EMS SCHOOL

Risk Theory and Related Topics

Bedlewo, Poland. september 29th- october 8th, 2008

Enlargement of filtrations and Credit Risk

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Let $\mathbf{F}$ be a given filtration and $\mathbf{G}$ be a larger filtration, i.e., $\mathcal{F}_t \subset \mathcal{G}_t, \ \forall t$.

Questions:

1) Find conditions such that any $\mathbf{F}$-martingale $M^{\mathbf{F}}$ remains a $\mathbf{G}$-semi-martingale.

2) Under these conditions, find the canonical decomposition of $M^{\mathbf{F}}$ as a $\mathbf{G}$-semimartingale:

$$M_t^{\mathbf{F}} = M_t^{\mathbf{G}} + A_t$$

where A is a $\mathbf{G}$-predictable process with bounded variation and $M^{\mathbf{G}}$ a $\mathbf{G}$-martingale.

We shall study three cases of enlargement of filtrations

Immersion of filtration where $\mathbf{F}$-martingales remain $\mathbf{G}$-martingales

Initial enlargement where L is a given random variable and $\mathcal{G}_t = \mathcal{F}_t^{(L)} := \mathcal{F}_t \vee \sigma(L)$

Progressive enlargement where τ is a given random time and $\mathcal{G}_t = \mathcal{F}_t^{\tau} := \mathcal{F}_t \vee \mathcal{H}_t$ where $(\mathcal{H}_t, t \geq 0)$ is the natural filtration of the process $H_t = 1\!\!1_{\{\tau \leq t\}}$

Immersion property

The filtration **F** is said to be **immersed** in **G** if any (square-integrable) **F**-martingale is a **G**-martingale. It is also referred to as the $(\mathcal{H})$ hypothesis.

In the particular case of progressive enlargement, hypothesis $(\mathcal{H})$ is equivalent to

$$\mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\tau \leq t|\mathcal{F}_\infty)$$

In particular,

$$F_t := \mathbb{P}(\tau \leq t|\mathcal{F}_t)$$

is increasing.

If the sub-martingale

$$F_t := \mathbb{P}(\tau \leq t | \mathcal{F}_t)$$

is increasing, then for any $\mathbf{F}$-martingale m, the process $(m_{t\wedge\tau}, t \geq 0)$ is a local $\mathcal{F}^\tau$-martingale.

This is equivalent to: for any bounded $\mathcal{F}$-martingale m, $\mathbb{E}(m_\tau) = m_0$.

Initial enlargements

Let $\mathbf{F}$ be a Brownian filtration generated by W and $\mathcal{F}_t^{(L)} = \mathcal{F}_t \vee \sigma(L)$.
Let $\lambda_t(\omega, dx)$ be the family of regular conditional distribution,

$$\lambda_t(\cdot, A) = \mathbb{E}(\mathbb{1}_{\{L \in A\}} | \mathcal{F}_t)$$

(Jacod's criterion.) Suppose that, for each t, $\lambda_t(\omega, dx) << \eta(dx)$ where η is the law of L. **Then, every F- martingale X is an $\mathbf{F}^{(L)}$-semi-martingale.**

Moreover, if $\lambda_t(\omega, dx) = f_t(\omega, x)\eta(dx)$, then any $\mathbf{F}$-martingale X admits the decomposition

$$X_t = \widetilde{X}_t + \int_0^t \frac{d\langle f(L; \cdot), X\rangle_s}{f_s(L)}$$

where $\widetilde{X}$ is an $\mathbf{F}^{(L)}$-martingale.

Grorud and Pontier result

Let

$$d\mathbb{Q}|_{\mathcal{F}_T^{(L)}} = 1/f_T(L)\, d\mathbb{P}|_{\mathcal{F}_T^{(L)}}$$

Then, L and $(W_t, t \leq T)$ are independent under $\mathbb{Q}$.

