I. Maxima and Worst Cases

1. Limiting Behaviour of Sums and Maxima

2. Extreme Value Distributions

3. The Fisher–Tippett Theorem

4. The Block Maxima Method

5. S&P Example

I1. Limiting Behaviour of Maxima

Let $X_1, X_2, \ldots$ be iid random variables with distribution function (df) F. In risk management applications these could represent financial losses, operational losses or insurance losses.

Let $M_n = \max(X_1, \ldots, X_n)$ be worst–case loss in a sample of n losses. Clearly

$$P(M_n \leq x) = P(X_1 \leq x, \ldots, X_n \leq x) = F^n(x).$$

It can be shown that, almost surely, $M_n \overset{n \to \infty}{\longrightarrow} x_F$, where $x_F := \sup\{x \in \mathbb{R} : F(x) < 1\} \leq \infty$ is the right endpoint of F.

But what about normalized maxima?

Limiting Behaviour of Sums or Averages

(See [Embrechts et al., 1997], Chapter 2.)

We are familiar with the central limit theorem.

Let $X_1, X_2, \ldots$ be iid with finite mean μ and finite variance σ^2. Let $S_n = X_1 + X_2 + \ldots + X_n$. Then

$$P\left((S_n - n\mu)\,/\sqrt{n\sigma^2} \le x\right) \stackrel{n\to\infty}{\longrightarrow} \Phi(x)\,,$$

where Φ is the distribution function of the standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du\,.$$

Note, more generally, the limiting distributions for appropriately normalized sample sums are the class of α–stable distributions; Gaussian distribution is a special case.

Limiting Behaviour of Sample Extrema

(See [Embrechts et al., 1997], Chapter 3.)

Let $X_1, X_2, \ldots$ be iid from F and let $M_n = \max(X_1, \ldots, X_n)$.

Suppose we can find sequences of real numbers $a_n > 0$ and b_n such that $(M_n - b_n)/a_n$, the sequence of normalized maxima, converges in distribution, i.e.

$$P\left((M_n - b_n)/a_n \le x\right) = F^n(a_n x + b_n) \stackrel{n\to\infty}{\longrightarrow} H(x),$$

for some non–degenerate df $H(x)$.

If this condition holds we say that F is in the maximum domain of attraction of H, abbreviated $F \in \text{MDA}(H)$. Note that such an H is determined up to location and scale, i.e. will specify a unique type of distribution.

I2. Generalized Extreme Value Distribution

The GEV has df

$$H_\xi(x) = \begin{cases} \exp\left(-(1+\xi x)^{-1/\xi}\right) & \xi \neq 0\,, \\ \exp\left(-e^{-x}\right) & \xi = 0, \end{cases}$$

where $1 + \xi x > 0$ and ξ is the shape parameter. Note, this parametrization is continuous in ξ. For

