Fractional Brownian Motion and Applications

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Outline

Outline (cont.)

Standard Fractional Brownian Motion

$(B(t), t \geq 0)$ is a standard fractional Brownian motion with $H \in (0,1)$ if it is a Gaussian process with continuous sample paths that satisfies

$$\mathbb{E}\left[B(t)\right] = 0$$
$$\mathbb{E}\left[B(s)B(t)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

for all $s, t \in \mathbb{R}_+$.

The formal derivative $\frac{dB}{dt}$ is called fractional Gaussian noise.

Some properties

1. Self-similarity

$$(B^H(\alpha t), t \geq 0) \overset{L}{\sim} (\alpha^H B^H(t), t \geq 0)$$

   for $\alpha > 0$

2. Long range dependence for $H \in (\frac{1}{2}, 1)$

$$r(n) = \mathbb{E}[B^H(1)(B^H(n+1) - B^H(n))]$$

$$\Sigma_{n=0}^{\infty} r(n) = \infty$$

3. A sample path property

$(B^H(t), t \geq 0)$ is of unbounded variation so the sample paths are not differentiable a.s.

$$\Sigma_i |B^H(t_{i+1}^{(n)}) - B^H(t_i^{(n)})|^p \to \begin{cases} 0 & pH > 1 \\ c(p) & pH = 1 \\ +\infty & pH < 1 \end{cases}$$

$c(p) = \mathbb{E}|B^H(1)|^p$

$(t_i^{(n)}, i = 0, 1, \cdots, n)$ is a sequence of nested partitions of $[0, 1]$

- $p = 2$ and $H \in (\frac{1}{2}, 1) \Longrightarrow$ zero quadratic variation
- $p = 2$ and $H \in (0, \frac{1}{2}) \Longrightarrow$ infinite quadratic variation
- FBM is not a semimartingale for $H \neq \frac{1}{2}$

Law of the Iterated Logarithm for a fractional Brownian motion

$$\limsup_{t\to\infty} \frac{B^H(t)}{(2\ t^{2H}\log\log t)^{\frac{1}{2}}} = k_H \quad a.s.$$

where k_H is a constant that only depends on H

Some Development of Fractional Brownian Motion

- I. J. Schonberg 1937
- A. N. Kolmogorov 1940
- H. E. Hurst 1951
- B. Mandelbrot 1965

Rainfall in the Nile River Valley

(Genesis 41:29-30) "Joseph said unto Pharoah.
There come seven years of great plenty throughout all the land of Egypt; and there shall arise after them seven years of famine."

(Genesis 7:11-12) "In the six hundredth year of Noah's life, the windows of heaven were opened, and the rain was upon the earth forty days and forty nights."

Hurst considered the data of annual rainfall in the Nile valley for about 850 years (622-1469). For n successive years, compute the deviation of the cumulative rainfall in the first k years from the linear function whose slope is the empirical (sample) mean for the n years. Hurst noted that the range of these deviations normalized by the (sample) standard deviation behaved as cn^H for H $= 0.7$ and the sum of n successive years of rainfall had approximately the same law as $n^H x_1$ where x_1 was the first year's rainfall in these n years.

Hurst advanced the idea of the Aswan High Dam that was constructed 1959-1970. He was nicknamed Abu Nil (Father of the NIle). The reservoir is Lake Nasser that stretches 300 miles upstream of the dam into northern Sudan.

B. Mandelbrot studied self similar "pathological" curves (fractals). A fractal curve retains some general pattern of irregularity no matter how it is magnified, e.g. snowflakes, tree barks.

Telecommunications

Traffic occurs in bursts with idle periods in between, e.g. lots of very short bursts, many long bursts and some very long bursts. (1994, Leland, Taqqu, Willinger and Wilson, On the self-similar nature of ethernet traffic)

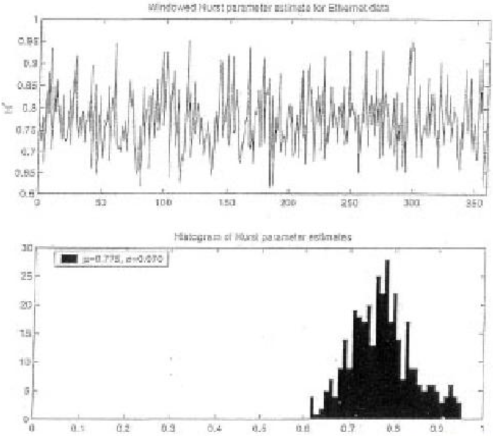
Epilepsy

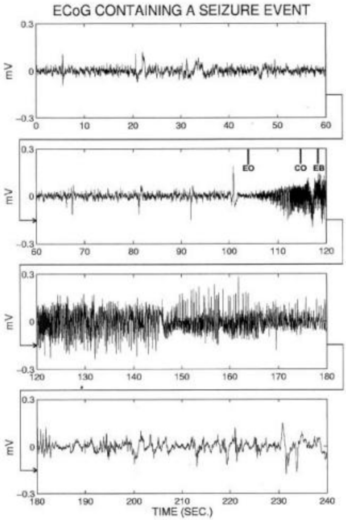
One percent of populations in industrialized countries
Ten percent of populations in developing countries

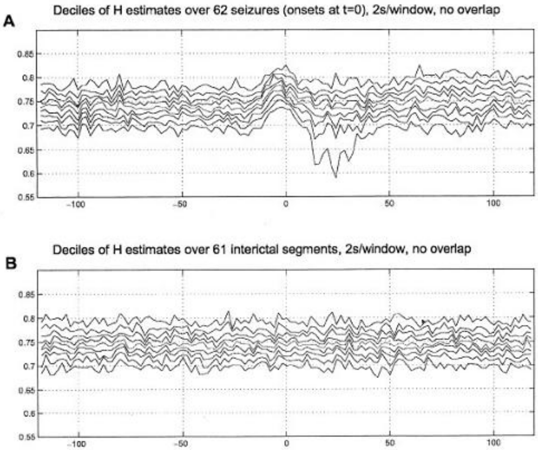
1/f Noise

Let X be a stationary stochastic process with spectral density
$S(f) = \frac{c}{f^a}$
where $a > 0$.
The Fourier transform of $\frac{1}{t^b}$ is $\frac{c}{f^{b+1}}$

Electronic devices
Flicker noise
Economic data


Windowed Hurst parameter estimate for Ethernet data
Histogram of Hurst parameter estimates


ECoG CONTAINING A SEIZURE EVENT
mV
0.3
0
−0.3
0 10 20 30 40 50 60
60 70 80 90 100 110 120
120 130 140 150 160 170 180
180 190 200 210 220 230 240
EO
CO
EB
TIME (SEC.)


A
Deciles of H estimates over 62 seizures (onsets at t=0), 2s/window, no overlap
0.85
0.8
0.75
0.7
0.65
0.6
0.55
-100
-50
0
50
100
B
Deciles of H estimates over 61 interictal segments, 2s/window, no overlap
0.85
0.8
0.75
0.7
0.65
0.6
0.55
-100
-50
0
50
100

Let $(V, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space and let $\alpha \in (0,1)$. If $\varphi \in L^1([0,T], V)$ then the left-sided and the right-sided fractional (Riemann-Liouville) integrals of φ are defined (for almost all $t \in [0,T]$) by

$$\left(I_{0+}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)\, ds$$

and

$$\left(I_{T-}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \varphi(s)\, ds$$

respectively, where $\Gamma(\cdot)$ is the gamma function.

