Risk Theory and Related Topics

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Credit Default Swaps

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References

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Objectives

1. Valuation of Credit Default Swaps

2. Hedging of Defaultable Claims with a CDS

3. Hedging of First-to-Default Claims with CDSs

4. Hedging of Basket Credit Derivatives with CDSs

Credit Default Swaps

Defaultable Claims

A generic defaultable claim (X, A, Z, τ) consists of:

1. A **promised contingent claim** X representing the payoff received by the owner of the claim at time T, if there was no default prior to or at maturity date T.
2. A process A representing the **dividends stream** prior to default.
3. A **recovery process** Z representing the recovery payoff at time of default, if default occurs prior to or at maturity date T.
4. A **default time** τ, where the use of the term **default** is merely a convention.

Dividend Process

The dividend process D describes all cash flows associated with a defaultable claim (except for the initial price of a claim at time 0).

Definition 1 *The* ***dividend process*** D *of a defaultable claim* (X, A, Z, τ) *maturing at* T *equals, for every* $t \in [0, T]$,

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T,\infty[}(t) + \int_{]0,t]} (1 - H_u)\, dA_u + \int_{]0,t]} Z_u\, dH_u.$$

Note that the process D *has finite variation on* $[0, T]$.

Ex-dividend Price

The ex-dividend price S_t of a defaultable claim is aimed to represent the current value at time t of all dividend payments occurring during the time period $]t, T]$.

Let the process B represent the savings account.

Definition 2 *The* ***ex-dividend price process*** S *associated with the dividend process* D *satisfies, for every* $t \in [0, T]$,

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big)$$

where $\mathbb{Q}^*$ *is a spot martingale measure.*

Cumulative Price

The cumulative price $\widehat{S}_t$ is aimed to represent the current value at time t of all dividend payments occurring during the period $]t, T]$ under the convention that they were immediately reinvested in the savings account.

Definition 3 *The cumulative price process* S *associated with the dividend process* D *satisfies, for every* $t \in [0, T]$,

$$\widehat{S}_t = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big) = S_t + \widehat{D}_t$$

where $\widehat{D}_t$ *equals*

$$\widehat{D}_t = B_t \int_{]0,t]} B_u^{-1} \, dD_u, \quad \forall \, t \in [0, T].$$

Credit Default Swap

Definition 4 *A (stylized)* ***credit default swap*** *with a constant rate* κ *and* ***recovery at default*** *is a claim* $(0, A, Z, \tau)$, *where* $Z = \delta$ *and* $A_t = -\kappa t$.

- An $\mathbb{F}$-predictable process $\delta : [0, T] \to \mathbb{R}$ represents the **default protection stream**.
- A constant κ represents the **CDS spread**. It defines the fee leg, also known as the **survival annuity stream**.

Lemma 1 *The ex-dividend price of a CDS maturing at* T *equals*

$$S_t(\kappa) = \mathbb{E}_{\mathbb{Q}^*}\left(\mathbb{1}_{\{t<\tau\leq T\}}\delta_\tau \,\Big|\, \mathcal{G}_t\right) - \mathbb{E}_{\mathbb{Q}^*}\left(\mathbb{1}_{\{t<\tau\}}\kappa\big((\tau \wedge T) - t\big) \,\Big|\, \mathcal{G}_t\right)$$

where $\mathbb{Q}^*$ *is a spot martingale measure and* $B = 1$.

Hazard Process Approach

Standing assumptions:

- The default time τ is a non-negative random variable on $(\Omega, \mathcal{G}, \mathbb{Q}^*)$, where $\mathbb{Q}^*$ is a spot martingale measure.
- The default process $H_t = \mathbb{1}_{\{\tau \le t\}}$ generates the filtration $\mathbb{H}$.
- We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathbb{F}$ is a reference filtration.
- We define the risk-neutral survival process G_t as

$$G_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\}.$$

- We assume that the hazard process Γ equals

$$\Gamma_t = -\ln G_t = \int_0^t \gamma_u \, du$$

where γ is the default intensity.