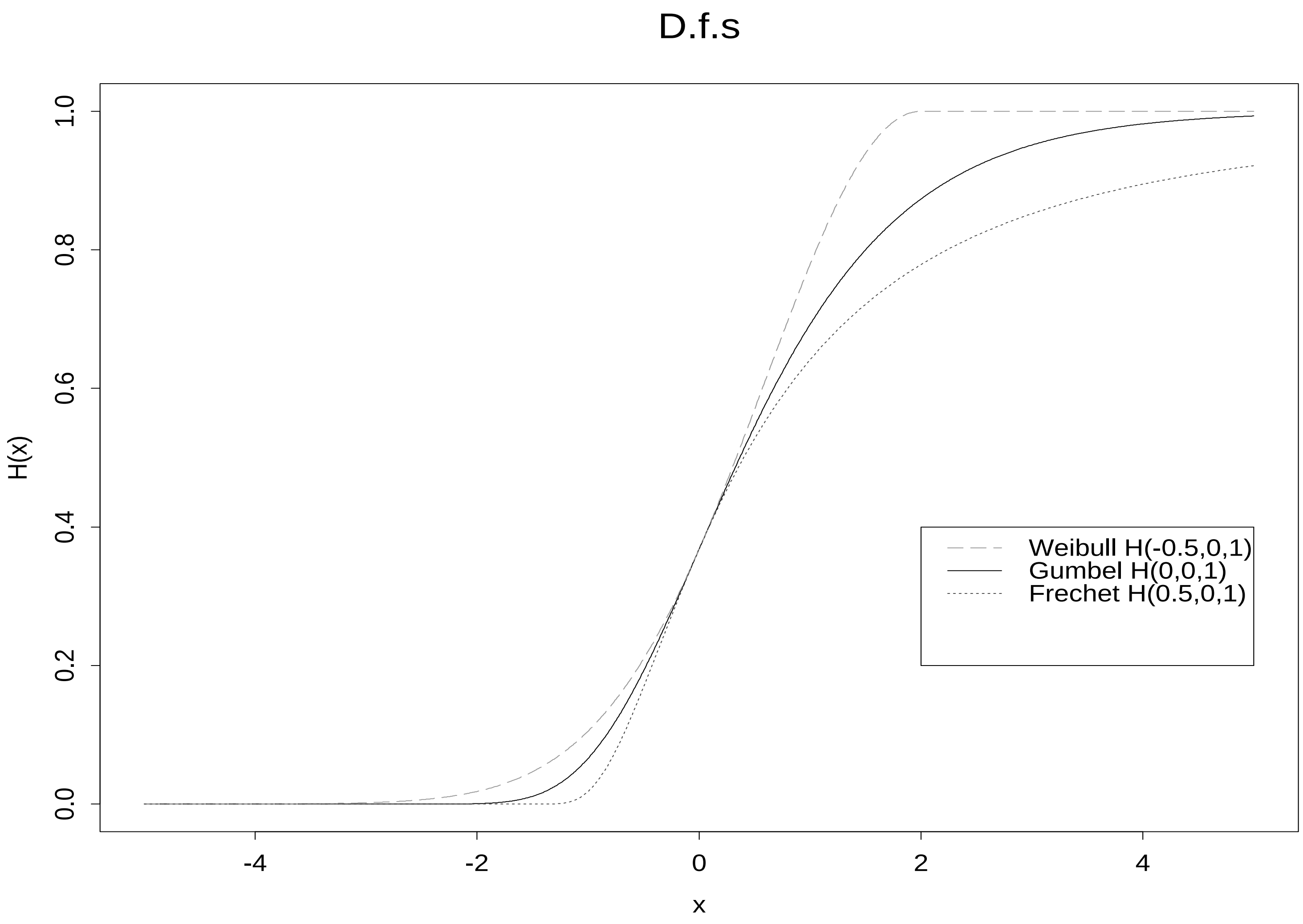
$\xi > 0$ H_ξ is equal in type to classical Fréchet df

