Risk Sensitive Portfolio Optimization - Dynamical Models

Łukasz Stettner, Institute of Mathematics Polish Acad. Sci.

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How to model asset prices?

discrete or continuous time discrete time:

$\frac{S_i(t+1)}{S_i(t)} = 1 + \zeta_i(z(t), \xi(t+1))$

$\frac{S_i(t+s)}{S_i(t)} = 1 + \zeta_i(t, s, z(.), \xi(.))$

$\xi(.)$ independent (increments) "noise" process

stationarity

$(1 + \zeta_i(t, s, z(.), \xi(.))(\omega)) = (1 + \zeta_i(s, z(.), \xi(.))(\theta_t \omega))$ is a multiplicative functional

$1 + \zeta_i(t + s, z(.), \xi(.))(\omega) =$
$(1 + \zeta_i(t, z(.), \xi(.))(\omega))(1 + \zeta_i(s, z(.), \xi(.))(\theta_t \omega))$.

$1 + \zeta_i(s, z(.), \xi) := e^{X_i(s)}$

$dX(t) = \alpha(z(t))dt + \sigma(z(t))dB(t) + \int_{R^d} \gamma(z(t), u)\tilde{N}(dt, du)$

$X(0) = 0$ and $\tilde{N}$ is a compensated Poisson measure

$z(t)$ Markov process (ergodic) of economic factors

key problem: Markovianity $(S(t), z(t))$ is a Markov process

problem: replace $(B(t))$ by a more general Gaussian (e.g. fractional Gaussian) processes

Wealth process and portfolio strategies

no transaction costs

$N(t) = (N_1(t), \ldots, N_d(t))^T$ number of assets in portfolio at time t.

$V(t)$ wealth process at time t

$\sum_{i=1}^{d} N_i(t)S_i(t) = V(t)$

$\pi_i(t) = \frac{N_i(t)S_i(t)}{V(t)}$

Wealth and portfolio dynamics:

$V(t+s) = V(t)\pi(t) \cdot (1 + \zeta(t, s, z(.), \xi))$

$\pi(t+s) = g(\pi(t) \diamond e^{X(t+s)-X(t)})$

where

$\pi(t) \diamond e^{X(t+s)-X(t)} = (\pi_i(t)(1 + \zeta_i(t, s, z(.), \xi)) =$
$(\pi_i(t)e^{X_i(t+s)-X_i(t)})$

$g_i(\pi) = \frac{\pi_i}{\sum_{j=1}^{m} \pi_j}$

Terminal utility maximization - discrete time

$E_{z,V}\left\{\sum_{t=0}^{T-1} f(V(t)) + h(V(T))\right\} \to \max$

strategy $\Pi = (\pi(t))$, e.g. $h(V) = V - \lambda(V-z)^-$

Bellman equations

$w_T(z,V) = h(V)$

$w_{T-1}(z,V) = f(V) + \sup_\pi E_{z,V}\left\{h(V\pi \cdot (1+\zeta(1,z,\xi))\right\}$

$\ldots$

$w_n(z,V) = f(V) + \sup_\pi E_{z,V}\left\{w_{n+1}(z(1), V\pi \cdot (1+\zeta(z,\xi(1)))\right\}$

$w_0(z,V) = f(V) + \sup_\pi E_{z,V}\left\{w_1(z(1), V\pi \cdot (1+\zeta(z,\xi(1)))\right\}$

Theorem

$w_0(z,V) = \sup_\Pi E_{z,V}\left\{\sum_{t=0}^{T-1} f(V(t)) + h(V(T))\right\}$

$\hat{\pi}(t) = \tilde{\pi}_t(z(t), V(t))$

Other functionals

infinite horizon models:

discounted reward:

$w^{\beta}(z, V) = \sup_{\Pi} E_{z,V} \{\sum_{t=0}^{\infty} \beta^t f(V(t))\}$

discounted Bellman equation:

$w^{\beta}(z, V) = f(V) + \beta \sup_{\pi} E_{z,V} \left\{ w^{\beta}(z(1), V\pi \cdot (1 + \zeta(z, \xi(1))) \right\}$

long time horizon:

$w(z, V) = \sup_{\Pi} \liminf_{T\to\infty} \frac{1}{T} E_{z,V} \left\{ \sum_{t=0}^{T-1} f(V(t)) \right\}$

long time Bellman equations:

$r(z, V) = f(V) - \gamma + \sup_{\pi} E_{z,V} \{ r(z(1), V\pi \cdot (1 + \zeta(z, \xi(1))) \}$

Theorem: $w(z, V) = \gamma$

example: growth optimal portfolio - risk neutral case

$\liminf_{T\to\infty} \frac{1}{T} E_{z,V} \{\ln(V(T))\} = J_0(\Pi)$

Risk sensitive functionals

finite horizon

$w(z,V) =$

$\sup_{\Pi} \frac{-1}{\lambda} \ln E_{z,V} \left\{ \exp\left\{ -\lambda (\sum_{t=0}^{T-1} f(V(t)) + h(V(T))) \right\} \right\}$

$w(z,V) = \sup_{\Pi} \frac{-1}{\lambda} \ln E_{z,V} \{ \exp\{ -\lambda \ln(V(T)) \} \} =$

$= \sup_{\Pi} \frac{-1}{\lambda} \ln E_{z,V} \left\{ \left\{ (V(T))^{-\lambda} \right\} \right\}$

Bellman equations:

$r_T(z,V) = h(V)$

$r_{T-1}(z,V) =$

$f(V) + \inf_{\pi} \ln E_{z,V} \{ \exp\{ r_T(z(1), V\pi \cdot (1 + \zeta(z, \xi(1))) \} \}$

$r_n(z,V) =$

$f(V) + \inf_{\pi} \ln E_{z,V} \{ \exp\{ r_{n+1}(z(1), V\pi \cdot (1 + \zeta(z, \xi(1))) \} \}$

$r_0(z,V) = f(V) + \inf_{\pi} \ln E_{z,V} \{ \exp\{ r_1(z(1), V\pi \cdot (1 + \zeta(z, \xi(1))) \} \}$

$w(z,V) = \frac{-1}{\lambda} r_0(z,V)$

Long Run Risk Sensitive Functionals

$w(z,V) =$
$\sup_\Pi \liminf_{T\to\infty} \frac{-1}{\lambda T} \ln E_{z,V} \left\{ \exp \left\{ -\lambda (\sum_{t=0}^{T-1} f(V(t)) + h(V(T)) \right\} \right\}$
$w(z,V) = \sup_\Pi \liminf_{T\to\infty} \frac{-1}{\lambda T} \ln E_{z,V} \{ \exp \{ -\lambda \ln(V(T)) \} \} =$
$\sup_\Pi J_\lambda(\Pi)$

Bellman equation:

$r(z,V) =$
$\lambda f(V) - \gamma_\lambda \lambda + \inf_\pi \ln E_{z,V} \{ \exp \{ r(z(1), V\pi \cdot (1 + \zeta(z, \xi(1)) \} \}$
$w(z,V) = \gamma_\lambda$
$\lim_{\lambda\to 0} J_\lambda(\Pi) = J_0(\Pi)$
$\lim_{\lambda\to 0} \gamma_\lambda = \gamma$

Problems with Transaction Costs

proportional transaction costs

$S_i(t),\ t \geq 0,\ i = 1, 2 \ldots, d$

$V(t),\ V^-(t)$ wealth processes after and before transaction at time t, $N_i(t),\ N_i^-(t)$ number of assets at time t in portfolio after and before transaction

transaction cost at t (assuming we make a transaction $N(t) \neq N^-(t)$)

$C(N(t), N^-(t), S(t)) =$
$\sum_{i=1}^d c_{1i}((N_i(t) - N_i^-(t))S_i(t))^+ + \sum_{i=1}^d c_{2i}((N_i(t) - N_i^-(t))S_i(t))^-$

$C(N, N^-, S) = \sum_{i=1}^d c_{1i}((N_i - N_i^-)S_i)^+ + \sum_{i=1}^d c_{2i}((N_i - N_i^-)S_i)^-$

selffinancing property $V^- = V + C(N, N^-, S)$,

$N^- \cdot S = N \cdot S + C(N, N^-, S)$

$V^-(t)$ the wealth before transaction, $V(t)$ after transaction,

$\pi_i^-(t) = \frac{N_i^-(t)S_i(t)}{V^-(t)}$, $\pi_i(t) = \frac{N_i(t)S_i(t)}{V(t)}$, $\pi(t) \in \mathcal{S}$

$V^-(t) = V(t) + V^-(t)c(\pi' - \pi^-(t))$, where

Problems with Transaction Costs (cont.)