Ex-dividend Price of a CDS

Recall that the survival process G_t satisfies

$$G_t = \mathbb{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} = \exp\Big(- \int_0^t \gamma_u \, du \Big).$$

We make the standing assumption that $\mathbb{E}_{\mathbb{Q}^*} |\delta_\tau| < \infty$.

Lemma 2 *The ex-dividend price at time $t \in [s, T]$ of a credit default swap started at s, with rate κ and protection payment δ_τ at default, equals*

$$S_t(\kappa) = \frac{1}{G_t} \, \mathbb{E}_{\mathbb{Q}^*} \left(- \int_t^T \delta_u \, dG_u - \kappa \int_t^T G_u \, du \, \Big| \, \mathcal{F}_t \right) = \mathbb{1}_{\{t<\tau\}} \, \widetilde{S}_t(\kappa)$$

where $\widetilde{S}_t(\kappa)$ is the pre-default ex-dividend price.

Market CDS Rate

The market CDS rate is defined similarly as the forward swap rate in a (default-free) interest rate swap.

Definition 5 *The T-maturity market CDS rate $\kappa(t, T)$ at time t is the level of a CDS rate κ for which the values of the two legs of a CDS are equal at time t.*

By assumption, κ is an $\mathcal{F}_t$-measurable random variable.

The T-maturity market CDS rate $\kappa(t, T)$ is given by the formula

$$\kappa(t, T) = -\frac{\mathbb{E}_{\mathbb{Q}^*}\left(\int_t^T \delta_u \, dG_u \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{Q}^*}\left(\int_t^T G_u \, du \,\middle|\, \mathcal{F}_t\right)}, \quad \forall \, t \in [0, T].$$

We fix a maturity date T, and we shall frequently write κ_t instead of $\kappa(t, T)$.

Single Name: Deterministic Default Intensity

Ex-dividend Price of a CDS

Standing assumptions:

- Assume that $\mathbb{F}$ is trivial, and the survival function $G(t)$ satisfies

$$G(t) = \mathbb{Q}^*\{\tau > t\} = \exp\Big(-\int_0^t \gamma(u)\, du \Big).$$

- We postulate that the default protection $\delta : [0,T] \to \mathbb{R}$ is deterministic.

In that case, the ex-dividend price at time $t \in [0,T]$ of a CDS with the spread κ and protection payment $\delta(\tau)$ at default equals

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G(t)} \left(-\int_t^T \delta(u)\, dG(u) - \kappa \int_t^T G(u)\, du \right) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa).$$

Market CDS Rate

- The T-maturity market CDS rate $\kappa(t,T)$ solves the following equation

$$\int_t^T \delta(u)\,dG(u) + \kappa(t,T)\int_t^T G(u)\,du = 0.$$

- We thus have, for every $t \in [0,T]$,

$$\kappa(t,T) = -\frac{\int_t^T \delta(u)\,dG(u)}{\int_t^T G(u)\,du}.$$

- We fix a maturity date T, and we write briefly $\kappa(t)$ instead of $\kappa(t,T)$.

- In addition, we assume that all CDSs with different starting dates have a common recovery function δ.

Market CDS Rate: Special Case

- Assume that $\delta(t) = \delta$ is constant, and $F(t) = 1 - e^{-\gamma t}$ for some constant default intensity $\gamma > 0$ under $\mathbb{Q}^*$.
- The ex-dividend price of a (spot) CDS with rate κ equals, for every $t \in [0, T]$,

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}}(\delta\gamma - \kappa)\gamma^{-1}\Big(1 - e^{-\gamma(T-t)}\Big).$$

- The last formula yields $\kappa(s, T) = \delta\gamma$ for every $s < T$, so that the market rate $\kappa(s, T)$ is here independent of s.
- As a consequence, the ex-dividend price of a market CDS started at s equals zero not only at the inception date s, but indeed at any time $t \in [s, T]$, both prior to and after default).
- Hence, this process follows a trivial martingale under $\mathbb{Q}^*$.