The inverse operators of these fractional integrals are called fractional derivatives and can be given by their respective Weyl representations

$$\left(D^{\alpha}_{0+}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{t^{\alpha}} + \alpha\int_0^t \frac{\psi(t)-\psi(s)}{(t-s)^{\alpha+1}}\,ds\right)$$

and

$$\left(D^{\alpha}_{T-}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{\psi(t)}{(T-t)^{\alpha}} + \alpha\int_t^T \frac{\psi(s)-\psi(t)}{(s-t)^{\alpha+1}}\,ds\right)$$

where $\psi \in I^{\alpha}_{0+}\left(L^1([0,T],V)\right)$ and $\psi \in I^{\alpha}_{T-}\left(L^1([0,T],V)\right)$ respectively.

FBM covariance in terms of fractional calculus

Let $s, t \in [0, T]$ and $u_a(r) = r^a$ for $a \in \mathbb{R}$.

$$\mathbb{E}[B^H(s)B^H(t)] = \rho(H) \int_0^T u^2_{\frac{1}{2}-H}(r)$$
$$(I^{H-\frac{1}{2}}_{T-} u_{H-\frac{1}{2}} 1_{[0,s]})(r)(I^{H-\frac{1}{2}}_{T-} u_{H-\frac{1}{2}} 1_{[0,t]})(r)dr$$

where $\rho(H) = \frac{\Gamma^2(H-\frac{1}{2})H(2H-1)}{\beta(H-\frac{1}{2},2-2H)}$ and $\beta(\cdot,\cdot)$ is

the beta function

$$\beta(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

so $\rho(H) = \frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}$.

<u>Remark</u> $\mathbb{E}[B^H(s)B^H(t)]$ is an analytic function of H.

Relation between Fractional Brownian Motion and Brownian Motion

$$W(t) = \int_0^t c_H^{-1} s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}} (u_{\frac{1}{2}-H} 1_{[0,t]})(s) dB(s)$$

is a standard Wiener process (Brownian motion) and inversely

$$B(t) = \int_0^t c_H s^{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} (u_{\frac{1}{2}-H} 1_{[0,t]})(s) dW(s)$$

is a standard fractional Brownian motion.
Formally the two expressions can be written as

$$\frac{dW}{dt} = \tilde{K}_H^{-1} \frac{dB}{dt}$$

and

$$\frac{dB}{dt} = \tilde{K}_H \frac{dW}{dt}$$

If $H \in (1/2, 1)$ then it is easily verified that $\mathcal{H} \supset \tilde{\mathcal{H}}$ where $\tilde{\mathcal{H}}$ is the Banach space of Borel measurable functions with the norm $|\cdot|_{\tilde{\mathcal{H}}}$ given by

$$|\varphi|^2_{\tilde{\mathcal{H}}} := \int_0^T \int_0^T \|\varphi(u)\| \, \|\varphi(v)\| \, \phi_H(u-v)\, du\, dv$$

where $\phi_H(u) = H(2H-1)|u|^{2H-2}$ and it is elementary to verify that $\tilde{\mathcal{H}} \supset L^p([0,T], V)$ for $p > 1/H$ and, in particular, for $p = 2$.

If $\varphi \in \tilde{\mathcal{H}}$ and $H > 1/2$, then

$$\mathbb{E}\left\|\int_0^T \varphi \, d\beta\right\|^2 = \int_0^T \int_0^T \langle \varphi(u), \varphi(v) \rangle \, \phi_H(u-v) \, du \, dv \ .$$

Deterministic linear functionals for FBM

If $H \in (\frac{1}{2}, 1)$, the paths of an FBM are 'smoother' than the paths of a BM, so the Hilbert space of deterministic linear functionals for this FBM contains 'rougher' functionals than any for a BM.

If $H \in (0, \frac{1}{2})$, the paths of an FBM are 'rougher' than the paths of a BM, so the Hilbert space of deterministic linear functionals for this FBM has 'smoother' functionals than any nonsmooth functional for a BM.

A Hilbert space of deterministic integrands

Let $H \in (0,1)$ be fixed and $u_a(s) = s^a$ for $a \in \mathbb{R}$.

$\tilde{L}^2_H([0,T])$ is the linear space of functions (for $H \in (0, \frac{1}{2})$) and the linear space of distributions (for $H \in (\frac{1}{2}, 1)$) such that $F \in \tilde{L}^2_H$ if

$$\begin{aligned}\langle F, F\rangle_{\tilde{L}} &= |F|^2_{\tilde{L}^2_H} \\ &= \rho(H)\int_0^T u^2_{\frac{1}{2}-H}(s)((I^{H-\frac{1}{2}}_{T-}u_{H-\frac{1}{2}}F)(s))^2 ds \\ &< \infty,\end{aligned}$$

where $\rho(H) = \frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}$.

Wiener integrals

If $f, g \in \tilde{L}^2_H$, then $\int f dB^H$ and $\int g dB^H$ are zero mean Gaussian random variables with

$$\mathbb{E}[\int f dB^H \int g dB^H] = \langle f, g \rangle_{\tilde{H}}$$

Let $H \in (\frac{1}{2}, 1)$.

$L^2_H([0, T])$ is the linear space of measurable generalized processes such that $X \in L^2_H$ if

$$\begin{aligned} \langle X, X \rangle_H &= |X|^2_{L^2_H} \\ &= \mathbb{E}[\int_0^T u^2_{\frac{1}{2}-H}(s)((I^{H-\frac{1}{2}}_{T-} u_{H-\frac{1}{2}} X)(s))^2 ds] \\ &< \infty \end{aligned}$$

The linear space $\mathcal{S}$ is the family of smooth, cylindrical, V-valued random variables on $(\Omega, \mathcal{F}, P)$ such that if $F \in \mathcal{S}$, then it has the form

$$F = \sum_{j=1}^{n} f_j \left(\int_0^1 \gamma_{1j} dB^H, \ldots, \int_0^1 \gamma_{n_j j} dB^H \right) \eta_j \qquad (1)$$

where $\eta_j \in V$, $\gamma_{kj} \in L^2_{\phi_H}([0,1], \mathcal{L}_2(U, \mathbb{R}))$, $f_j \in C_p^\infty(\mathbb{R}^{n_j})$ for $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, n_j\}$ and

$$C_p^\infty(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \in C^\infty \text{ and } f \text{ and all of its derivatives have polynomial growth}\}.$$

Path derivative to construct a stochastic integral

The derivative $D : S \to L^2_H$ is the linear operator acting on smooth cylindrical random variables F as

$$D_t F = \Sigma_{j=1}^{n} \frac{\partial F}{\partial x_j}\Big(\int \gamma_1 dB^H, \cdots, \int \gamma_n dB^H\Big)\gamma_j(t)$$

Remark The derivative operator D can be extended in $L^2(\Omega)$ to a closed linear operator

$$D : D_H^{1,2} \to L^2_H$$

where $D_H^{1,2} = Dom(D)$

Let $X \in L^2_H$. The real-valued generalized process X is integrable with respect to B^H if

$$F \to \langle X, DF \rangle_{L^2_H}$$

is continuous on S with the $L^2(\Omega)$ norm topology.