$\xi = 0$ H_ξ is equal in type to classical Gumbel df

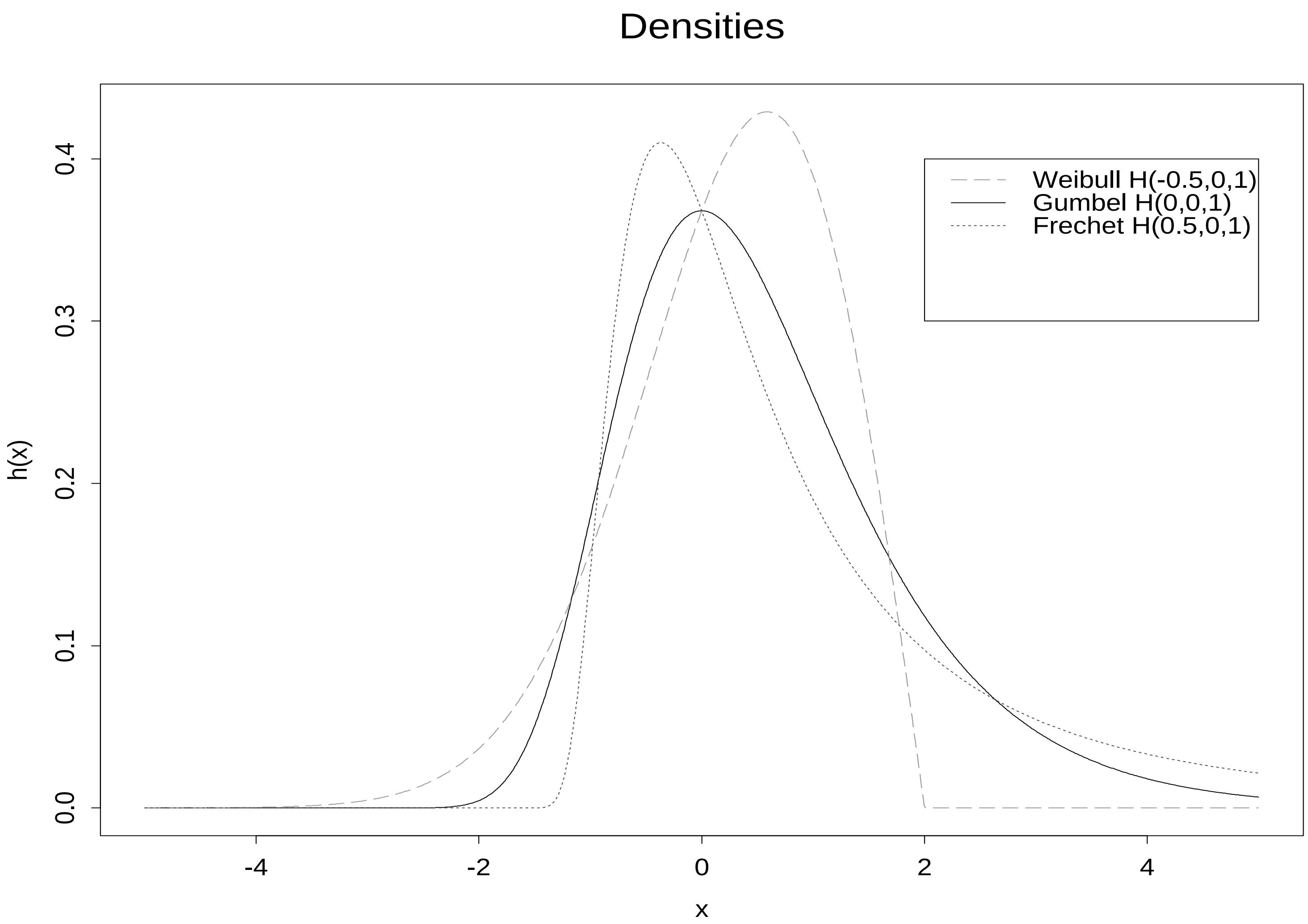
$\xi < 0$ H_ξ is equal in type to classical Weibull df.

We introduce location and scale parameters μ and $\sigma > 0$ and work with $H_{\xi,\mu,\sigma}(x) := H_\xi((x-\mu)/\sigma)$. Clearly $H_{\xi,\mu,\sigma}$ is of type H_ξ.

GEV: distribution functions for various ξ



GEV: densities for various ξ



I3. Fisher–Tippett Theorem (1928)

Theorem: If $F \in \text{MDA}(H)$ then H is of the type H_ξ for some ξ.

"If suitably normalized maxima converge in distribution to a non–degenerate limit, then the limit distribution must be an extreme value distribution."

Remark 1: Essentially all commonly encountered continuous distributions are in the maximum domain of attraction of an extreme value distribution.

Remark 2: We can always choose normalizing sequences a_n and b_n so that the limit law H_ξ appears in standard form (without relocation or rescaling).

Fisher-Tippett: Examples

Recall: $F \in \text{MDA}(H_\xi)$, iff there are sequences a_n and b_n with

$$P\left((M_n - b_n)/a_n \le x\right) = F^n(a_n x + b_n) \stackrel{n\to\infty}{\longrightarrow} H(x).$$

We have the following examples:

- The exponential distribution, $F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$, $x \ge 0$, is in $\text{MDA}(H_0)$ (Gumbel-case). Take $a_n = 1/\lambda$, $b_n = (\log n)/\lambda$.

- The Pareto distribution,

$$F(x) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^\alpha, \quad \alpha, \kappa > 0, \quad x \ge 0,$$

  is in $\text{MDA}(H_{1/\alpha})$ (Fréchet case). Take $a_n = \kappa n^{1/\alpha}/\alpha$, $b_n = \kappa n^{1/\alpha} - \kappa$.

I4. Using Fisher–Tippett: Block Maxima Method

Assume that we have a large enough block of n iid random variables so that the limit result is more or less exact, i.e. $\exists a_n > 0$, $b_n \in \mathbb{R}$ such that, for some ξ,

$$P\left(\frac{M_n - b_n}{a_n} \leq x\right) \approx H_\xi(x)\,.$$

Now set $y = a_n x + b_n$. $\quad P\left(M_n \leq y\right) \approx H_\xi\left(\frac{y-b_n}{a_n}\right) = H_{\xi,b_n,a_n}(y)$.
We wish to estimate ξ, b_n and a_n.

