J. The Peaks–over–Thresholds (POT) Method

1. The Generalized Pareto Distribution (GPD)
2. The POT Method: Theoretical Foundations
3. Modelling Tails and Quantiles of Distributions
4. The Danish Fire Loss Analysis
5. Expected Shortfall and Mean Excess Plot

J1. Generalized Pareto Distribution

The GPD is a two parameter distribution with df

$$G_{\xi,\beta}(x) = \begin{cases} 1 - (1 + \xi x/\beta)^{-1/\xi} & \xi \neq 0\,, \\ 1 - \exp(-x/\beta) & \xi = 0\,, \end{cases}$$

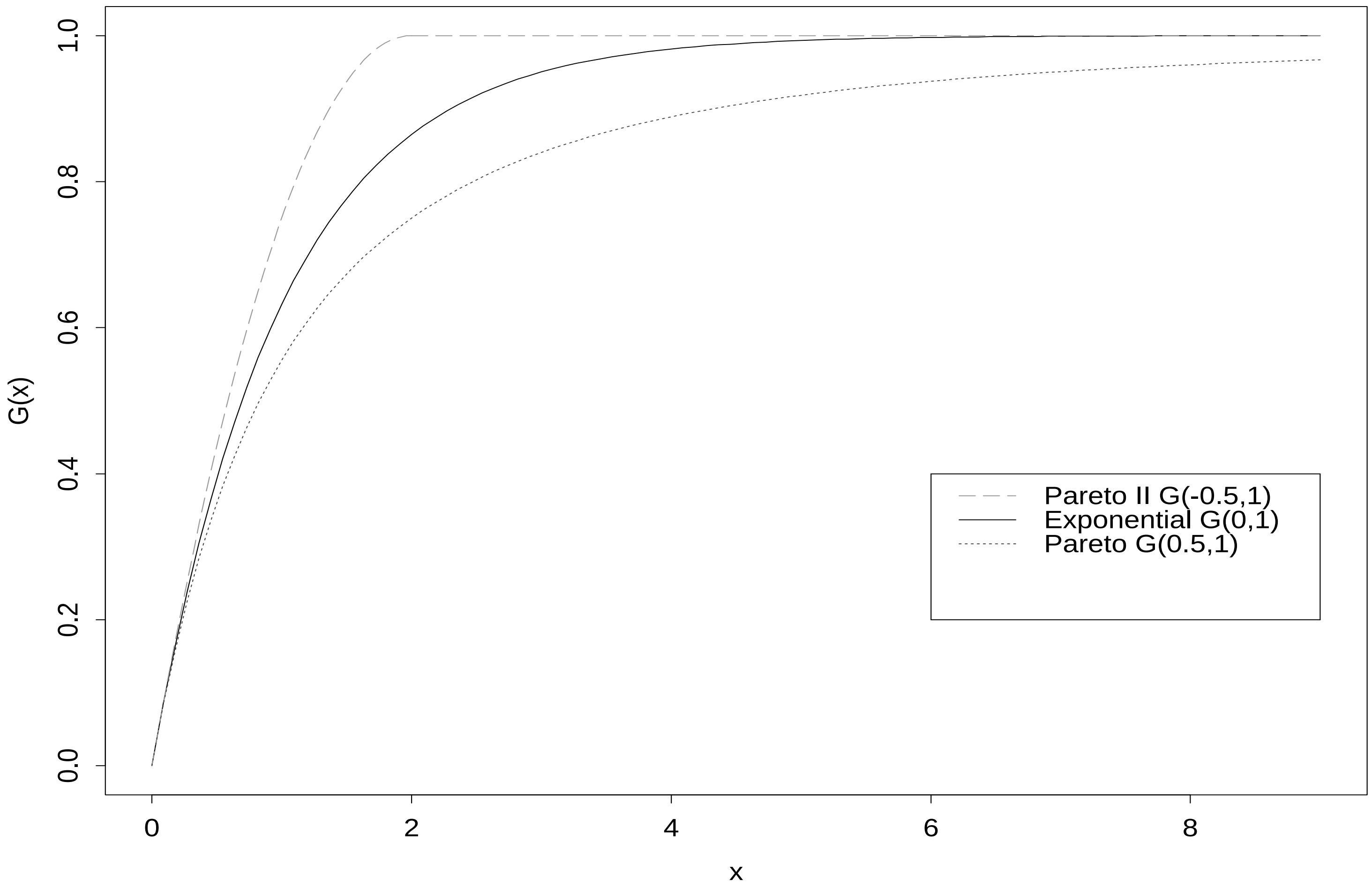
where $\beta > 0$, and the support is $x \geq 0$ when $\xi \geq 0$ and $0 \leq x \leq -\beta/\xi$ when $\xi < 0$.

This subsumes:

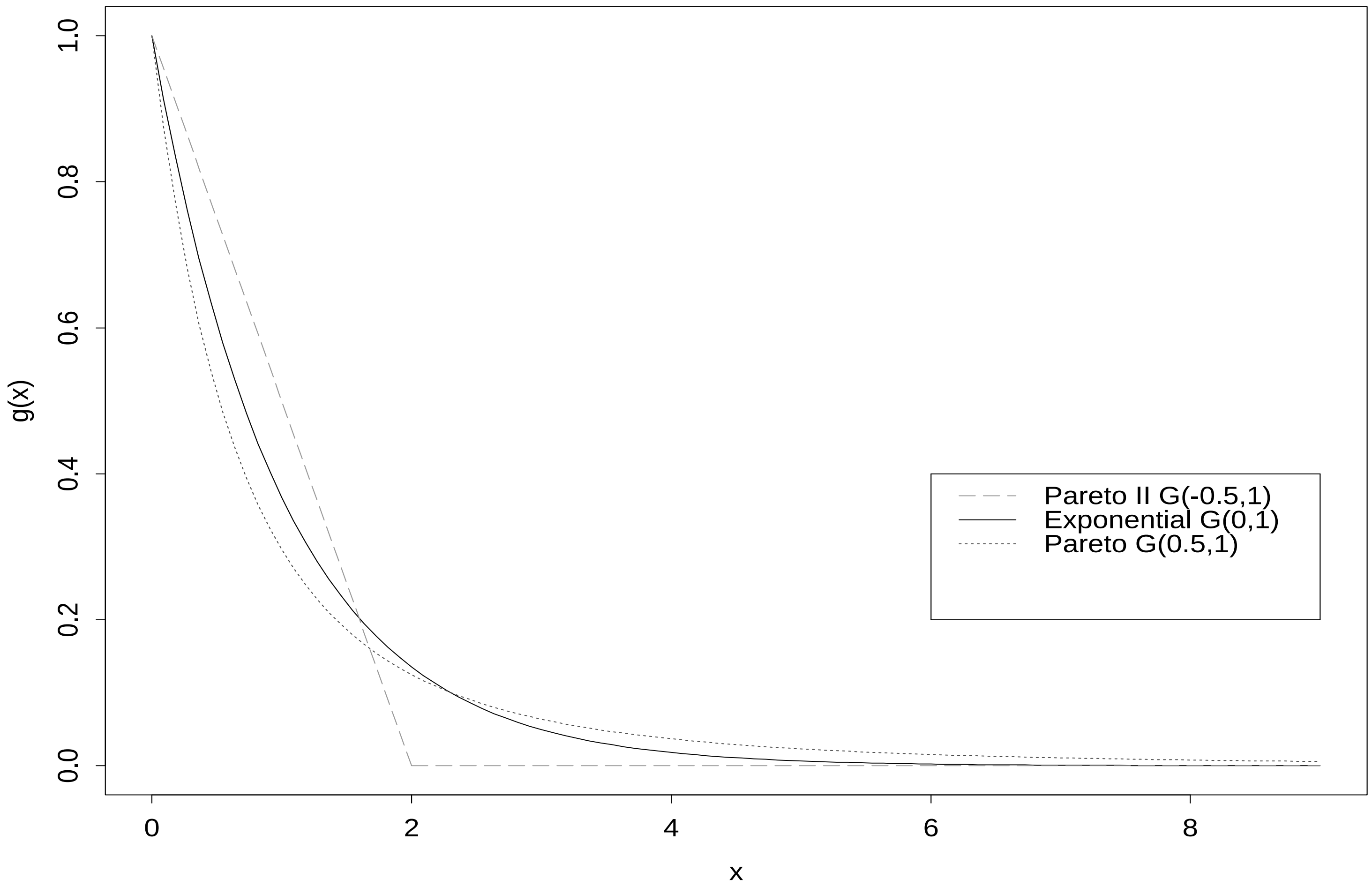
$\xi > 0$	Pareto (reparametrized version)
$\xi = 0$	exponential
$\xi < 0$	Pareto type II.

Moments. For $\xi > 0$ distribution is heavy tailed. $E\left(X^k\right)$ does not exist for $k \geq 1/\xi$.

GPD: distribution functions for various ξ



GPD: densities for various ξ



J2. POT Method: Theoretical Foundations

The excess distribution: Given that a loss exceeds a high threshold, by how much can the threshold be exceeded?

Let u be the high threshold and define the excess distribution above the threshold u to have the df

$$F_u(x) = P\left(X - u \leq x \mid X > u\right) = \frac{F(x+u) - F(u)}{1 - F(u)},$$

for $0 \leq x < x_F - u$ where $x_F \leq \infty$ is the right endpoint of F.

Extreme value theory suggests the GPD is a natural approximation for this distribution.

Examples

1. Exponential. $F(x) = 1 - e^{\lambda x}$, $\lambda > 0$, $x \geq 0$.

$$F_u(x) = F(x)\,, \quad x \geq 0\,.$$

The "lack–of–memory" property.

2. GPD. $F(x) = G_{\xi,\beta}(x)$.

$$F_u(x) = G_{\xi,\beta+\xi u}(x)\,,$$

where $0 \leq x < \infty$ if $\xi \geq 0$ and $0 \leq x < -\frac{\beta}{\xi} - u$ if $\xi < 0$.

The excess distribution of a GPD remains a GPD with the same shape parameter; only the scaling changes.

Asymptotics of Excess Distribution

Theorem. (Pickands–Balkema–de Haan (1974/75)) We can find a function $\beta(u)$ such that

$$\lim_{u \to x_F} \sup_{0 \le x < x_F - u} \left| F_u(x) - G_{\xi,\beta(u)}(x) \right| = 0 \,,$$

if and only if $F \in \text{MDA}\,(H_\xi)$, $\xi \in \mathbb{R}$.

Essentially all the common continuous distributions used in risk management or insurance mathematics are in $\text{MDA}\,(H_\xi)$ for some value of ξ, as we will see below.

Exploiting Pickands–Balkema–de Haan

"For a wide class of distributions, the distribution of the excesses over high thresholds can be approximated by the GPD."

This result suggests we choose u high and assume the limit result is more or less exact

$$F_u(x) \approx G_{\xi,\beta}(x)\,,$$

for some ξ and β. To estimate these parameters we fit the GPD to the excess amounts over the threshold u. Standard properties of maximum likelihood estimators apply if $\xi > -0.5$.

To implement the POT method we must choose a suitable threshold u. There are data–analytic tools (e.g. mean excess plot) to help us here, although later simulations will suggest that inference is often robust to choice of threshold.

When does $F \in$ MDA (H_ξ) hold?

1. Fréchet Case: ($\xi > 0$)

Gnedenko (1943) showed that for $\xi > 0$

$$F \in \text{MDA}\,(H_\xi) \iff 1 - F(x) = x^{-1/\xi} L(x)\,,$$

for some slowly varying function $L(x)$.