$\int XdB^H$ is a zero mean random variable such that

$$\langle X, DF \rangle_{L^2_H} = \mathbb{E}[\int XdB^H F]$$

for each $F \in S$

The stochastic integral is a dual of D.

A Hilbert space of integrands

Let $H \in (\frac{1}{2}, 1)$. The linear space $L_H^{1,2}([0, T])$ is the family of generalized processes
$(X(t), t \in [0, T])$ such that

1. $X \in L_H^2$;
2. DX exists and is jointly measurable;
3. $$\begin{aligned} |X|_{L_H^{1,2}} &= \langle X, X \rangle_H \\ &= \langle X, X \rangle_{L_H^2} + \mathbb{E} \int_0^T \int_0^T u^2_{\frac{1}{2}-H}(s)\ u^2_{\frac{1}{2}-H}(t) \\ &(I_{T-}^{H-\frac{1}{2}} u_{H-\frac{1}{2}}(q)\ (I_{T-}^{H-\frac{1}{2}} u_{H-\frac{1}{2}}(r) \\ &D_r X(q))(s))(t))^2\ dsdt < \infty. \end{aligned}$$

Stochastic integrals

If $X \in L_H^{1,2}([0,T])$, then X is integrable with respect to B^H and $\int_0^T X dB^H$ is a zero mean random variable with finite second moment.

If $X, Y \in L_H^{1,2}$, then

$$\mathbb{E}[\int_0^T X dB^H \int_0^T Y dB^H] = \langle X, Y \rangle$$

If $G \in D_H^{1,2}$ and $\gamma \in \bar{L}_H^2$, then

$$\int_0^1 G \, \gamma \, dB^H = G \int_0^1 \gamma \, dB^H - \int_0^1 \int_0^1 D_s G \, \gamma(t) \, \phi_H(s-t) ds dt.$$

Remark This equality can be used to define a Stratonovich integral (the first integral on the RHS) and give its relation to the corresponding Itô integral (the integral on the LHS).

An elementary Itô formula for $H \in (\frac{1}{2}, 1)$

If $f \in C^2$, then

$$f(B^H(T)) - f(B^H(0)) = \int_0^T f'(B^H(s))dB^H(s) + H\int_0^T s^{2H-1} f''(B^H(s))ds$$

- Formally letting $H = \frac{1}{2}$ recovers the usual Itô formula.
- Recall that an FBM with $H \in (\frac{1}{2}, 1)$ has zero quadratic variation.

Stochastic Differential Equations

$$dX(t) = a(t, X(t))dt + b(t, X(t))dB^H(t)$$

$$X(0) = X_0$$

Find a solution.

$$dX(t) = aX(t)dt + bX(t)dB^H(t), \quad X(0) = X_0 > 0$$

for $a \in \mathbb{R}, \quad b \in \mathbb{R} \setminus \{0\}, \quad H \in (\frac{1}{2}, 1)$

$$X(t) = X_0 \exp(at + bB^H(t) - \frac{1}{2}b^2 t^{2H})$$

$\lim_{t\to\infty} X(t) = 0, \quad a.s.$, so the FBM always stabilizes the deterministic equation. Letting $H = \frac{1}{2}$ gives the solution for BM.

$$dX(t) = X(t)(Adt + CdB^H(t)), \quad X(0) = X_0$$

Assume that $[A, C] = 0, \quad A, C \in L(\mathbb{R}^n)$

$$X(t) = X_0 \exp(At + C\ B^H(t) - \frac{1}{2} C^2\ t^{2H})$$

Bilinear equations with some noncommuting operators

$$dX(t) = X(t)(A\ dt + C\ dB^H(t)), \quad X(0) = I$$

$A, C, X(t) \in L(\mathbb{R}^n)$, $(B^H(t), t \geq 0)$ is a real-valued standard fractional Brownian motion with $H \in (\frac{1}{2}, 1)$

Assume $[A, [A, C]] = [C, [A, C]] = 0$. Find an explicit solution.

A special case of the Baker-Campbell-Hausdorff formula

Let K and L be linear operators in R^n
If $[K,[K,L]] = [L,[K,L]] = 0$ then
$e^K e^L = e^{K+L+\frac{1}{2}[K,L]} = e^{K+L} e^{\frac{1}{2}[K,L]}$
or equivalently
$e^K e^L e^{\frac{-1}{2}[K,L]} = e^{K+L}$
where $[K,L] = KL - LK$.

For the bilinear equation described above, a solution is the process $(\hat{X}(t), t \geq 0)$ where

$$\hat{X}(t) = \prod_{j=1}^{6} e^{Z_j(t)}$$

$$Z_1(t) = tA \qquad Z_2(t) = C\ B^H(t)$$

$$Z_3(t) = -[A, C] \int_0^t B^H(s)ds \qquad Z_4(t) = \frac{1}{2}\ C\ [A, C]\ t^{2H+1}$$

$$Z_5(t) = -\frac{1}{2(2H+2)}\ [A, C]^2\ t^{2H+2} \qquad Z_6(t) = -\frac{1}{2}\ C^2\ t^{2H}$$

Two examples of bilinear equations

$$dX(t) = X(t)(A\ dt + C\ dB^H(t)), \quad X(0) = I$$

Let

$$X(t), A, C \in L(\mathbb{R}^3)$$

$$A_0 = C_0 = I$$

$$A_1 = E_{12} + E_{13}$$

$$C_1 = E_{23}$$

where E_{ij} is the elementary matrix with 1 in the (i, j) position and zeros elsewhere.

Case 1: Let $A = A_0 + A_1$, $C = C_0 + C_1$, then

$$\lim_{t\to\infty} X(t) = 0.$$

Case 2: Let $A = A_1$, $C = C_1$, then

$$\limsup_{t\to\infty} \hat{X}_{13}(t) = +\infty \quad a.s.$$

$$\liminf_{t\to\infty} \hat{X}_{13}(t) = -\infty \quad a.s.$$

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A cylindrical process $\langle B, \cdot \rangle \colon \Omega \times \mathbb{R}_+ \times V \to \mathbb{R}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a standard cylindrical fractional Brownian motion with the Hurst parameter $H \in (0, 1)$ if

1. For each $x \in V \setminus \{0\}$, $\frac{1}{\|x\|}\langle B(\cdot), x \rangle$ is a standard scalar fractional Brownian motion with the Hurst parameter H.
2. For $\alpha, \beta \in \mathbb{R}$ and $x, y \in V$

$$\langle B(t), \alpha x + \beta y \rangle = \alpha \langle \beta(t), x \rangle + \beta \langle B(t), y \rangle \quad \text{a.s. } \mathbb{P}.$$

Definition

Let $G\colon [0,T] \to \mathcal{L}(V)$ be Borel measurable, $(e_n, n \in \mathbb{N})$ be a complete orthonormal basis in V, $G(\cdot)e_n \in \mathcal{H}$ for $n \in \mathbb{N}$, and B be a standard cylindrical fractional Brownian motion. The stochastic integral $\int_0^T G\,dB$ is defined as

$$\int_0^T G\,dB := \sum_{n=1}^{\infty} \int_0^T Ge_n\,d\beta_n$$

provided the infinite series converges in $L^2(\Omega)$.