Implication: We collect data on block maxima and fit the three–parameter form of the GEV. For this we require a lot of raw data so that we can form sufficiently many, sufficiently large blocks.

ML Inference for Maxima

We have block maxima data $\mathbf{y} = \left(M_n^{(1)}, \ldots, M_n^{(m)}\right)'$ from m blocks of size n. We wish to estimate $\boldsymbol{\theta} = (\xi, \mu, \sigma)'$. We construct a log–likelihood by assuming we have independent observations from a GEV with density $h_{\boldsymbol{\theta}}$,

$$l(\boldsymbol{\theta}; \mathbf{y}) = \log \left(\prod_{i=1}^{m} h_{\boldsymbol{\theta}} \left(M_n^{(i)}\right) 1_{\left\{1+\xi\left(M_n^{(i)}-\mu\right)/\sigma>0\right\}} \right),$$

and maximize this w.r.t. $\boldsymbol{\theta}$ to obtain the MLE $\widehat{\boldsymbol{\theta}} = (\widehat{\xi}, \widehat{\mu}, \widehat{\sigma})'$.

Clearly, in defining blocks, bias and variance must be traded off. We reduce bias by increasing the block size n; we reduce variance by increasing the number of blocks m.

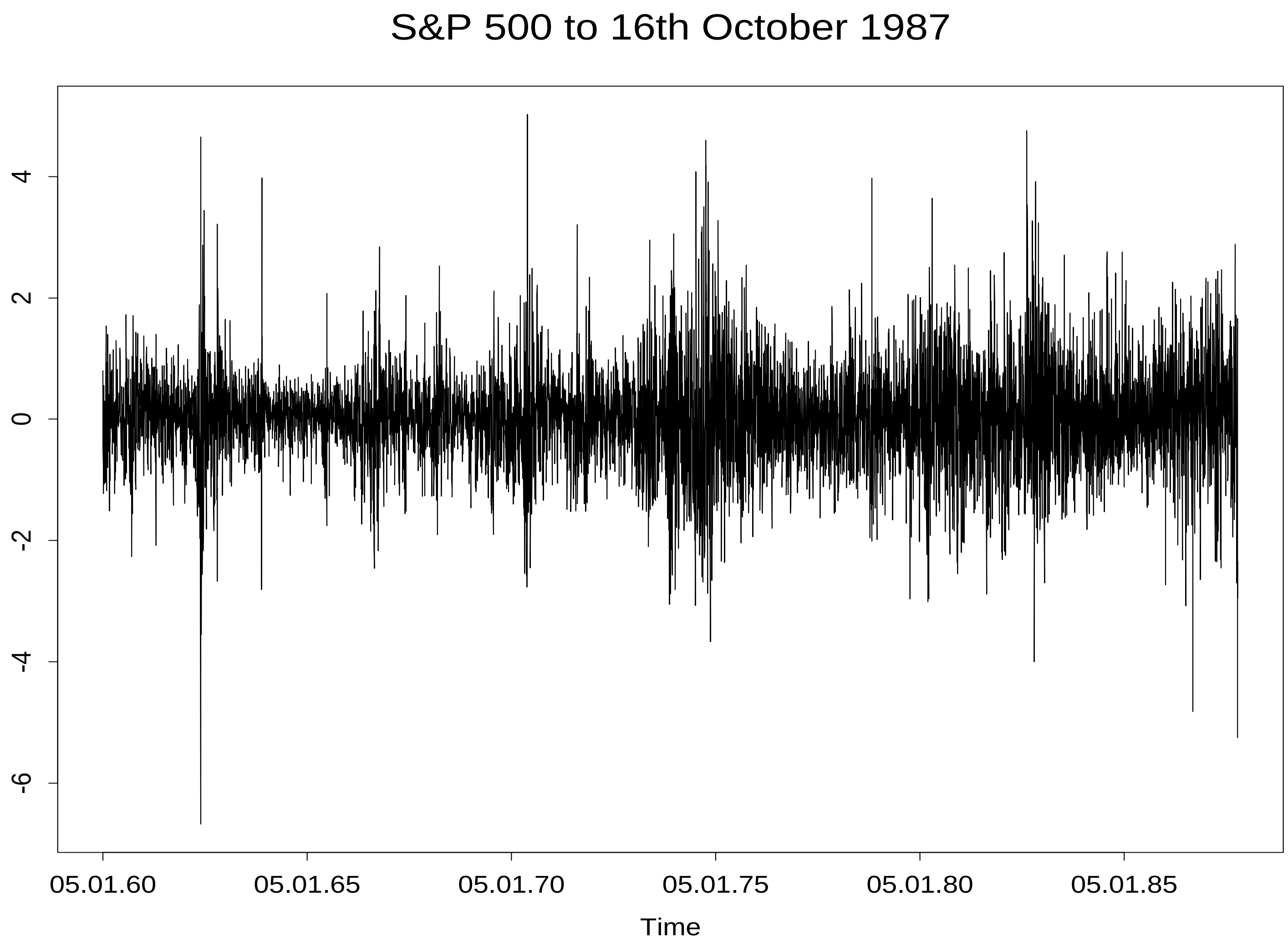
I5. An Example: S&P 500

It is the early evening of Friday the 16th October 1987. In the equity markets it has been an unusually turbulent week, which has seen the S&P 500 index fall by 9.21%. On that Friday alone the index is down 5.25% on the previous day, the largest one–day fall since 1962. At our disposal are all daily closing values of the index since 1960.

We analyse annual maxima of daily percentage falls in the index. These values $M_{260}^{(1)}, \ldots, M_{260}^{(28)}$ are assumed to be iid from $H_{\xi,\mu,\sigma}$.

Remark. Although we have only justified this choice of limiting distribution for maxima of iid data, it turns out that the GEV is also the correct limit for maxima of stationary time series, under some technical conditions on the nature of the dependence. These conditions are fulfilled, for example, by GARCH processes.