Price Dynamics of a CDS

The following result furnishes risk-neutral dynamics of the ex-dividend price of a CDS with spread κ and maturity T.

Proposition 1 *The dynamics of the ex-dividend price $S_t(\kappa)$ on $[s, T]$ are*

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt$$

where the $\mathbb{H}$-martingale M under a spot martingale measure $\mathbb{Q}^*$ is given by the formula

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u)\, du, \quad \forall\, t \in \mathbb{R}_+.$$

Prior to default, we have

$$d\widetilde{S}_t(\kappa) = \widetilde{S}_t(\kappa)\gamma(t)\, dt + (\kappa - \delta(t)\gamma(t))\, dt.$$

At default, the ex-dividend price process jumps to 0.

Replication of Defaultable Claims

We assume that the following two assets are traded:

- a CDS with maturity $U \geq T$,
- the constant savings account $B = 1$ (this is not restrictive).

Let ϕ^0, ϕ^1 be an $\mathbb{H}$-predictable processes and let $C : [0, T] \to \mathbb{R}$ be a function of finite variation with $C_0 = 0$.

Definition 6 *We say that $(\phi, C) = (\phi^0, \phi^1, C)$ is a* ***trading strategy with financing cost*** C *if the wealth process $V(\phi, C)$, defined as*

$$V_t(\phi, C) = \phi_t^0 + \phi_t^1 S_t(\kappa),$$

where $S_t(\kappa)$ is the ex-dividend price of a CDS at time t, satisfies

$$dV_t(\phi, C) = \phi_t^1 \big(dS_t(\kappa) + dD_t \big) - dC(t)$$

where D is the dividend process of a CDS.

Recall that a generic defaultable claim (X, A, Z, τ) consists of

1. a promised claim X,
2. a function A representing dividends stream,
3. a recovery function Z,
4. a default time τ.

Definition 7 *A trading strategy* (ϕ, C) *replicates a defaultable claim* (X, A, Z, τ) *if:*

1. *the processes* $\phi = (\phi^0, \phi^1)$ *and* $V(\phi, C)$ *are stopped at* $\tau \wedge T$*,*
2. $C(\tau \wedge t) = A(\tau \wedge t)$ *for every* $t \in [0, T]$*,*
3. *we have* $V_{\tau \wedge T}(\phi, C) = Y$*, where the random variable* Y *equals*

$$Y = X \mathbb{1}_{\{\tau > T\}} + Z(\tau) \mathbb{1}_{\{\tau \leq T\}}.$$

Risk-Neutral Valuation of a Defaultable Claim

- Let us denote, for every $t \in [0,T]$,

$$\widetilde{Z}(t) = \frac{1}{G(t)} \left(XG(T) - \int_t^T Z(u)\, dG(u) \right)$$

and

$$\widetilde{A}(t) = \frac{1}{G(t)} \int_{]t,T]} G(u)\, dA(u).$$

- Let π and $\widetilde{\pi}$ be the risk-neutral value and the pre-default risk-neutral value of a defaultable claim, so that $\pi_t = \mathbb{1}_{\{t<\tau\}} \widetilde{\pi}(t)$ for $t \in [0,T]$.

- Let $\widehat{\pi}$ stand for the risk-neutral cumulative value

$$\widehat{\pi}_t = B_t \, \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big).$$

- It is clear that $\pi(0) = \widetilde{\pi}(0) = \widehat{\pi}(0)$.

Price Dynamics of a Defaultable Claim

Proposition 2 *The pre-default risk-neutral value of a defaultable claim* (X, A, Z, τ) *equals*

$$\widetilde{\pi}(t) = \widetilde{Z}(t) + \widetilde{A}(t)$$

and thus

$$d\widetilde{\pi}(t) = \gamma(t)(\widetilde{\pi}(t) - Z(t))\, dt - dA(t).$$

Moreover

$$d\pi_t = -\widetilde{\pi}(t-)\, dM_t - \gamma(t)(1 - H_t)Z(t)\, dt - dA(t \wedge \tau)$$

and

$$d\widehat{\pi}_t = (Z(t) - \widetilde{\pi}(t-))\, dM_t.$$

Replication of a Defaultable Claim

Proposition 3 *Assume that the inequality $\widetilde{S}_t(\kappa) \neq \delta(t)$ holds for every $t \in [0, T]$.*