A function L on $(0, \infty)$ is slowly varying if
$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1\,, \quad t > 0\,.$

Summary:

If the tail of the df F decays like a power function, then the distribution is in MDA (H_ξ) for $\xi > 0$.

When does $F \in$ MDA (H_ξ) hold? (II)

Examples of Fréchet case: Heavy-tailed distributions such as Pareto, Burr, loggamma, Cauchy and t–distributions as well as various mixture models. Not all moments are finite.

2. Gumbel Case: $F \in$ MDA (H_0)

The characterization of this class is more complicated. Essentially it contains distributions whose tails decay roughly exponentially and we call these distributions light–tailed. All moments exist for distributions in the Gumbel class.

Examples are the Normal, lognormal, exponential and gamma.

J3. Estimating Tails of Distributions

R Smith (1987) proposed a tail estimator based on GPD approximation to excess distribution. Let $N_u = \sum_{i=1}^{n} 1_{\{\{X_i > u\}\}}$ be the random number of exceedances of u from iid sample $X_1, \ldots, X_n$.

Note that for $x > u$ we may write $\overline{F}(x) = \overline{F}(u)\overline{F}_u(x-u)$.

We estimate $\overline{F}(u)$ empirically by N_u/n and and $\overline{F}_u(x-u)$ using a GPD approximation to obtain the tail estimator

$$\widehat{\overline{F}}(x) = \frac{N_u}{n}\left(1 + \hat{\xi}\,\frac{x-u}{\hat{\beta}}\right)^{-1/\hat{\xi}} ;$$

this estimator is only valid for $x > u$. A high u reduces bias in estimating excess function. A low u reduces variance in estimating excess function and $F(u)$.

Estimating Quantiles in Tail

Recall the qth quantile of F

$$x_q = F^{\leftarrow}(q) = \inf\{x \in \mathbb{R} : F(x) \geq q\}\,.$$

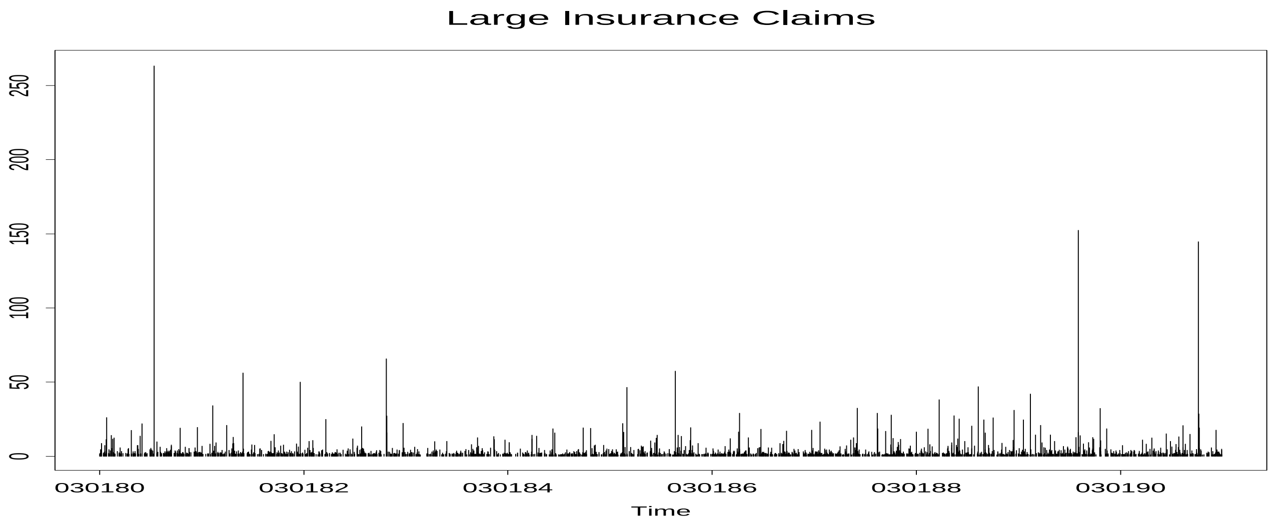
Suppose $x_q > u$ or equivalently $q > F(u)$. By inverting the tail estimation formula we get

$$\widehat{x}_q = u + \frac{\hat{\beta}}{\hat{\xi}}\left(\left(\frac{n}{N_u}(1-q)\right)^{-\hat{\xi}} - 1\right)\,.$$

Asymmetric confidence interval for x_q can be constructed using profile likelihood method.

J4. Danish Fire Loss Example

The Danish data consist of 2167 losses exceeding one million Danish Krone from the years 1980 to 1990. The loss figure is a total loss for the event concerned and includes damage to buildings, damage to contents of buildings as well as loss of profits. The data have been adjusted for inflation to reflect 1985 values.