Linear Stochastic Equations in a Hilbert Space

Some results are reviewed for a linear stochastic differential equation with a cylindrical fractional Brownian motion whose solution is often called a fractional Ornstein-Uhlenbeck process. This process is a mild solution of the linear stochastic equation

$$dZ(t) = AZ(t)\,dt + \Phi\,dB(t)$$
$$Z(0) = x$$

where $Z(t), x \in V$, $(B(t), t \geq 0)$ is a standard cylindrical fractional Brownian with $H \in (0,1)$, $\Phi \in \mathcal{L}(V)$, A: $\mathrm{Dom}(A) \to V$, $Dom(A) \subset V$, and A is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$ on V.

A mild solution is

$$\begin{aligned} Z(t) &= S(t)x + \int_0^t S(t-r)\Phi\, dB(r) \\ &= S(t)x + \hat{Z}(t)\ . \end{aligned}$$

Theorem

If $H \in (1/2, 1)$*,* $S(t)\Phi \in \mathcal{L}_2(V)$ *for each* $t > 0$ *and*

$$\int_0^{T_0}\int_0^{T_0} u^{-\alpha} v^{-\alpha} |S(u)\Phi|_{\mathcal{L}_2(V)} |S(v)\Phi|_{\mathcal{L}_2(V)} \phi_h(u-v)\, du\, dv < \infty$$

for some $T_0 > 0$ *and* $\alpha > 0$ *then there is a Hölder continuous* V*-valued version of the process* $(\hat{Z}(t), t \geq 0)$ *with Hölder exponent* $\beta < \alpha$ *where* $\hat{Z}$ *is the stochastic convolution. If* $(S(t), t \geq 0)$ *is an analytic semigroup then there is a version of the process* $(\hat{Z}(t), t \in [0, T])$ *with* $C^{\beta}([0, T], V_{\delta})$ *sample paths for each* $T > 0$ *and* $\beta + \delta < \alpha$*.*

Theorem

Let $(S(t), t \geq 0)$ *be an analytic semigroup,* $H \in (0, 1/2)$ *and*

$$|S(t)\Phi|_{\mathcal{L}_2(V)} \leq ct^{-\gamma}$$

for $t \in [0, T]$*, some* $c > 0$*, and* $\gamma \in [0, H]$*. Let* $\alpha \geq 0$ *and* $\delta \geq 0$ *satisfy*

$$\alpha + \beta + \gamma < H \ ,$$

then there is a version of the process $(\hat{Z}(t), t \in [0, T])$ *with* $C^{\alpha}([0, T], V_{\delta})$ *sample paths. If it is assumed instead that* $\Phi \in \mathcal{L}_2(V)$ *and* $\alpha + \delta < H$ *then the process* $\left(\hat{Z}(t), t \in [0, T]\right)$ *has a* $C^{\alpha}([0, T], V_{\delta})$ *version for all* $\alpha \geq 0$ *and* $\delta \geq 0$*. In particular, there is a* $C^{\alpha}([0, T], V)$ *version for* $0 < \alpha < H$*.*

Let $K_H(t,s)$ for $0 \le s \le t \le T$ be the real-valued kernel function

$$K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2} - H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right) du$$

where

$$c_H = \left[\frac{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(2-2H\right)}\right]^{\frac{1}{2}}$$

and $H \in (0,1)$. If $H \in (1/2,1)$, then K_H has a simpler form as

$$K_H(t,s) = c_H\left(H - \frac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\, du \, .$$

Define the integral operator $\mathbb{K}_H$ induced from the kernel K_H by

$$\mathbb{K}_H h(t) = \int_0^t K_H(t,s)h(s)\,ds$$

for $h \in L^2([0,T],V)$. It is well-known that

$$\mathbb{K}_H \colon L^2([0,T],V) \to I_{0+}^{H+\frac{1}{2}}\left(L^2([0,T],V)\right)$$

is a bijection.

$\mathbb{K}_H$ can be described as

$$\mathbb{K}_H h(s) = \overline{c}_H I_{0+}^{2H} \left(u_{\frac{1}{2}-H} I_{0+}^{\frac{1}{2}-H} \left(u_{H-\frac{1}{2}} h \right) \right) (s) \tag{2}$$

for $H \in (0, 1/2]$ and

$$\mathbb{K}_H h(s) = c_H I_{0+}^{1} \left(u_{H-\frac{1}{2}} I_{0+}^{H-\frac{1}{2}} \left(u_{\frac{1}{2}-H} h \right) \right) (s) \tag{3}$$

for $H \in [1/2, 1)$ where $u_a(s) = s^a$ for $s \geq 0$ and $a \in \mathbb{R}$.

$$c_H = \left[\frac{2H\Gamma\left(H + \frac{1}{2}\right)\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(2 - 2H\right)} \right]^{\frac{1}{2}} , \tag{4}$$

$$\overline{c}_H = c_H \Gamma(2H) ,$$

The inverse operator, $\mathbb{K}_H^{-1}$ for the two cases is given by

$$\mathbb{K}_H^{-1}\varphi(s) = \overline{c}_H^{-1} s^{\frac{1}{2}-H} D_{0+}^{\frac{1}{2}-H}\left(u_{H-\frac{1}{2}} D_{0+}^{2H}\varphi\right)(s)$$

for $H \in (0, 1/2]$ and

$$\mathbb{K}_H^{-1}\varphi(s) = c_H^{-1} s^{H-\frac{1}{2}} D_{0+}^{H-\frac{1}{2}}\left(u_{\frac{1}{2}-H} D_{0+}^{2H}\varphi\right)(s)$$

for $H \in [1/2, 1)$ and $\varphi \in I_{0+}^{H+\frac{1}{2}}\left(L^2([0, T], V)\right)$.
Note that if $\varphi \in H^1([0, T], V)$, the Sobolev space, then

$$\mathbb{K}_H^{-1}\varphi(s) = \bar{c}_H^{-1} s^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H}\left(u_{\frac{1}{2}-H}\varphi'\right)(s) \tag{5}$$

Theorem

Let $H \in (0,1)$ *and* $T > 0$ *be fixed and let* $(u(t), t \in [0,T])$ *be a* V*-valued,* $(\mathcal{F}_t)$*-adapted process such that*

1. $$\int_0^T \|u(t)\| \, dt < \infty \quad a.s.\ \mathbb{P}$$

   and

2. $$U(t) := \int_0^t u(s)\, ds \in I_{0+}^{H+\frac{1}{2}} \left(L^2 \left([0,T], V \right) \right) \quad a.s.\ \mathbb{P}\ .$$

Furthermore, it is assumed that

$$\mathbb{E}\xi(T) = 1$$

where

$$\xi(T) = \exp\left[\int_0^T \left\langle \mathbb{K}_H^{-1}(U)(t), dW(t) \right\rangle - \frac{1}{2}\int_0^T \left\| \mathbb{K}_H^{-1}(U)(t) \right\|^2 dt\right]$$

where $(W(t), t \in [0, T])$ is a standard cylindrical Wiener process in V.