Let $\phi^1_t = \widetilde{\phi}_1(\tau \wedge t)$, where the function $\widetilde{\phi}_1 : [0, T] \to \mathbb{R}$ is given by the formula

$$\widetilde{\phi}_1(t) = \frac{Z(t) - \widetilde{\pi}(t-)}{\delta(t) - \widetilde{S}_t(\kappa)}, \quad \forall\, t \in [0, T],$$

and let $\phi^0_t = V_t(\phi, A) - \phi^1_t S_t(\kappa)$, where the process $V(\phi, A)$ is given by the formula

$$V_t(\phi, A) = \widetilde{\pi}(0) + \int_{]0, \tau \wedge t]} \widetilde{\phi}_1(u)\, d\widehat{S}_u(\kappa) - A(t \wedge \tau).$$

Then the strategy (ϕ^0, ϕ^1, A) replicates a defaultable claim (X, A, Z, τ).

Several Names: Deterministic Default Intensities

First-to-Default Intensities and Martingales

Assumptions and Objectives

Let $\tau_1, \ldots, \tau_n$ be default times of n reference entities.

Assume that:

1. The joint distribution of default times $(\tau_1, \ldots, \tau_n)$ is known.
2. The protection payments at default are known functions of time, number of defaults and names of defaulted entities.
3. Single-name CDSs for n reference entities are traded.

We will argue that it is possible to replicate a basket CDS with single-name CDSs under mild technical assumptions of non-degeneracy (a system of linear equations).

It suffices to consider the case of a first-to-default claim and then to use the backward induction.

Default Times and Filtrations

- Let $\tau_1, \tau_2, \dots, \tau_n$ be the default times associated with n names, respectively.
- Let
$$F(t_1, t_2, \dots, t_n) = \mathbb{Q}^*(\tau_1 \le t_1, \tau_2 \le t_2, \dots, \tau_n \le t_2)$$
denote the joint distribution function of the default times associated with the n names.
- For each $i = 1, 2, \dots, n$ we define the default indicator process for the ith credit name as $H_t^i = 1\!\!1_{\{\tau_i \le t\}}$ and the σ-field $\mathbb{H}_t^i = \sigma(H_u^i : u \le t)$.
- We write
$$\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2 \vee \cdots \vee \mathbb{H}^n$$
and
$$\mathbb{G}^i = \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^{i-1} \vee \mathbb{H}^{i+1} \vee \cdots \vee \mathbb{H}^n$$
so that $\mathbb{G} = \mathbb{G}^i \vee \mathbb{H}^i$ for $i = 1, 2, \dots, n$.

First-to-Default Intensities

Definition 8 *The ith first-to-default intensity is the function*

$$\widetilde{\lambda}_i(t) = \lim_{h \downarrow 0} \frac{1}{h} \frac{\mathbb{Q}^*(t < \tau_i \leq t + h \,|\, \tau_1 > t, \ldots, \tau_{i-1} > t, \tau_{i+1} > t, \ldots, \tau_n > t)}{\mathbb{Q}^*(\tau_i > t \,|\, \tau_1 > t, \ldots, \tau_{i-1} > t, \tau_{i+1} > t, \ldots, \tau_n > t)}$$

$$= \lim_{h \downarrow 0} \frac{1}{h} \mathbb{Q}^*(t < \tau_i \leq t + h \,|\, \tau_{(1)} > t).$$

Definition 9 *The first-to-default intensity* $\widetilde{\lambda}$ *is defined as the sum* $\widetilde{\lambda} = \sum_{i=1}^{n} \widetilde{\lambda}_i$*, or equivalently, as the intensity function of the random time* $\tau_{(1)}$ *modeling the moment of the first default.*

First-to-Default Martingales

- Let λ^i be the $\mathbb{G}^i$-intensity of the ith default time. The process M^i given by the formula

$$M_t^i = H_t^i - \int_0^t (1 - H_u^i)\lambda_u^i \, du, \quad \forall\, t \in \mathbb{R}_+,$$

is known to be a $\mathbb{G}$-martingale under $\mathbb{Q}^*$.