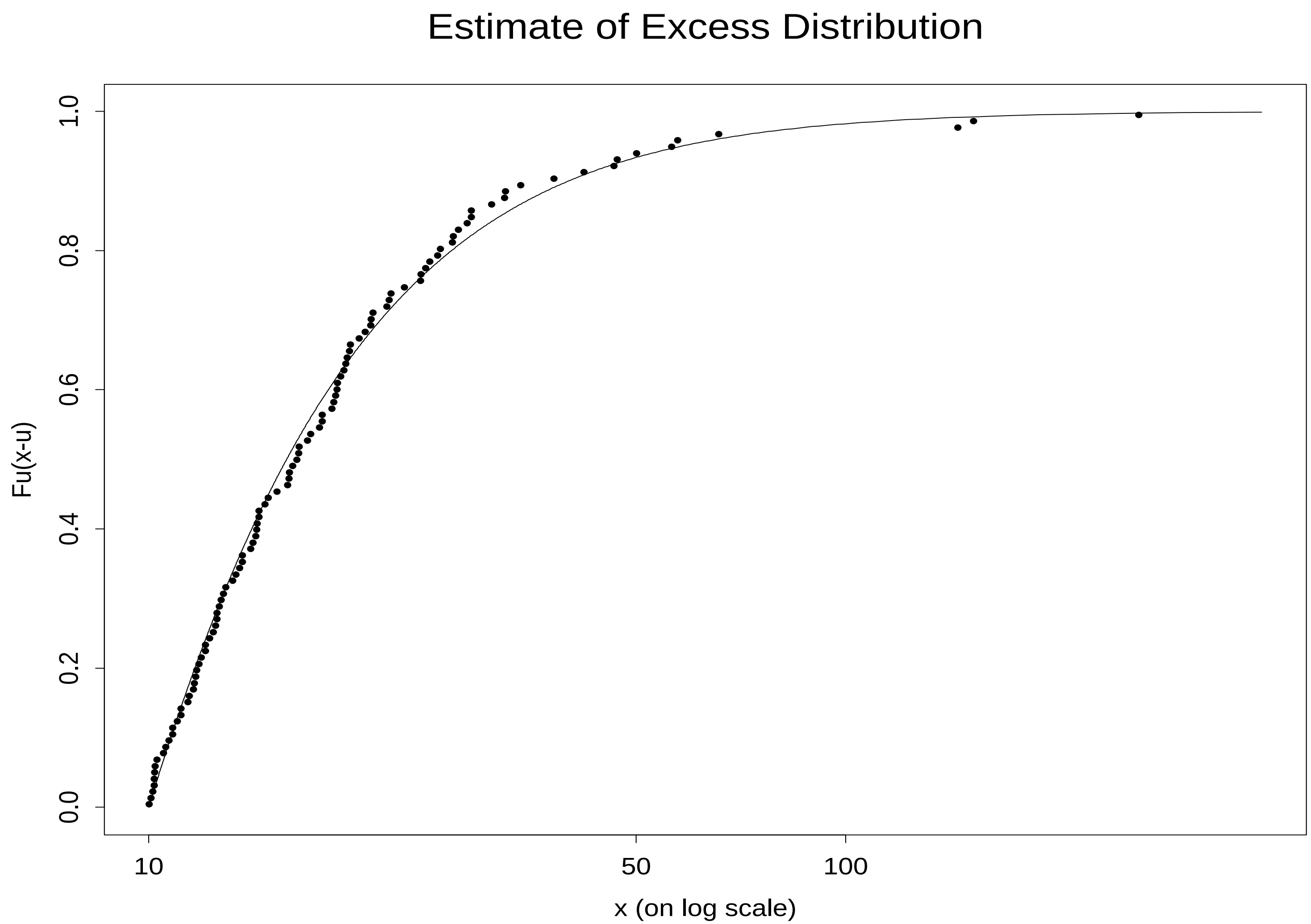


EVIS POT Analysis

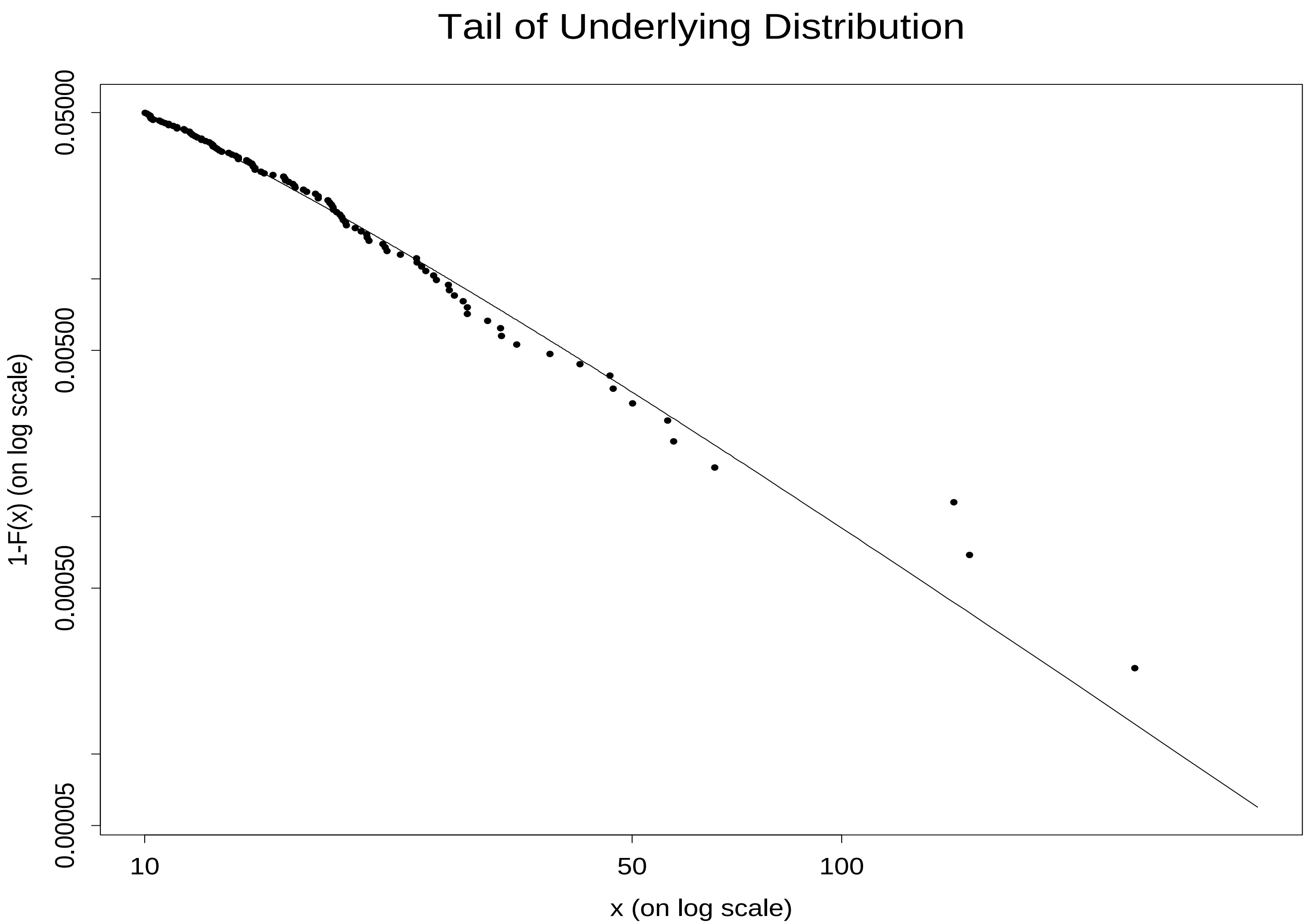
```
> out <- gpd(danish,10)
> out
$n:
[1] 2167

$data:
  [1]  11.37482  26.21464  14.12208
  [4]  11.71303  12.46559  17.56955
  [7]  13.62079  21.96193 263.25037
...etc...
[106] 144.65759  28.63036  19.26568
[109]  17.73927

$threshold:
[1] 10

$p.less.thresh:
[1] 0.9497

$n.exceed:
[1] 109
```

```
$par.ests:
        xi     beta
 0.4969857 6.975468

$par.ses:
        xi    beta
 0.1362838 1.11349

$varcov:
            [,1]        [,2]
[1,]  0.01857326 -0.08194611
[2,] -0.08194611  1.23986096

$information:
[1] "observed"

$converged:
[1] T

$nllh.final:
[1] 374.893

$
```