Progressive enlargement

Let τ be a random time on $(\Omega, \mathbf{F}, \mathbb{P})$ and $\mathbf{G} = \mathbf{F} \vee \mathbf{H} = \mathbf{F}^\tau$. The supermartingale $G_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$ admits a Doob-Meyer decomposition as $G_t = Z_t - A_t$.

Then, if X is an $\mathbf{F}$-martingale, the process X^τ defined as

$$X_t^\tau = X_{t \wedge \tau}$$

is a $\mathbf{G}$-semi-martingale and its $\mathbf{G}$-decomposition is

$$X_t^\tau = \widetilde{X}_t + \int_0^{t \wedge \tau} \frac{d\langle X, G \rangle_s}{G_{s-}}$$

where $\widetilde{X}$ is a $\mathbf{G}$-martingale (Jeulin's result).

The random time τ is **honest** if τ is equal, on $\{\tau < t\}$ to an $\mathcal{F}_t$-measurable random variable. In particular, τ is $\mathcal{F}_\infty$-measurable.

Example: if X is a transient diffusion, the last passage time Λ_a is honest.

A key point is the following description of $\mathbf{F}^\tau$-predictable processes: if τ is honest, and if Z is an $\mathbf{F}^\tau$-predictable process, then there exist two $\mathbf{F}$-predictable processes z and $\widetilde{z}$ such that

$$Z_t = z_t \mathbb{1}_{\{\tau > t\}} + \widetilde{z}_t \mathbb{1}_{\{\tau \leq t\}} \,.$$

We assume that τ is honest and avoids $\mathbf{F}$-stopping times.

Then, if X is an $\mathbf{F}$-local martingale, there exists an $\mathbf{F}^\tau$-local martingale $\widetilde{X}$ such that

$$X_t = \widetilde{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X, G\rangle_s}{G_{s-}} - \int_{t\wedge\tau}^{t} \frac{d\langle X, G\rangle_s}{1 - G_{s-}} \,.$$

Initial times

Let $(\Omega, \mathbf{F}, \mathbb{P})$ be a given filtered probability space, τ a random time and

$$H_t = 1\!\!1_{\tau \leq t}$$

Let $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ where $\mathcal{H}_t = \sigma(H_s, s \leq t)$ and, for any (t, θ),

$$G_t(\theta) = \mathbb{P}(\tau > \theta | \mathcal{F}_t)$$

the conditional survival process.

We assume that $G_t := G_t(t)$ is continuous.

The positive random time τ is called an **initial time** if it satisfies Jacod's criterion. Then,

$$G_t(\theta) = \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty f_t(u)\eta(du)\,.$$

From $G_s(\theta) = \mathbb{E}(G_t(\theta)|\mathcal{F}_s)$ for any $s \leq t$, it follows that for any $u \geq 0$, $(f_t(u))_t$ is a non-negative $\mathbb{F}$-martingale.

The Doob-Meyer decomposition of $G_t(t)$ is

$$G_t = G_t(t) = G_0(t) + \int_0^t g_s(s)dW_s - \int_0^t f_s(s)\eta(ds)$$

where $G_t(\theta) = G_0(\theta) + \int_0^t g_s(\theta)dW_s$.

- Under the condition that the initial time τ avoids the $\mathbb{F}$-stopping times, there is equivalence between $\mathbb{F}$ is immersed in $\mathbb{G}$ and for any $u \geq 0$, the martingale $(f_t(u), t \geq 0)$ is constant after u.