- A random time $\tau_{(1)}$ is manifestly a $\mathbb{G}$-stopping time. Therefore, for each $i = 1, 2, \ldots, n$, the process $\widehat{M}^i$, given by the formula

$$\widehat{M}_t^i := M_{t\wedge\tau_{(1)}}^i = H_{t\wedge\tau_{(1)}}^i - \int_0^t \mathbb{1}_{\{\tau_{(1)}>u\}} \widetilde{\lambda}_i(u) \, du, \quad \forall\, t \in \mathbb{R}_+,$$

also follows a $\mathbb{G}$-martingale under $\mathbb{Q}^*$.

- Processes $\widehat{M}^i$ are referred to as the basic first-to-default martingales.

Traded Credit Default Swaps

- As traded assets, we take the constant savings account and a family of single-name CDSs with default protections δ_i and rates κ_i.
- For convenience, we assume that the CDSs have the same maturity T, but this assumption can be easily relaxed. The ith traded CDS is formally defined by its dividend process

$$D_t^i = \int_{(0,t]} \delta_i(u)\, dH_u^i - \kappa_i (t \wedge \tau_i), \quad \forall\, t \in [0, T].$$

- Consequently, the price at time t of the ith CDS equals

$$S_t^i(\kappa_i) = \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{t<\tau_i \le T\}} \delta_i(\tau_i) \,\big|\, \mathcal{G}_t\big) - \kappa_i\, \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{t<\tau_i\}}\big((\tau_i \wedge T) - t\big) \,\big|\, \mathcal{G}_t\big).$$

- To replicate a first-to-default claim, we only need to examine the dynamics of each CDS on the interval $[0, \tau_{(1)} \wedge T]$.

The Value of a CDS at Default

- For any $j \neq i$, we define a function $S^i_{t|j}(\kappa_i), t \in [0,T]$, which represents the ex-dividend price of the ith CDS at time t on the event $\{\tau_{(1)} = \tau_j = t\}$.
- Formally, this quantity is defined as the unique function satisfying

$$\mathbb{1}_{\{\tau_{(1)}=\tau_j \leq T\}} S^i_{\tau_{(1)}|j}(\kappa_i) = \mathbb{1}_{\{\tau_{(1)}=\tau_j \leq T\}} S^i_{\tau_{(1)}}(\kappa_i)$$

so that

$$\mathbb{1}_{\{\tau_{(1)} \leq T\}} S^i_{\tau_{(1)}}(\kappa_i) = \sum_{j \neq i} \mathbb{1}_{\{\tau_{(1)}=\tau_j \leq T\}} S^i_{\tau_{(1)}|j}(\kappa_i).$$

- Let $m = 2$. Then the function $S^1_{t|2}(\kappa_1)$, $t \in [0,T]$, is the price of the first CDS at time t on the event $\{\tau_{(1)} = \tau_2 = t\}$.

Lemma 3 *The function* $S^1_{v|2}(\kappa_1)$, $v \in [0,T]$, *equals*

$$S^1_{v|2}(\kappa_1) = \frac{\int_v^T \delta_1(u) f(u,v) du}{\int_v^\infty f(u,v)\, du} - \kappa_1 \frac{\int_v^T du \int_u^\infty f(z,v)\, dz}{\int_v^\infty f(u,v)\, du}.$$

Price Dynamics of the ith CDS

Proposition 4 *The dynamics of the pre-default ex-dividend price* $\widetilde{S}^i_t(\kappa_i)$ *are*

$$d\widetilde{S}^i_t(\kappa_i) = \widetilde{\lambda}_i(t)\big(\widetilde{S}^i_t(\kappa_i) - \delta_i(t)\big)\,dt + \sum_{j\neq i}\widetilde{\lambda}_j(t)\big(\widetilde{S}^i_t(\kappa_i) - S^i_{t|j}(\kappa_i)\big)\,dt + \kappa_i\,dt.$$