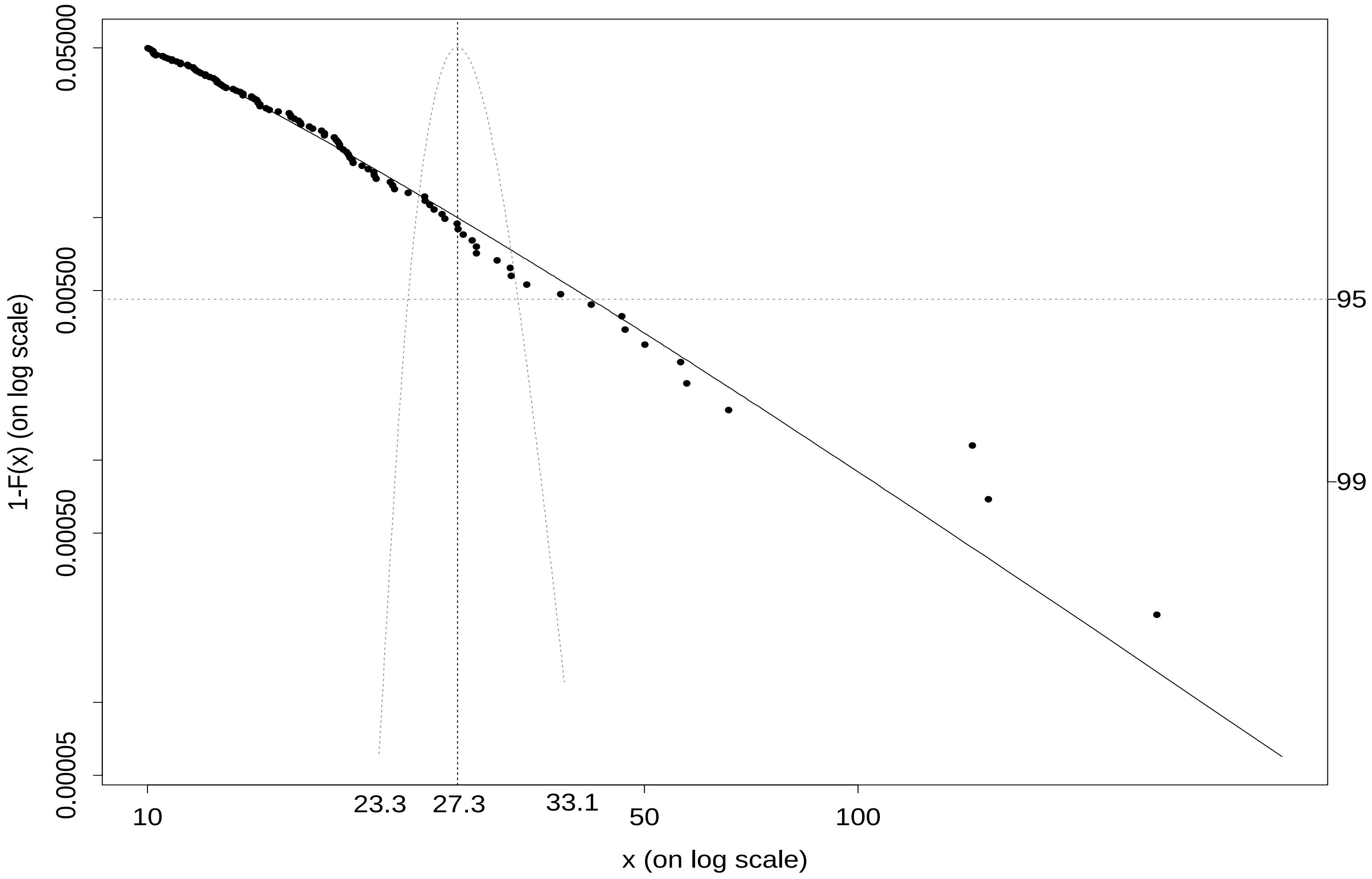
Estimating Excess df



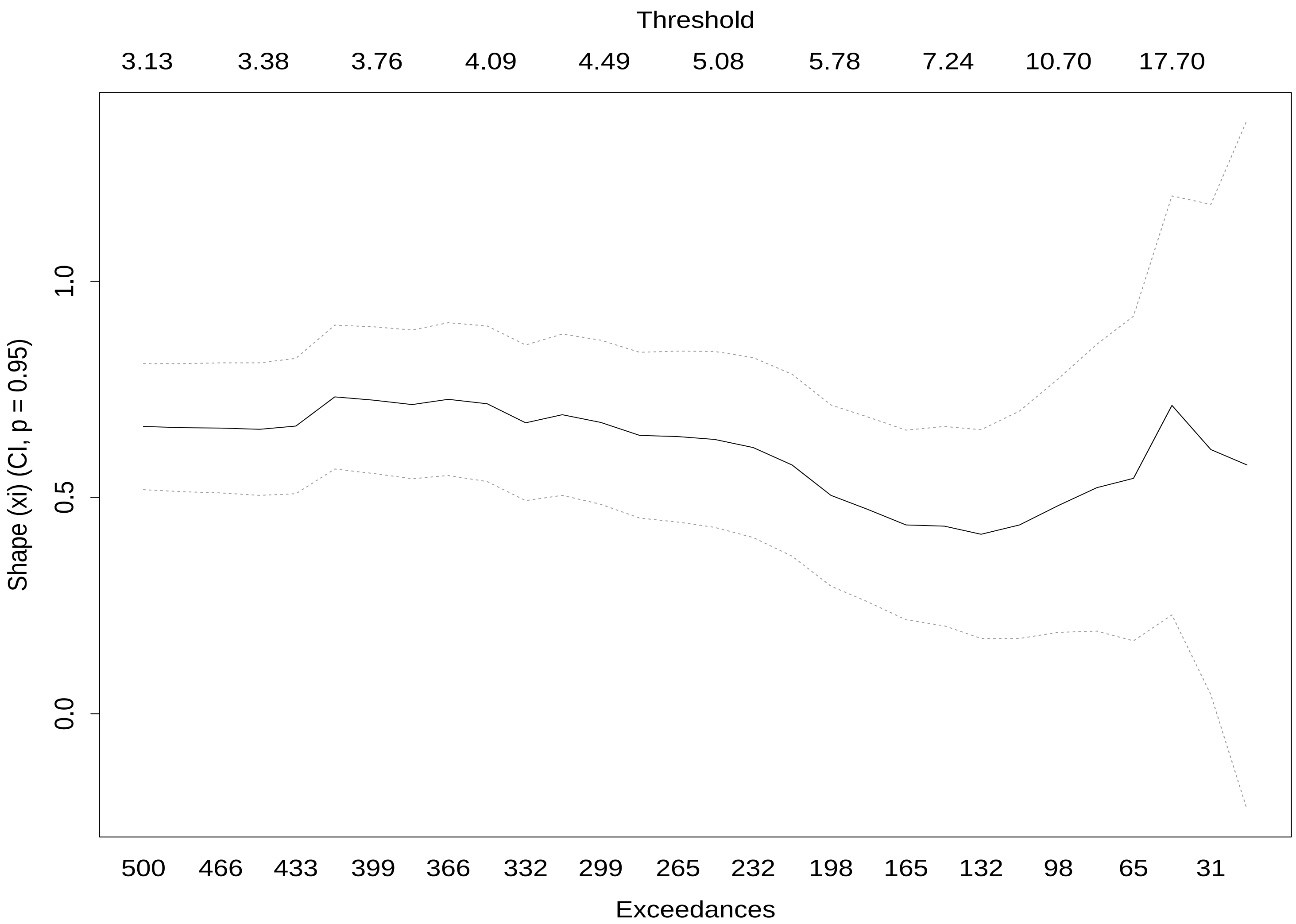