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• Let $(K_t(u))_{t \geq 0}$ be a family of $\mathbb{F}$-predictable processes indexed by $u \geq 0$. Then

$$\mathbb{E}\left(K_t(\tau) \middle| \mathcal{F}_t \right) = \int_0^\infty K_t(u) f_t(u) \eta\,(du) \qquad (*)$$

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$$\mathbb{E}\left(K_t(\tau) | \mathcal{F}_t \right) = \int_0^\infty K_t(u) f_t(u) \eta\,(du) \qquad (*)$$

• If X is an $\mathbb{F}$-martingale

$$\widehat{X}_t := X_t - \int_0^{t\wedge\tau} \frac{d\,\langle X, G\rangle_s}{G_s} - \int_{t\wedge\tau}^{t} \frac{d\,\langle X, f.(\theta)\rangle_s}{f_s(\theta)}\bigg|_{\theta=\tau} \in \mathcal{M}(\mathbb{G}).$$

Proof. We prove that $\widehat{X}$ is a **G**-martingale. Let us consider a $\mathcal{G}_s$-measurable random variable of the form $F_s h(\tau \wedge s)$ with F_s a bounded $\mathcal{F}_s$-measurable random variable and $h : \mathbb{R}^+ \to \mathbb{R}$ a bounded Borel function. Then,

$$\begin{aligned} \mathbb{E}\left(F_s h(\tau \wedge s)\left(\widehat{X}_t - \widehat{X}_s\right)\right) &= \mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s}\left(\widehat{X}_t - \widehat{X}_s\right)\right) \\ &\quad + \mathbb{E}\left(F_s h(s) 1_{s < \tau}\left(\widehat{X}_t - \widehat{X}_s\right)\right) \\ &= a + b \end{aligned}$$

and we can compute each part of the right hand side member:

Computation of $a = \mathbb{E}\left(F_s h(\tau) 1_{\tau \le s}\left(\widehat{X}_t - \widehat{X}_s\right)\right), \quad s < t$**.**

On $\{\tau \le s\}$, $t \wedge \tau = s \wedge \tau = \tau$ hence

$$1_{\tau \le s}\left(\int_0^{t\wedge\tau} \frac{d\left\langle X, G\right\rangle_u}{G_u} - \int_0^{s\wedge\tau} \frac{d\left\langle X, G\right\rangle_u}{G_u}\right) = 0,$$

and it follows that

$$a = \mathbb{E}\left(F_s h(\tau) 1_{\tau \le s}\left(X_t - X_s\right)\right) - \mathbb{E}\left(F_s h(\tau) 1_{\tau \le s}\left(\int_s^t \frac{d\langle X, f_\cdot(\theta)\rangle_u}{f_u(\theta)}|_{\theta=\tau}\right)\right)$$

We prove that $a = 0$

$$\begin{aligned}
\mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s}\left(X_t - X_s\right)\right) &= \mathbb{E}\left(F_s\left(X_t - X_s\right)\int_0^s h(\theta) f_t(\theta)\eta(d\theta)\right) \\
&= \int_0^s h(\theta)\mathbb{E}\left(F_s\left(X_t f_t(\theta) - X_s f_s(\theta)\right)\right)\eta(d\theta) \\
&= \int_0^s h(\theta)\mathbb{E}\left(F_s \int_s^t d\left\langle X, f_.(\theta)\right\rangle_v\right)\eta(d\theta)
\end{aligned}$$

where the first equality comes from a conditioning w.r.t. $\mathcal{F}_t$, the second from the martingale property of $f_.(\theta)$, and the third from integration by parts and the fact that X and $f_.(\theta)$ are martingales.

Moreover, for $dK_v(\theta) = d\left\langle X, f.(\theta)\right\rangle_v / f_v(\theta)$

$$\begin{aligned}\mathbb{E}\left(F_s h(\tau) 1_{\tau\leq s} \int_s^t dK_v(\tau)\right) &= \mathbb{E}\left(F_s \int_0^s h(\theta) \int_s^t dK_v(\theta) f_t(\theta) \eta(d\theta)\right) \\ &= \int_0^s h(\theta) \mathbb{E}\left(F_s \int_s^t f_v(\theta) dK_v(\theta)\right) \eta(d\theta)\end{aligned}$$

where the first equality comes from (*) applied to the $\mathbb{F}$-predictable process indexed by u $J_t^u = h(u)1_{u\leq s} \int_s^t dK_v(u)$ (F_s is F_t-measurable) and the second from the martingale property of $f.(\theta)$