Then the process $\left(\tilde{B}(t), t \in [0, T]\right)$ given by

$$\tilde{B}(t) := B(t) - U(t)$$

is a standard cylindrical fractional Brownian motion in V with the Hurst parameter H on the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ where

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \xi(T) \quad \text{a.s.}$$

$$\tilde{W}(t) := W(t) - \int_0^t \mathbb{K}_H^{-1}(U)(s)\, ds$$

is a standard cylindrical Wiener process in V. Let

$$\tilde{\beta}_n(t) := \langle B(t), e_n \rangle - \langle U(t), e_n \rangle$$

and

$$\tilde{w}_n(t) = \langle W(t), e_n \rangle - \left\langle \int_0^t \mathbb{K}_H^{-1}(U)(s)\, ds, e_n \right\rangle .$$

It follows that

$$\int_0^t K_H(t,s)\, d\tilde{w}_n(s) = \int_0^t K_H(t,s)\, dw_n(s) - \int_0^t K_H(t,s) \langle \mathbb{K}_H^{-1}(U)(s), e_n \tag{6}$$

$$= \beta_n(t) - \left\langle \int_0^t K_H(t,s) \left(\mathbb{K}_H^{-1}(U)(s) \right) ds, e_n \right\rangle$$

$$= \beta_n(t) - \langle \mathbb{K}_H \mathbb{K}_H^{-1}(U)(t), e_n \rangle$$

$$= \beta_n(t) - \langle U(t), e_n \rangle = \tilde{\beta}_n(t) .$$

The following semilinear stochastic equation is considered:

$$dX(t) = (AX(t) + F(X(t)))\, dt + \Phi\, dB(t)$$

where $t \in \mathbb{R}_+$, $X(t), X_0 \in V$, $(B(t), t \geq 0)$ is a standard cylindrical fractional Brownian motion with the Hurst parameter $H \in (0,1)$, $\Phi \in \mathcal{L}(V)$, $A\colon Dom(A) \to V$, $Dom(A) \subset V$, and A is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$ on V. The function $F\colon V \to V$ is nonlinear but for the applications to stochastic partial differential equations it is more useful to assume that F is only defined on a (dense) subspace of V. So, let $(E, \|\cdot\|_E)$ be a separable Banach space that is continuously embedded in V and $F\colon E \to E$ with $X_0 \in E$.

It is assumed that $F \colon E \to E$ is Borel measurable, $im(F) \subset im(\Phi)$, for $G := \Phi^{-1} F$, $G \colon E \to V$, and

$$\|G(x)\| \leq \hat{k} \left(1 + \|x\|_E^p\right)$$

and

$$\|F(x)\|_E \leq \hat{k} \left(1 + \|x\|_E^p\right)$$

for each $x \in E$ and some $p \geq 1$.

Furthermore, it is assumed that there is a constant $\bar{K}$ such that for each pair (x, y) in $Dom(A)$, there is a $z^* \in \partial\|z\|_E$ such that

$$\langle Ax - Ay + F(x) - F(y), z^* \rangle_{E,E^*} \leq \bar{K}\|x - y\|_E$$

where $\partial\|z\|_E$ is the subdifferential of the norm $\|z\|_E$ at the point $z = x - y$ and $\langle \cdot, \cdot, \rangle_{E,E^*}$ is the pairing between E and E^*. This inequality is a one-sided growth condition that ensures the absence of explosions.

Definition

A weak solution is a triple $(X(t), B(t), (\Omega, \mathcal{F}, \mathbb{P}), t \geq 0)$ where $(B(t), t \geq 0)$ is a standard cylindrical fractional Brownian motion in V that is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(X(t), t \geq 0)$ is an E-valued process satisfying

$$X(t) = S(t)X_0 + \int_0^t S(t-r)F(X(r))\,dr + \int_0^t S(t-r)\Phi\,dB(r)\,.$$

A mild solution, $(X(t), t \geq 0)$ of the equation is an E-valued process on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a given standard cylindrical fractional Brownian motion that is the fractional Brownian motion, and the process $(X(t), t \geq 0)$ satisfies the equation.

Definition

The equation has a unique weak solution if for any two weak solutions, $(X(t), B(t), (\Omega, \mathcal{F}, \mathbb{P}), t \geq 0)$, and $(\tilde{X}(t), \tilde{B}(t), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), t \geq 0)$, the processes $(X(t), t \geq 0)$ and $(\tilde{X}(t), t \geq 0)$ have the same probability law.

The equation has a unique mild solution if for any two processes $(X_1(t), t \geq 0)$ and $(X_2(t), t \geq 0)$ that satisfy the equation on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the same standard cylindrical fractional Brownian motion, $\mathbb{P}(X_1(t) = X_2(t), t \geq 0) = 1$.

The following three assumptions are made to construct a solution:

(H1). The semigroup $(S(t), t \geq 0)$ generated by A is analytic on V and for each $t \geq 0$, $S(t)|_E \in \mathcal{L}(E)$ and $\|S(t)\|_{\mathcal{L}(E)}$ is bounded on compact time intervals.

(H2). $\Phi \in \mathcal{L}(V)$ is injective and for $T > 0$, the stochastic convolution process

$$\left(\int_0^t S(t-r)\Phi \, dB(r), t \in [0, T] \right)$$

has $C([0, T], E)$ sample paths.

(H3). The function $F\colon E \to V$ in (59) is Borel measurable, $im(F) \subset im(\Phi)$ and the function $G = \Phi^{-1}F\colon E \to V$ satisfies

$$\|G(x)\| \leq k\,(1 + \|x\|_E)$$

for some $k > 0$ and all $x \in E$.

Theorem

If $H \in (0, 1/2)$ and conditions (H1)-(H3) are satisfied, then the equation has a weak solution. If additionally $F : E \to E$ and

$$\|F(x)\|_E \leq k_1 \left(1 + \|x\|_E\right)$$

for some $k_1 > 0$ and all $x \in E$, then the weak solution is unique.