Estimating Tail of Underlying df



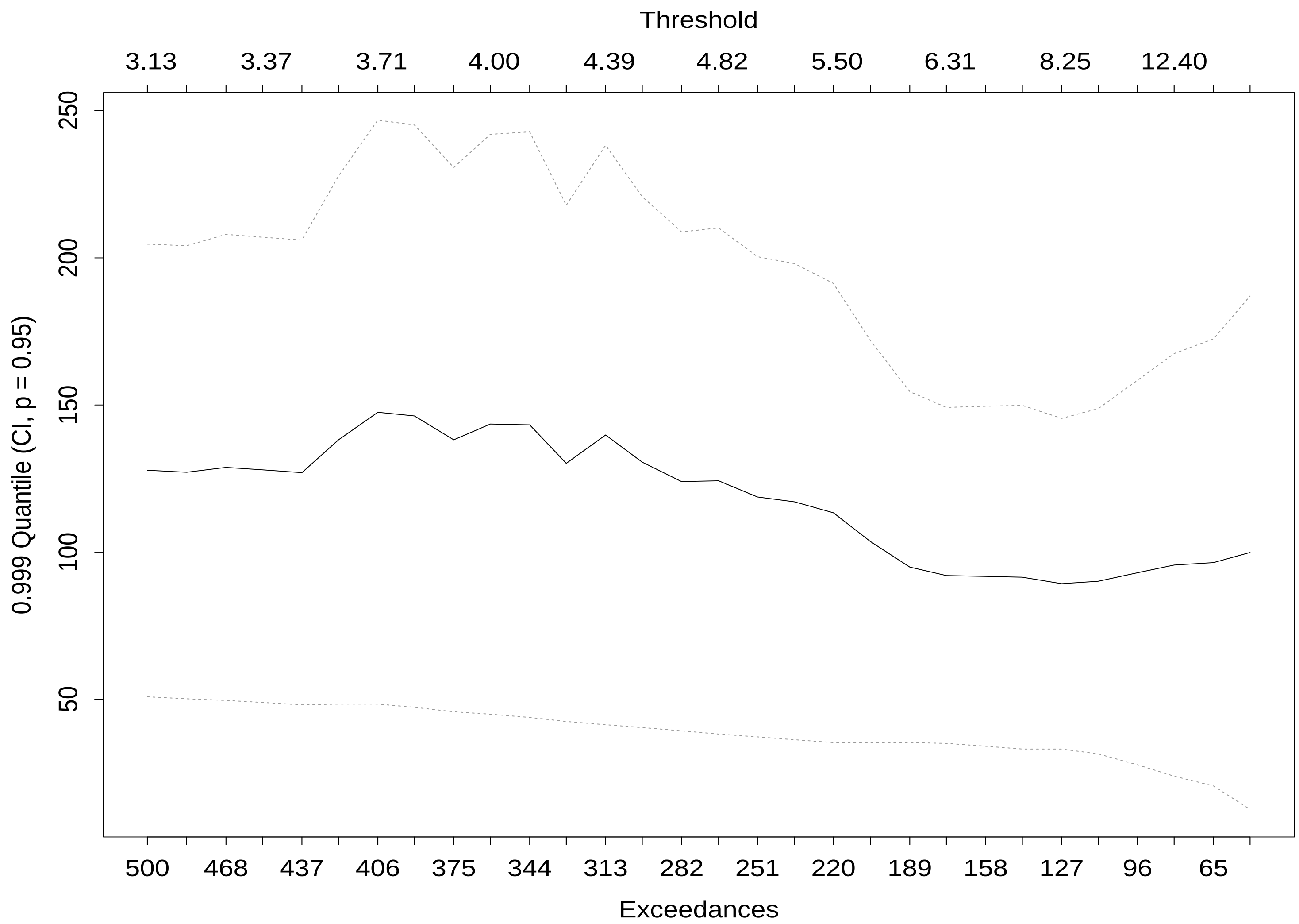
Estimating a Quantile (99%)