The cumulative ex-dividend price of the i*th CDS stopped at* $\tau_{(1)}$ *satisfies*

$$\widehat{S}^i_t(\kappa_i) = S^i_t(\kappa_i) + \int_0^t \delta_i(u)\,dH^i_{u\wedge\tau_{(1)}} + \sum_{j\neq i}\int_0^t S^i_{u|j}(\kappa_i)\,dH^j_{u\wedge\tau_{(1)}} - \kappa_i(\tau_{(1)}\wedge t),$$

and thus

$$d\widehat{S}^i_t(\kappa_i) = \big(\delta_i(t) - \widetilde{S}^i_{t-}(\kappa_i)\big)\,d\widehat{M}^i_t + \sum_{j\neq i}\big(S^i_{t|j}(\kappa_i) - \widetilde{S}^i_{t-}(\kappa_i)\big)\,d\widehat{M}^j_t.$$

Replication of First-to-Default Claims

Definition 10 *A first-to-default claim (an FTDC, for short) on a basket of n credit names is a defaultable claim $(X, A, Z, \tau_{(1)})$, where*

1. *X is a constant amount payable at maturity if no default occurs,*
2. *$A : [0, T] \to \mathbb{R}$ with $A_0 = 0$ is a function of bounded variation representing the dividend stream up to $\tau_{(1)}$,*
3. *$Z = (Z_1, Z_2, \ldots, Z_n)$, where a function $Z_i : [0, T] \to \mathbb{R}$ specifies the recovery payment made at the time τ_i if the ith firm was the first defaulted firm, that is, on the event $\{\tau_i = \tau_{(1)} \leq T\}$.*

Pricing of an FTDC

Proposition 5 *The pre-default risk-neutral value of an FTDC equals*

$$\widetilde{\pi}(t) = \sum_{i=1}^{n} \frac{\Psi_i(t)}{G_{(1)}(t)} + \frac{1}{G_{(1)}(t)} \int_t^T G_{(1)}(u)\, dA(u) + X \frac{G_{(1)}(T)}{G_{(1)}(t)}$$

where

$$\begin{aligned} \Psi_i(t) \quad = \quad & \int_{u_i=t}^{T} \int_{u_1=u_i}^{\infty} \cdots \int_{u_{i-1}=u_i}^{\infty} \int_{u_{i+1}=u_i}^{\infty} \cdots \int_{u_n=u_i}^{\infty} \\ & Z_i(u_i) F(du_1, \ldots, du_{i-1}, du_i, du_{i+1}, \ldots, du_n). \end{aligned}$$

Price Dynamics of an FTDC

Proposition 6 *The pre-default risk-neutral value of an FTDC satisfies*

$$d\widetilde{\pi}(t) = \sum_{i=1} \widetilde{\lambda}_i(t)\big(\widetilde{\pi}(t) - Z_i(t)\big)\,dt - dA(t).$$

Moreover, the risk-neutral value of an FTDC satisfies

$$d\pi_t = \sum_{i=1}^{n} \left(Z_i(t) - \widetilde{\pi}(t-)\right)\, d\widehat{M}^i_u - dA(\tau_{(1)} \wedge t),$$

and the risk-neutral cumulative value $\widehat{\pi}$ of an FTDC satisfies

$$d\widehat{\pi}_t = \sum_{i=1}^{n} \left(Z_i(t) - \widetilde{\pi}(t-)\right)\, d\widehat{M}^i_u.$$

Self-financing Strategies with CDSs

- Consider a family of single-name CDSs with default protections δ_i and rates κ_i.
- For convenience, we assume that they have the same maturity T; this assumption can be easily relaxed.