Hence, $a = 0$.

b : We rewrite b as

$$\begin{aligned} b &= \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_t - X_{t\wedge\tau}\right)\right) + \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_{t\wedge\tau} - X_s\right)\right) \\ & -\mathbb{E}\left(F_s h(s) 1_{s<\tau}\int_s^{t\wedge\tau} \frac{d\left\langle X, G\right\rangle_u}{G_u}\right) - \mathbb{E}\left(F_s h(s) 1_{s<\tau}\int_{t\wedge\tau}^t \frac{d\langle X, f.(\tau)\rangle_u}{f_u(\tau)}\right). \end{aligned}$$

Using Jeulin's formula before default, we have

$$\begin{aligned} \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_{t\wedge\tau} - X_s\right)\right) &= \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_{t\wedge\tau} - X_{s\wedge\tau}\right)\right) \\ &= \mathbb{E}\left(F_s h(s) 1_{s<\tau}\int_s^{t\wedge\tau} \frac{d\langle X, G\rangle_u}{G_u}\right), \end{aligned}$$

and it follows

$$\begin{aligned} b &= \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_t - X_{t\wedge\tau}\right)\right) - \mathbb{E}\left(F_s h(s) 1_{s<\tau}\int_{t\wedge\tau}^t \frac{d\langle X, f.(\tau)\rangle_u}{f_u(\tau)}\right) \\ &= \mathbb{E}\left(F_s h(s) 1_{s<\tau\le t}\left(X_t - X_\tau\right)\right) - \mathbb{E}\left(F_s h(s) 1_{s<\tau\le t}\int_\tau^t \frac{d\langle X, f.(\tau)\rangle_u}{f(\tau, u)}\right). \end{aligned}$$

Moreover, we can write the decomposition:

$$\begin{aligned}
\mathbb{E}\left(F_s h(s) 1_{s<\tau\leq t} X_\tau\right) &= \mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} X_v dH_v\right) \\
&= \mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} X_v dA_v\right) \\
&= \mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} X_v f_v(v) \eta(dv)\right)
\end{aligned}$$

where the second equality comes from the definition of the predictable dual projection, and the third from the computation of the Doob Meyer decomposition of G.

It follows

$$
\begin{aligned}
b &= \mathbb{E}\left(F_s h(s) X_t \int_{v\in]s,t]} f_t(v)\eta(dv)\right) - \mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} X_v f_v(v)\eta(dv)\right) \\
&\quad -\mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} \int_{u\in]v,t]} \frac{d\left\langle X, f_{\cdot}(v)\right\rangle_u}{f_u(v)} f_t(v)\eta(dv)\right) \\
&= \mathbb{E}\left(F_s h(s) \int_{v\in]s,t]} \left((X_t f_t(v) - X_v f_v(v)) - \int_{u\in]v,t]} d\left\langle X, f_{\cdot}(v)\right\rangle_u\right)\eta(dv)\right)
\end{aligned}
$$

where the second equality comes from integration by parts formula.

The proof is done.

Example: "Cox-like" construction. Here

- λ is a non-negative $\mathbb{F}$-adapted process, $\Lambda_t = \int_0^t \lambda_s ds$
- Θ is a given r.v. independent of $\mathcal{F}_\infty$ with unit exponential law
- V is a $\mathcal{F}_\infty$ -measurable non-negative random variable
- $\tau = \inf\{t : \Lambda_t \geq \Theta V\}$.

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For any θ and t,

$$G_t(\theta) = \mathbb{P}(\tau \geq \theta|\mathcal{F}_t) = \mathbb{P}(\Lambda_\theta \leq \Theta V|\mathcal{F}_t) = \mathbb{P}\left(\exp -\frac{\Lambda_\theta}{V} \geq e^{-\Theta}\middle|\mathcal{F}_t\right).$$

Let us denote $\exp(-\Lambda_t/V) = 1 - \int_0^t \psi_s ds$, with

$$\psi_s = (\lambda_s/V)\,\exp -\int_0^s (\lambda_u/V)\,du,$$

and define $\gamma_t(s) = \mathbb{E}\,(\,\psi_s|\,\mathcal{F}_t)$. Then, $f_t(s) = \gamma_t(s)/\gamma_0(s)$.