Theorem

If $H \in (1/2, 1)$, *(H1)–(H3) are satisfied and*

$$\|G(x) - G(y)\| \leq k_G \|x - y\|^{\gamma} \tag{7}$$

for all $x, y \in E$, *some* $\gamma \in (0, 1]$, $k_G > 0$ *and* $\tilde{Z} \in C^{\beta}([0, T], V)$ *for some* β *satisfying*

$$\beta > \frac{H - \frac{1}{2}}{\gamma} \tag{8}$$

where $\tilde{Z}$ *is the stochastic convolution process in (H2), then the semilinear equation has a weak solution. If, additionally, the inequality in the above theorem is satisfied, then the weak solution is unique.*

Theorem

Let $H \in (0, 1/2)$ and (H1) and (H2) be satisfied. Let $\Phi \in \mathcal{L}(V)$ be injective, $\Phi^{-1} \in \mathcal{L}(E, V)$ and $(S(t)|_E, t \geq 0)$ be a strongly continuous semigroup on E such that

$$|S(t)|_E|_{\mathcal{L}(E)} \leq e^{\tilde{w}t} \tag{9}$$

for $t \geq 0$ and some $\tilde{w} \in \mathbb{R}$. Let $F \colon E \to E$ be continuous and satisfy

$$\|F(x)\|_E \leq k_1 \left(1 + \|x\|_E^\rho\right) \tag{10}$$

for $x \in E$ for some $k_1 \geq 0$ and $\rho \geq 1$ and for each pair $x, y \in E$, there is a $z^* \in \partial\|x - y\|_E$ where $\partial\|z\|_E$ is the subdifferential of the norm $\|\cdot\|_E$ at $z \in E$ such that

$$\langle F(x) - F(y), z^* \rangle_{E,E^*} \leq k_2\|x - y\|_E \tag{11}$$

for some $k_2 \in \mathbb{R}$, that is, $F - k_2 I$ is dissipative on E.

Then there is one and only one mild solution of the semilinear equation and its probability law on the Borel σ-algebra of $\check{\Omega} = C([0, T], E)$ is mutually absolutely continuous with respect to the probability law of the fractional Ornstein-Uhlenbeck process on Ω.

Theorem

Let $H \in (1/2, 1)$ and the other assumptions in the above theorem be satisfied. Let $\Phi^{-1} \in \mathcal{L}(V)$, $\tilde{Z} \in C^{\beta}([0, T], V)$ for some $\beta \in (0, 1)$,

$$\langle F(x) - F(y), x - y \rangle \leq k_2 \|x - y\|^2 \tag{12}$$

for each pair $x, y \in E$ and a $k_2 \in \mathbb{R}_+$ (that is, $F - k_2 I$ is dissipative on E with respect to the norm on V) and

$$\|F(x) - F(y)\| \leq k_3 \left(1 + \|x\|_E^q + \|y\|_E^q\right) \|x - y\|^{\gamma} \tag{13}$$

for each $x, y \in E$, with some $k_3 > 0$, $q \geq 1$, and $\gamma \in (0, 1]$ such that

$$\gamma\beta > H - \frac{1}{2} \,. \tag{14}$$

Then there is one and only one mild solution to the semilinear equation and its probability law is mutually absolutely continuous with respect to the probability law of the fractional Ornstein-Uhlenbeck process on $\check{\Omega}$.

Consider the equation

$$dX(t) = f(X(t))\,dt + \Phi\,dB(t)$$

where $f\colon \mathbb{R}^n \to \mathbb{R}^n$, $\Phi \in \mathcal{L}(\mathbb{R}^n)$ and $(B(t), t \geq 0)$ is an $\mathbb{R}^n$-valued standard fractional Brownian motion with Hurst parameter $H \in (0,1)$. Let $E = V = \mathbb{R}^n$, $S(t) = I$ for $t \in \mathbb{R}_+$ and assume that $Q = \Phi\Phi^*$ is positive definite. The process

$$\left(\int_0^t \Phi\,dB, t \in [0, T]\right)$$

has sample paths in $C^\beta([0, T], \mathbb{R}^n)$ for $0 < \beta < H$.

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is Borel measurable and

$$\|f(x)\| \leq k_1(1 + \|x\|)$$

for some $k_1 > 0$ and all $x \in \mathbb{R}^n$ then for $H < \frac{1}{2}$ there is one and only one weak solution.

If, additionally, it is assumed that

$$\|f(x) - f(y)\| \leq k\|x - y\|^{\gamma}$$

for all $x, y \in \mathbb{R}^n$ and some $\gamma > 1 - \frac{1}{2H}$, then for $H > \frac{1}{2}$, there is one and only one weak solution. In each of these cases, the probability measure of the solution is mutually absolutely continuous with respect to the probability measure of the process $(\Phi B(t), t \in [0, T])$.

Now, replace the inequality of linear growth by

$$\|f(x)\| \leq k_1(1 + \|x\|^p)$$

for some $p \geq 1$ and $k_1 > 0$. Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies

$$\langle f(x) - f(y), x - y \rangle \leq k_3 \|x - y\|^2$$

for some $k_3 > 0$ and all $x, y \in \mathbb{R}^n$.

If $H > \frac{1}{2}$, then assume that

$$\|f(x) - f(y)\| \leq k_4 (1 + \|x\|^q + \|y\|^q) \|x - y\|^\gamma$$

for some $q \geq 1$, $k_4 > 0$, $\gamma > 1 - \frac{1}{2H}$. Then, the probability law of the solution is mutually absolutely continuous with respect to the probability law of $(\Phi B(t), t \in [0, T])$. Furthermore, there is one and only one mild solution and this mild solution is a strong solution because the state space is finite dimensional.

Stochastic Partial Differential Equations

Consider a $2m$th order stochastic parabolic equation

$$\frac{\partial u}{\partial t}(t,\xi) = [L_{2m}u](t,\xi) + \eta(t,\xi)$$

for $(t,\xi) \in [0,T] \times \mathcal{O}$ with the initial condition

$$u(0,\xi) = x(\xi)$$

for $\xi \in \mathcal{O}$ and the Dirichlet boundary condition

$$\frac{\partial^k u}{\partial v^k}(t,\xi) = 0$$

for $(t,\xi) \in [0,T] \times \partial\mathcal{O}$, $k \in \{0,\ldots,m-1\}$.

$\frac{\partial}{\partial v}$ denotes the conormal derivative, $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ with a smooth boundary and L_{2m} is a $2m$th order uniformly elliptic operator

$$L_{2m} = \sum_{|\alpha| \leq 2m} a_\alpha(\xi) D^\alpha$$

and $a_\alpha \in C_b^\infty(\mathcal{O})$.

For example, if $m = 1$ then this equation is called the stochastic heat equation. The process η denotes a space dependent noise process that is fractional in time with the Hurst parameter $H \in (0,1)$ and, possibly, in space. The system is modeled as

$$\begin{aligned} dZ(t) &= AZ(t)\,dt + \Phi\,dB(t) \\ Z(0) &= x \end{aligned}$$

in the space $V = L^2(\mathcal{O})$ where $A = L_{2m}$,

$$Dom(A) = \left\{ \varphi \in H^{2m}(\mathcal{O}) \mid \frac{\partial^k}{\partial \nu^k}\varphi = 0 \text{ on } \partial D \text{ for } k \in \{0, \dots, m-1\} \right\},$$

$\Phi \in \mathcal{L}(V)$ defines the space correlation of the noise process and $(B(t), t \geq 0)$ is a cylindrical standard fractional Brownian motion in V.

For $\Phi = I$, the noise process is uncorrelated in space. It is well known that A generates an analytic semigroup $(S(t), t \geq 0)$. Furthermore

$$|S(t)\Phi|_{\mathcal{L}_2(V)} \leq |S(t)|_{\mathcal{L}_2(V)}|\Phi|_{\mathcal{L}(V)} \leq ct^{-\frac{d}{4m}}$$

for $t \in [0, T]$.