Definition 11 *A trading strategy $\phi = (\phi^0, \phi^1, \ldots, \phi^n)$, in assets $(B, S^1(\kappa_1), \ldots, S^n(\kappa_n))$ is self-financing with financing cost C if its wealth process $V(\phi)$, defined as*

$$V_t(\phi) = \phi_t^0 + \sum_{i=1}^{n} \phi_t^i S_t^i(\kappa_i),$$

satisfies

$$dV_t(\phi) = \sum_{i=1}^{n} \phi_t^i \big(dS_t^i(\kappa_i) + dD_t^i \big) - dC_t,$$

where $S^i(\kappa_i)$ is the ex-dividend price of the ith CDS.

Standing Assumption

- We assume that $\det N(t) \neq 0$ for every $t \in [0, T]$, where

$$N(t) = \begin{bmatrix} \delta_1(t) - \widetilde{S}^1_t(\kappa_1) & S^2_{t|1}(\kappa_2) - \widetilde{S}^2_t(\kappa_2) & . & S^n_{t|1}(\kappa_n) - \widetilde{S}^n_t(\kappa_n) \\ S^1_{t|2}(\kappa_1) - \widetilde{S}^1_t(\kappa_1) & \delta_2(t) - \widetilde{S}^2_t(\kappa_2) & . & S^n_{t|2}(\kappa_n) - \widetilde{S}^n_t(\kappa_n) \\ . & . & . & . \\ S^1_{t|n}(\kappa_1) - \widetilde{S}^1_t(\kappa_1) & S^2_{t|n}(\kappa_1) - \widetilde{S}^2_t(\kappa_1) & . & \delta_n(t) - \widetilde{S}^n_t(\kappa_n) \end{bmatrix}$$

- Let $\widetilde{\phi}(t) = (\widetilde{\phi}_1(t), \widetilde{\phi}_2(t), \ldots, \widetilde{\phi}_n(t))$ be the unique solution to the equation

$$N(t)\widetilde{\phi}(t) = h(t)$$

where $h(t) = (h_1(t), h_2(t), \ldots, h_n(t))$ with $h_i(t) = Z_i(t) - \widetilde{\pi}(t-)$.

Replication of an FTDC

Proposition 7 *Let the functions* $\widetilde{\phi}_1, \widetilde{\phi}_2, \dots, \widetilde{\phi}_n$ *satisfy for* $t \in [0, T]$

$$\widetilde{\phi}_i(t)\big(\delta_i(t) - \widetilde{S}^i_t(\kappa_i)\big) + \sum_{j \neq i} \widetilde{\phi}_j(t)\big(S^j_{t|i}(\kappa_j) - \widetilde{S}^j_t(\kappa_j)\big) = Z_i(t) - \widetilde{\pi}(t-).$$

Let $\phi^i_t = \widetilde{\phi}_i(\tau_{(1)} \wedge t)$ *for* $i = 1, 2, \dots, n$ *and*

$$\phi^0_t = V_t(\phi, A) - \sum_{i=1}^{n} \phi^i_t S^i_t(\kappa_i), \quad \forall\, t \in [0, T],$$

where the process $V(\phi, A)$ *is given by the formula*

$$V_t(\phi, A) = \widetilde{\pi}(0) + \sum_{i=1}^{n} \int_{]0, \tau_{(1)} \wedge t]} \widetilde{\phi}_i(u)\, d\widehat{S}^i_u(\kappa_i) - A(\tau_{(1)} \wedge t).$$

Then the trading strategy (ϕ, A) *replicates an FTDC* $(X, A, Z, \tau_{(1)})$*.*

Final Remarks

In a single-name case:

- we first considered the case of a default time with a deterministic intensity,
- we have shown that a generic defaultable claim can be replicated by dynamic trading in a CDS and the savings account,
- the extension to the case of non-trivial reference filtration was not presented.

In a multi-name case:

- we first considered the case of a finite family of default times with known joint distribution,
- the replicating strategy for a first-to-default claim was examined; the method can be extended to kth-to-default claims,
- in the next step, the approach was extended to the case of a reference filtration generated by a multi-dimensional Brownian motion.