mathbf{1}_{\{\tau>t\}} = \sum_{i=1}^{K-1}\mathbf{1}_{\{C_t=i\}}\left[\sum_{j=1}^{K-1}\left(\mathbf{E}\left(e^{-\int_t^T r_u du}\left[e^{-(\mathbb{I}-Q)\int_t^T \lambda_u du}\right]_{i,j}\middle|\mathcal{F}_t\right)\right.\right.$$
$$\left.\left. + \delta_j \int_t^T \mathbf{E}\left(e^{-\int_t^u r_v dv}\left[e^{-(\mathbb{I}-Q)\int_t^u \lambda_v dv}\right]_{i,j} p_{j,K}\lambda_u \middle|\mathcal{F}_t\right) du\right)\right].$$

We are also interested in pricing credit derivatives connected with such defaultable bond, for example in pricing of CDS on such bond. Credit Default Swap is an agreement between two parties protection seller and protection buyer. This contracts have two legs.

Premium Leg: Protection buyer agrees to pay fixed amount κ (CDS spread) at given dates $\mathcal{T} = \{T_1 < T_2 < \ldots < T_n\}$ provided that default didn't happened before or at T_n. For $t \leq T_1$ we have:

$$B_t \mathbf{E}\left(\sum_{k=1}^{n} \frac{\kappa}{B_{T_k}} 1_{\{\tau > T_k\}} | \mathcal{G}_t\right).$$

Default Leg: Protection seller agrees to cover all losses on bond provided that loss occurred before T_n, the protection horizon. For $t \leq T_1$ value of this leg is equal to

$$B_t \mathbf{E}\left(\frac{1-\delta_{C_{\tau-}}}{B_\tau}1_{\{t<\tau\leq T_n\}}|\mathcal{G}_t\right).$$

Provided that we know value of spread κ, at time t value of CDS is difference between premium leg and default leg:

$$\text{CDS}(t,\mathcal{T},\kappa) = B_t \mathbf{E}\left(\sum_{k=1}^{n}\frac{\kappa}{B_{T_k}}1_{\{\tau>T_k\}} - \frac{(1-\delta_{C_{\tau-}})}{B_\tau}1_{\{t<\tau\leq T_n\}}|\mathcal{G}_t\right).$$

CDS spread κ which is agreed at contracts inception (time $t \leq T_1$) is chosen in such a way that value of contract (at inception date) is equal to 0: $\text{CDS}(t,\mathcal{T},\kappa) = 0$.

Pricing of CDS is the mainly the issue of determining a CDS spread κ. To find κ we must compute value of two legs, since *fair* CDS spread is given as:

$$\kappa(t,\mathcal{T})1_{\{C(t)\neq K\}} = 1_{\{C(t)\neq K\}} \frac{\mathbf{E}\left(\frac{B_t}{B_\tau}(1-\delta_{C_{\tau-}})1_{\{t<\tau\leq T_n\}}|\mathcal{G}_t\right)}{\mathbf{E}\left(\sum_{k=1}^{n}\frac{B_t}{B_{T_k}}1_{\{\tau>T_k\}}|\mathcal{G}_t\right)}.$$

Theorem 27. *The value $V_D(t)$ of the default leg is equal to*

$$\sum_{i=1}^{K-1}\mathbf{1}_{\{C_t=i\}}\left(\sum_{j=1}^{K-1}(1-\delta_j)\int_t^{T_n}\mathbf{E}\left(e^{-\int_t^u r_v dv}\left[e^{-(\mathbb{I}-Q)\int_t^u \lambda_v dv}\right]_{i,j} p_{j,K}\lambda_u \middle| \mathcal{F}_t\right) du$$

The value of the premium leg is given by

$$V_P(t) = \sum_{i=1}^{K-1}\mathbf{1}_{\{C_t=i\}}\left(\sum_{k=1}^{n}\sum_{j=1}^{K-1}\mathbf{E}\left(e^{-\int_t^{T_k} r_u du}\left[e^{-(\mathbb{I}-Q)\int_t^{T_k}\lambda_u du}\right]_{i,j}\middle|\mathcal{F}_t\right)\right).$$

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Thank you for your attention.