It is assumed that there is a $\delta_1 > 0$ and $\hat{\beta} \in \mathbb{R}$ such that

$$im(\Phi) \subset Dom\left((\hat{\beta}I - A)^{\delta_1}\right)$$

so that for $r \geq 0$

$$\begin{aligned} &|S(t)\Phi|_{\mathcal{L}_2(V)} \\ &\leq |S(t)(\hat{\beta}I - A)^r|_{\mathcal{L}(V)}|(\hat{\beta}I - A)^{-r-\delta_1}|_{\mathcal{L}_2(V)}|(\hat{\beta}I - A)^{\delta_1}\Phi|_{\mathcal{L}(V)} \leq ct^{-r} \end{aligned} \tag{15}$$

for $t \in (0, T]$ assuming that the operator $(\hat{\beta}I - A)^{-r-\delta_1}$ is a Hilbert-Schmidt operator on V, which occurs if

$$r + \delta_1 > \frac{d}{4m}.$$

For the spaces $V_\delta = Dom\left(\left(\hat{\beta}I - A\right)^{\delta}\right)$, $\delta \geq 0$, if

$$H > \frac{d}{2m} \ ,$$

then for any $\Phi \in \mathcal{L}(V)$, the stochastic convolution process

$$\left(\int_0^t S(t-r)\Phi \, dB(r), t \in [0, T]\right)$$

is well-defined and has a version with $C^{\alpha}([0, T], V_\delta)$ sample paths for $\alpha \geq 0$, $\delta \geq 0$ satisfying

$$\alpha + \delta < H - \frac{d}{4m}.$$

Consider the equation

$$\frac{\partial y}{\partial t}(t,\xi) = \frac{\partial^2 y}{\partial \xi^2}(t,\xi) + f(y(t,\xi)) + \eta(t,\xi)$$

for $(t,\xi) \in (0,T) \times (0,1)$ and

$$y(0,\xi) = x_0(\xi)$$
$$\frac{\partial y}{\partial \xi}(t,0) = \frac{\partial y}{\partial \xi}(t,1) = 0$$

for $(t,\xi) \in (0,T) \times (0,1)$ where $f : \mathbb{R} \to \mathbb{R}$ and $(\eta(t,\xi), t \in [0,T])$ is a space dependent noise that is fractional in time.

The equation is rewritten in a standard infinite dimensional form letting $V = L^2([0,1])$, $A = \frac{\partial^2}{\partial \xi^2}$,

$$Dom(A) = \left\{ \phi \in H^2([0,1]) \mid \frac{\partial}{\partial \xi}\phi(0) = \frac{\partial}{\partial \xi}\phi(1) = 0 \right\}$$

and $F\colon V \to V$ where $F(x)(\xi) = f(x(\xi))$ for $x \in V$ and $\xi \in (0,1)$. It is assumed that $Q = \Phi\Phi'$ has a bounded inverse in $\mathcal{L}(V)$.

Some Other Applications

Mutual Information

Prediction of Some Processes Related to FBM

Model A: Fractional Brownian Motion

$$dY(t) = X(t)dt + dB(t)$$
$$Y(0) = 0$$

$t \in [0, t]$, $(B(t), t \in [0, T])$ is a real-valued standard fractional Brownian motion with the fixed Hurst parameter $H \in (0, 1)$ and $(X(t), t \in [0, T])$ is a real-valued process independent of $B(\cdot)$, $T > 0$ is fixed and both of these processes are defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

$$I(X, Y) = \int N \log N d(\mu_X \otimes \mu_Y)$$

where N is the Radon-Nikodym derivative,

$$N = \frac{d\mu_{XY}}{d(\mu_X \otimes \mu_Y)} = \frac{d\mu_{XY}}{d(\mu_X \otimes \mu_B)} \frac{d\mu_B}{d\mu_Y}.$$

If N does not exist then $I(X, Y) = +\infty$.

Theorem 1

Let $(X(t), t \in [0, T])$ satisfy C1 and C2. Then $\mu_{XY} \ll \mu_X \otimes \mu_B$ and the Radon-Nikodym derivative is

$$\frac{d\mu_{XY}}{d(\mu_X \otimes \mu_B)} = \tilde{M}(T)$$
$$= \exp\left(\int_0^T \mathbb{K}_H^{-1}(Z)(t)dW(t) - \frac{1}{2}\int_0^T |\mathbb{K}_H^{-1}(Z)(t)|^2 dt\right)$$

where $\mathbb{K}_H^{-1}$ is given above, $(W(t), t \in [0, T])$ is the standard Wiener process (Brownian motion) associated to $(B(t), t \in [0, T])$ and $(Z(t), t \in [0, T])$ is the process given by

$$Z(t) = \int_0^t X(s)ds.$$

Theorem 2

Let μ_B, μ_X and μ_{XY} be the measures for the processes $(B(t), t \in [0, T])$, $(X(t), t \in [0, T])$ and $((X(t), Y(t)), t \in [0, T])$ respectively. Then $\mu_Y \ll \mu_B$ and the associated Radon-Nikodym derivative is

$$M(T) = \mathbb{E}_X[\tilde{M}(T)] = \frac{d\mu_Y}{d\mu_B}$$
$$= \exp\left(\int_0^T \langle \widehat{\mathbb{K}_H^{-1}(Z)}(t,t), dW(t) \rangle - \frac{1}{2} \int_0^T |\widehat{\mathbb{K}_H^{-1}(Z)}(t,t)|^2 dt \right)$$

where $Z(t) = \int_0^t X(s)ds, \quad t \in [0, T]$ and

$$
\begin{aligned}
\widehat{\mathbb{K}_H^{-1}(Z)}(t,t) &= \mathbb{E}[\mathbb{K}_H^{-1}(Z)(t)|\mathcal{F}_t] \\
&= \frac{1}{\Gamma(\frac{3}{2}-H)}(t^{1-2H}\hat{X}(t,t) \\
&+ (H-\frac{1}{2})\int_0^t \frac{t^{\frac{1}{2}-H}\hat{X}(t,t) - s^{\frac{1}{2}-H}\hat{X}(s,t)}{(t-s)^{H+\frac{1}{2}}}ds)
\end{aligned}
$$

$$
\hat{X}(s,t) = \mathbb{E}[X(s)|\mathcal{F}_t]
$$

Theorem 3

Let X satisfy C1 and C2. The mutual information $I(X, Y)$ for $(X(t), t \in [0, T])$ $(Y(t), t \in [0, T])$ is given by

$$I(X,Y) = \frac{1}{2}\mathbb{E}[\int_0^T |\mathbb{K}_H^{-1}(Z)(t) - \widehat{\mathbb{K}_H^{-1}(Z)}(t,t)|^2 dt$$

$$= \frac{1}{2}\mathbb{E}[\int_0^T \frac{1}{\Gamma^2(\frac{3}{2}-H)}|t^{1-2H}(X(t) - \hat{X}(t,t))$$

$$+ (H - \frac{1}{2})\int_0^t \frac{t^{\frac{1}{2}-H}(X(t) - \hat{X}(t,t)) - s^{\frac{1}{2}-H}(X(s) - \hat{X}(s,t))}{(t-s)^{H+\frac{1}{2}}} ds|^2 dt]$$

$$Z(\cdot) = \int_0^{\cdot} X(s)ds$$

A Modification of Model A

Let $\alpha > 0$, let $T > 0$ be fixed and let $(Y_\alpha(t), t \in [0, T])$ be the process that satisfies

$$dY_\alpha(t) = \alpha X(t)dt + dB(t)$$
$$Y_\alpha(0) = 0.$$

Find the rate of change of $I(X, Y_\alpha)$ with respect to α.