Varying the Threshold I



Varying the Threshold II



J5. Expected Shortfall and Mean Excess Plot

The mean excess function of a rv X is

$$e(u) = E(X - u \mid X > u).$$

It is the mean of the excess distribution function above the threshold u expressed as a function of u.

Our Model Assumption:

Excess losses over threshold u are exactly GPD with $\xi < 1$, i.e. $X - u \mid X > u \sim \text{GPD}(\xi, \beta)$. It is easily shown that for any higher threshold $v \geq u$

$$e(v) = E(X - v \mid X > v) = \frac{\beta + \xi(v - u)}{1 - \xi},$$

so that mean excess function is linear in v above u.

Sample Mean Excess Plot

The sample mean excess plot estimates $e(u)$ in the region where we have data:

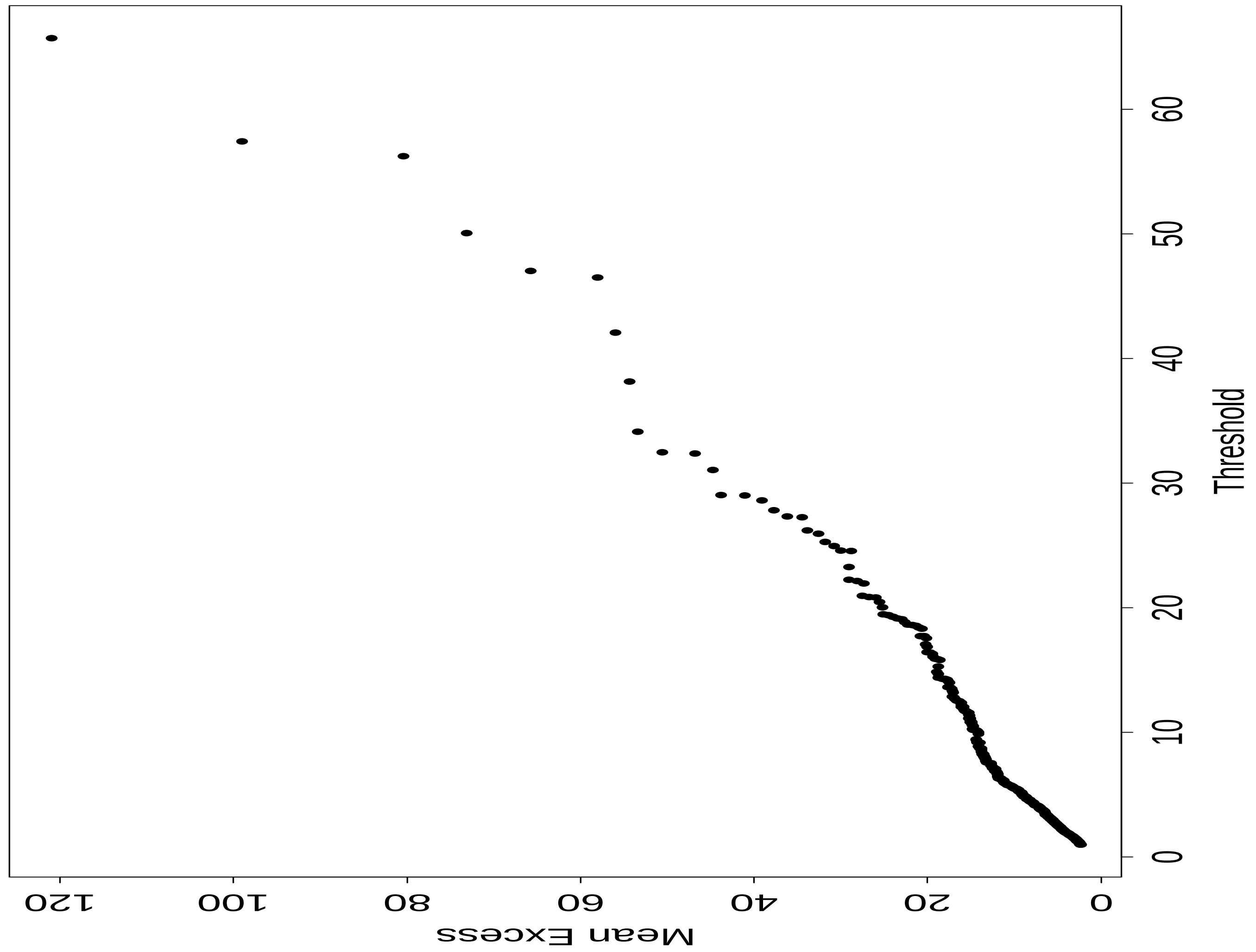
$$e_n(u) = \frac{\sum_{i=1}^{n}(X_i - u)^+}{\sum_{i=1}^{n} 1_{\{X_i > u\}}},$$

We seek a threshold u, above which the plot is roughly linear.