S&P 500 Return Data



Assessing the Risk in S&P

We will address the following two questions:

- What is the probability that next year's maximum exceeds all previous levels?

- What is the 40–year return level $R_{260,40}$?

In the first question we assess the probability of observing a new record. In the second problem we define and estimate a rare stress or scenario loss.

Return Levels

$R_{n,k}$, the k n–block return level, is defined by

$$P\left(M_n > R_{n,k}\right) = 1/k\,;$$

i.e. it is that level which is exceeded in one out of every k n–blocks, on average.

We use the approximation

$$R_{n,k} \approx H^{-1}_{\xi,\mu,\sigma}(1 - 1/k) \approx \mu + \sigma\left((-\log(1 - 1/k))^{-\xi} - 1\right)/\xi\,.$$

We wish to estimate this functional of the unknown parameters of our GEV model for maxima of n–blocks.

S–Plus Maxima Analysis with EVIS

```
> out <- gev(-sp,"year")
> out
$n.all: [1] 6985

$n: [1] 28

$data:
     1960     1961     1962     1963     1964     1965     1966     1967
 2.268191 2.083017 6.675635 2.806479 1.253012 1.757765 2.460411 1.558183
     1968     1969     1970     1971     1972     1973     1974     1975
 1.899367 1.903001 2.768166 1.522388 1.319013 3.051598 3.671256 2.362394
     1976     1977     1978     1979     1980     1981     1982     1983
 1.797353 1.625611 2.009257 2.957772 3.006734 2.886327 3.996544 2.697254
     1984     1985     1986     1987
 1.820587 1.455301 4.816644 5.253623

$par.ests:
        xi     sigma       mu
 0.3343843 0.6715922 1.974976

$par.ses:
     xi    sigma        mu
 0.2081 0.130821 0.1512828

$nllh.final:
[1] 38.33949
```

S&P Example (continued)

Answers:

- Probability is estimated by

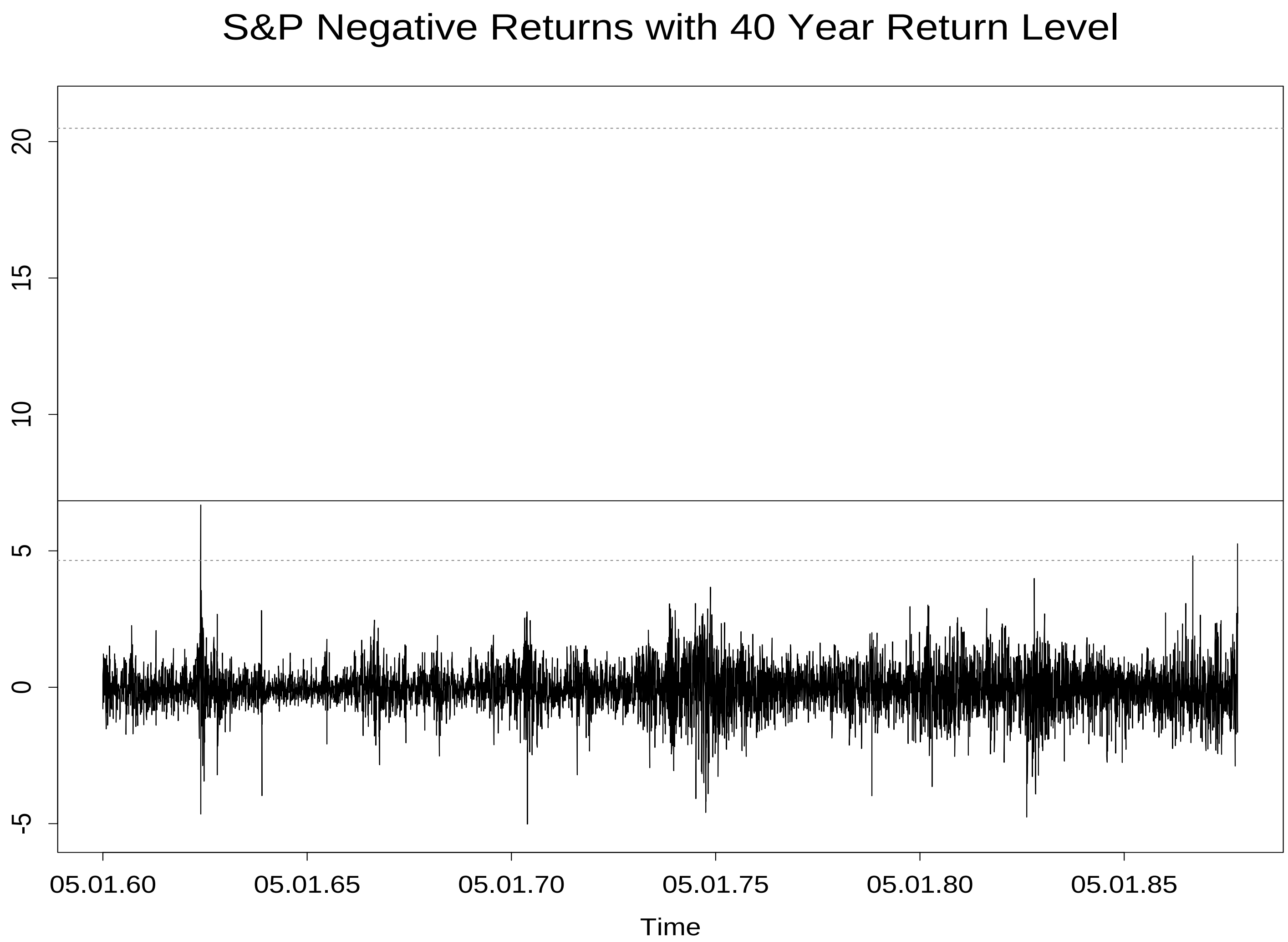
$$1 - H_{\hat{\xi},\hat{\mu},\hat{\sigma}}\left(\max\left(M_{260}^{(1)},\ldots,M_{260}^{(28)}\right)\right) = 0.027\,.$$

- $R_{260,40}$ is estimated by

$$H^{-1}_{\hat{\xi},\hat{\mu},\hat{\sigma}}(1 - 1/40) = 6.83\,.$$

It is important to construct confidence intervals for such statistics. We use asymptotic likelihood ratio ideas to construct asymmetric intervals – the so–called profile likelihood method.

Estimated 40–Year Return Level



References

On EVT in general:

- [Embrechts et al., 1997]
- [Reiss and Thomas, 1997]

On Fisher-Tippett Theorem:

- [Fisher and Tippett, 1928]
- [Gnedenko, 1943]

Application of Block Maxima Method to S&P Data:

- [McNeil, 1998]

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