Theorem 4

Let $(Y_\alpha(t), t \in [0, T])$ be the process above where $\alpha > 0$ is a parameter. It is assumed that X satisfies C1 and C2. Then the mutual information $I(X, Y_\alpha)$ satisfies the following equality

$$\frac{dI(X, Y_\alpha)}{d\alpha} = \alpha \mathbb{E}[\int_0^T |\mathbb{K}_H^{-1}(Z)(t) - \widehat{\mathbb{K}_H^{-1}(Z)}(t, T)|^2 dt]$$

$$= \alpha \mathbb{E}[\int_0^T \frac{1}{\Gamma^2(\frac{3}{2} - H)} | t^{1-2H}(X(t) - \hat{X}(t, T))$$

$$+ (H - \frac{1}{2}) \int_0^t \frac{t^{\frac{1}{2}-H}(X(t) - \hat{X}(t, T)) - s^{\frac{1}{2}-H}(X(s) - \hat{X}(s, T))}{(t-s)^{H+\frac{1}{2}}} ds|^2 dt]$$

$$\widehat{\mathbb{K}_H^{-1}(Z)}(t, T) = \mathbb{E}[\mathbb{K}_H^{-1}(Z)(t) | Y_\alpha(u), 0 \le u \le T]$$

and

$$\hat{X}(t, T) = \mathbb{E}[X(t) | Y_\alpha(u), 0 \le u \le T]$$

Prediction for Some Processes Related to FBM

Let $(X(t),\ t \geq 0)$ be the real-valued, Gaussian process that is the solution of the stochastic differential equation

$$\begin{aligned} dX(t) &= a(t)X(t)dt + dB(t) \\ X(0) &= x_0, \end{aligned} \tag{16}$$

where $x_0 \in \mathbb{R}$, $a : \mathbb{R}_+ \to \mathbb{R}$ is bounded and Borel measurable and $(B(t),\ t \geq 0)$ is a standard fractional Brownian motion with $H \in (0, 1)$, It is elementary to verify that $X(t)$ is given by

$$X(t) = e^{\int_0^t a} x_0 + \int_0^t e^{\int_s^t a} dB(s) \tag{17}$$

so it follows from the above that

$$\overline{\sigma(X(u),\ u \in [0, t])} = \overline{\sigma(B(u),\ u \in [0, t])}.$$

Lemma

If $0 < s < t$ *and* $c : [s, t] \to \mathbb{R}$ *is an element of* L_H^2*, then*

$$\mathbb{E}\left[\int_s^t c dB \mid B(r),\ r \in [0, s]\right] = \int_0^s u_{-(H-1/2)} \left(I_{s-}^{-(H-1/2)} \left(I_{t-}^{(H-1/2)} u_{H-1/2} c\right)\right) dB \quad (18)$$

Proposition:

Let $(X(t),\ t \geq 0)$ be the process given by (17) that is the solution of (16) and let $t > 0$ and $s \in (0,t)$ be fixed. Then the following equality is satisfied

$$
\begin{aligned}
&\mathbb{E}[X(t) \mid X(r),\ r \in [0,s]] \\
&\quad = \mathbb{E}[X(t) \mid \mathcal{F}_s] \\
&\quad = e^{\int_s^t a} X(s) + \int_0^s u_{-(H-1/2)} \left(I_{s-}^{-(H-1/2)} \left(I_{t-}^{H-1/2} u_{H-1/2} v 1_{[s,t)} \right) \right) dB \\
&\quad = e^{\int_s^t a} X(s) + \int_0^s u_{-(H-1/2)} \left(I_{s-}^{-(H-1/2)} \left(I_{t-}^{H-1/2} u_{H-1/2} v 1_{[s,t)} \right) \right) \\
&\qquad\qquad \times (dX - aXdr) \qquad\qquad (19)
\end{aligned}
$$

where $u_a(r) = r^a$ and $v(r) = e^{\int_r^t a}$.

Now consider the stochastic differential equation for a geometric fractional Brownian motion with $H \in (0,1)$

$$dX(t) = X(t)(a(t)dt + b(t)dB(t)) \tag{20}$$
$$X(0) = x_0$$

where $a : \mathbb{R}_+ \to \mathbb{R}$ and $b : \mathbb{R}_+ \to \mathbb{R}$ are bounded, Borel measurable functions and $x_0 > 0$. A solution of (20) is

$$X(t) = x_0 \exp\left[\int_0^t a ds + \int_0^t b dB - \frac{1}{2}|b 1_{[0,t]}|_H^2\right]. \tag{21}$$

This solution can also be expressed using the Wick exponential, $\exp^\diamond$, (e.g., [2]) as

$$X(t) = x_0 \exp\left[\int_0^t a\right] \exp^\diamond\left(\int_0^t b dB\right) \tag{22}$$

where

$$\exp^\diamond\left(\int_0^t b dB\right) := \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int_0^t b dB\right)^{\diamond n} \tag{23}$$

and $(\ )^{\diamond n}$ is the nth Wick product.

Wick Products and Expectation

$$\mathbb{E}[Y \diamond Z \mid G] = \mathbb{E}[Y \mid G] \diamond \mathbb{E}[Z \mid G] \tag{24}$$

where G is a sub-σ-algebra of $\mathcal{F}$.

Proposition:

Let $(X(t), t \geq 0)$ be the process given by (12) and let $t > 0$ and $s \in (0, t)$ be fixed. Then the following equality is satisfied

$$\mathbb{E}[X(t) \mid \mathcal{F}_s]$$

$$= \mathbb{E}\left[X(s) \exp\left[\int_s^t a + \int_s^t bdB - \frac{1}{2}\langle b1_{[s,t)}, b1_{[s,t)}\rangle_H\right] \middle| \mathcal{F}_s\right]$$

$$= X(s) \exp\left[\int_s^t a\right] \exp^{\diamond}\left(\mathbb{E}\left[\int_s^t bdB \mid \mathcal{F}_s\right]\right)$$

$$= X(s) \exp\left[\int_s^t a - \frac{1}{2}\langle b1_{[s,t)}, b1_{[s,t)}\rangle_H\right.$$

$$+ \int_0^s u_{-(H-1/2)}\left(I_{s-}^{-(H-1/2)}\left(I_{t-}^{H-1/2} u_{H-1/2} b1_{[s,t)}\right)\right) d \tag{25}$$