$\pi' \in \mathcal{S}^0$ is s.t. $\pi(t) = g(\pi')$ and
$c(\nu) = \sum_{i=1}^{d} c_{1i}(\nu_i)^+ + \sum_{i=1}^{d} c_{2i}(\nu_i)^-$
$\sum \pi'_i + c(\pi' - \pi^-(t)) = 1$ and $\pi(t) = g(\pi')$, $g_i(\pi) = \frac{\pi_i}{\sum_{j=1}^{m} \pi_j}$
Lemma 1. Given $\pi, \bar{\pi} \in \mathcal{S}$ there is a continuous $e(\pi, \bar{\pi}) \in (0, 1]$ s.t.
$F^{\pi,\bar{\pi}}(e(\pi, \bar{\pi})) = 1$ with $F^{\pi,\bar{\pi}}(\delta) = \delta + c(\delta\bar{\pi} - \pi)$
$V^{\Pi,v}(t) = e(\pi^-(t), \pi(t)) V^{\Pi,v-}(t)$
If there are no transactions in $(s, s+t)$
$V^{\Pi,v}(t+s)^- = V^{\Pi,v}(t)\pi(t) \cdot e^{X(t+s)-X(s)}$
$\pi^-(t+s) = g(\pi(t) \diamond e^{X(t+s)-X(t)})$
$\pi(t) \diamond e^{X(t+s)-X(s)} = (\pi_i(t) e^{X_i(t+s)-X_i(t)})$

Asymtotics of $\pi(t)$ (without transactions)

$1 + \zeta_i(s, z(.), \xi) := e^{X_i(s)}$
$dX(t) = \alpha(z(t))dt + \sigma(z(t))dB(t) + \int_{R^d} \gamma(z(t), u)\tilde{N}(dt, du)$
$X(0) = 0$
by the Law of Large Numbers
$\lim_{t\to\infty} \frac{1}{t} X_i(t) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \alpha_i(z(s))ds = \int \alpha_i(z)\mu(dz) = r_i$
μ a unique invariant measure for $(z(t))$
If $\exists_k \ r_k > r_i$ for $i \neq k$ then
$\pi(t) \to \pi(\infty) = \delta_k$, provided $\pi_k(0) > 0$

Asymptotics of the portfolio growth (without transactions)

$\ln(\pi \cdot e^{X(n)}) = \sum_{t=0}^{n-1} \ln \pi(t) \cdot e^{X(t+1)-X(t)}$ $(X(0) = 0)$

$\lim_{n\to\infty} \frac{1}{n} \left(\ln(\pi \cdot e^{X(n)}) - \sum_{t=0}^{n-1} w(\pi(t), z(t)) \right) = 0$

$w(\pi, z) = E_z \left[\ln \pi \cdot e^{X(1)} \right]$

$\lim_{n\to\infty} \frac{1}{n} \sum_{t=0}^{n-1} w(\pi(t), z(t)) = \int w(\pi(\infty), z)\mu(dz)$

$\lim_{n\to\infty} \frac{1}{n} \ln(\pi \cdot e^{X(n)}) = \int w(\pi(\infty), z)\mu(dz)$

$\pi \mapsto w(\pi, z)$ has at most one interior supremum point in $\mathcal{S}$, can we get better than $\int w(\pi(\infty), z)\mu(dz)$?

Continuous time models

feasible control impulsive control $\Pi = (\tau_i, \pi_{\tau_i})$
at time τ_i we choose portfolio π_{τ_i}
for models with fixed plus proportional transaction costs the class of impulse controls is optimal
how to solve such problems: quasi variational inequalities

Partial observation models in finance

$\frac{S_i(t+1)}{S_i(t)} = 1 + \zeta_i(z(t), \xi(t+1))$
$\frac{S_i(t+s)}{S_i(t)} = 1 + \zeta_i(t, s, z(.), \xi(.))$
we do not observe $(z(t))$
can we recover $z(t)$ from $\{S(u); u \leq t\}$?
how to calculate strategies: discretization (Runggaldier Stettner 1993), quantization,
problem with computational complexity
General question: how to calculate strategies for discrete or continuous time models?
in the worst cases the main tool is Monte Carlo Method