If we can find such a threshold, the result of Pickands-Balkema-De Haan could be applied above that threshold.

Note that the plot is erratic for large u, when the averaging is over very few excesses. It is often better to omit these from the plot.

Mean Excess Plot for Danish Data



Expected Shortfall: Estimation II

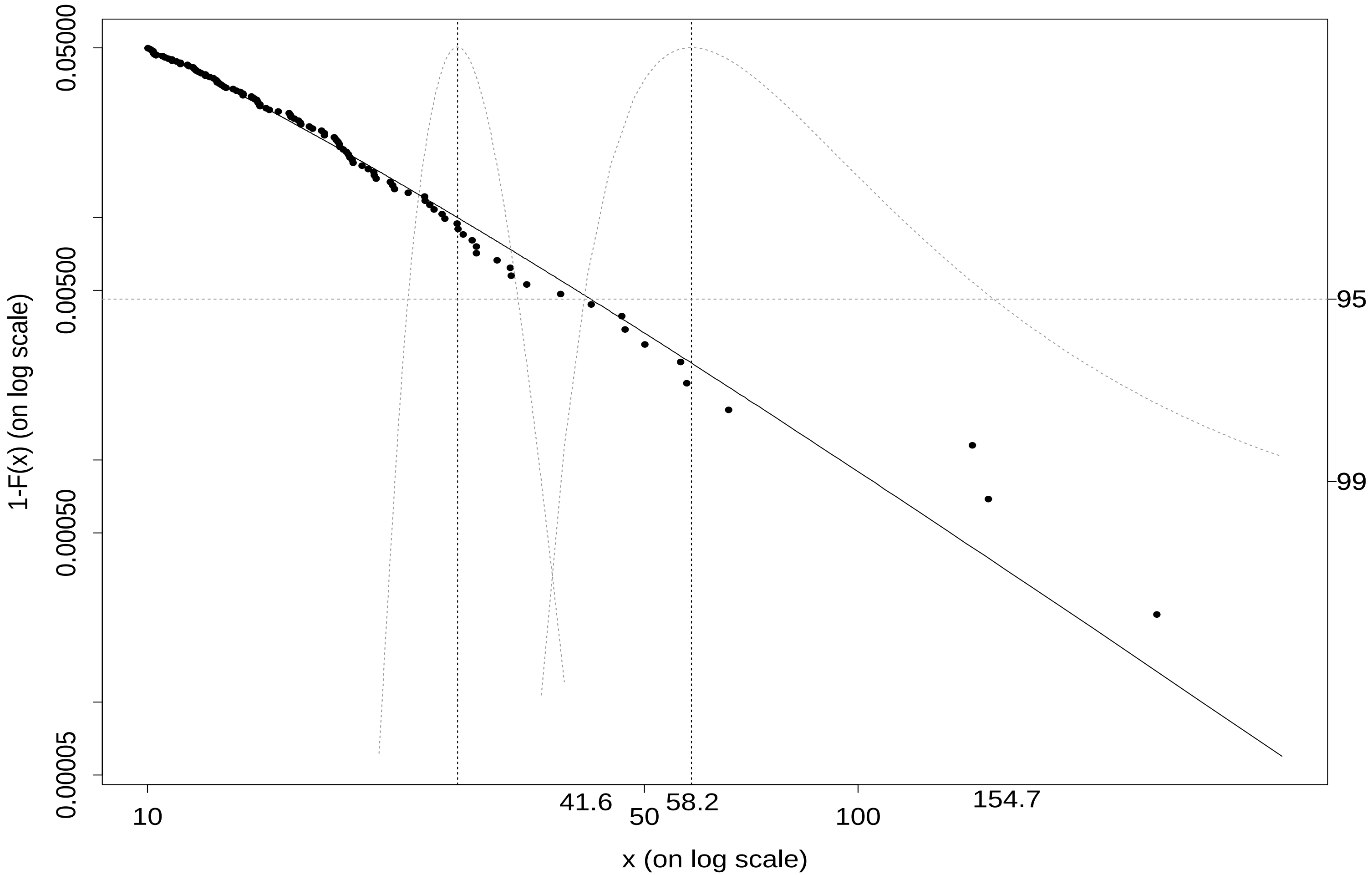
Now observe that for $x_q > u$

$$\begin{aligned} ES_q(X) &= E\left(X \mid X > x_q\right) \\ &= x_q + E\left(X - x_q \mid X > x_q\right) \\ &= x_q + \frac{\beta + \xi(x_q - u)}{1 - \xi}. \end{aligned}$$

This yields the estimator

$$\widehat{ES}_q(X) = \hat{x}_q \left(\frac{1}{1 - \hat{\xi}} + \frac{\hat{\beta} - \hat{\xi} u}{(1 - \hat{\xi})\hat{x}_q} \right).$$

Estimates of 99% VaR and ES (Danish Data)



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Pickands, Balkema, de Haan:

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GPD Tail estimation:

- [Smith, 1987]
- [McNeil, 1997] analysis of Danish data

POT method for risk managers:

- [McNeil, 1999]

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