HJM model

Assume that for any $\theta > 0$, the process $(G_t(\theta), 0 \le t)$ satisfies

$$\frac{dG_t(\theta)}{G_t(\theta)} = \Psi(t,\theta)dW_t$$

where $\Psi(t,\theta)$ is an $\mathbb{F}$-adapted process which is differentiable with respect to θ. Similar as in the interest rate modelling, we define the forward rate $\gamma_t(\theta) = -\frac{\partial}{\partial\theta}\ln G_t(\theta)$. If, in addition, $\psi(t,\theta) = \dfrac{\partial}{\partial\theta}\Psi(t,\theta)$ is bounded, then we have

1. $G_t(\theta) = G_0(\theta)\exp\left(\int_0^t \Psi(s,\theta)dW_s - \frac{1}{2}\int_0^t |\Psi(s,\theta)|^2 ds\right)$
2. $\gamma_t(\theta) = \gamma_0(\theta) - \int_0^t \psi(s,\theta)dW_s + \int_0^t \psi(s,\theta)\Psi(s,\theta)^* ds.$
3. $G_t = \exp\left(-\int_0^t \gamma_s(s)ds + \int_0^t \Psi(u,u)dW_u - \frac{1}{2}\int_0^t |\Psi(u,u)|^2 du\right).$

From Grorud and Pontier result

One can start with a model such that, under $\mathbb{P}^0$, τ and $\mathcal{F}_\infty$ are independent.

Let $f_\infty(u)$ be a family of non negative $\mathcal{F}_\infty$-measurable r.vs. such that $\int_0^\infty f_\infty(u)\eta(du) = 1$ where η is a probability law on $I\!R^+$. Let $f_t(u) = \mathbb{E}^0(f_\infty(u)|\mathcal{F}_t)$.

There exists a probability space $(\widehat{\Omega}, \mathbb{Q})$ and a random variable τ such that

(i) The law of τ is η

(ii) The restriction of $\mathbb{Q}$ to $\mathcal{F}_\infty$ is $\mathbb{P}$

(iii) $\mathbb{Q}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty f_t(u)\eta(du)$

In order to have a family of non negative $\mathcal{F}_\infty$-measurable r.vs. such that $\int_0^\infty f_\infty(u)\eta(du) = 1$: start with a family of densities on $\mathbb{R}^+$, $\varphi(\alpha, u)$ where α is a parameter and set $f_\infty(u) = \varphi(X, u)$ for some $\mathcal{F}_\infty$-measurable r.v.

Let $G_t = L^{\mathbb{F}}_t e^{-\int_0^t \lambda^{\mathbb{F}}_s \eta(ds)}$ be the multiplicative decomposition of G.

Any cadlag process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale if and only if there exist an $\mathbb{F}$-adapted cadlag process Y and an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $Y_t(.)$ such that $Y^{\mathbb{G}}_t = Y_t 1\!\!1_{\{\tau > t\}} + Y_t(\tau) 1\!\!1_{\{\tau \le t\}}$ and that

1. $(Y_t G_t + \int_0^t Y_s(s) f_s(s) \eta(ds),\ t \ge 0)$ or equivalently $(L^{\mathbb{F}}_t [Y_t + \int_0^t (Y_s(s) - Y_s) \lambda^{\mathbb{F}}_s \eta(ds)], t \ge 0)$ is an $\mathbb{F}$-local martingale;
2. $((Y_t(\theta) - Y_\theta(\theta)) f_t(\theta), t \ge \theta)$ is an $\mathbb{F}$-martingale on $[\theta, \zeta^\theta)$.

Application to credit risk

Let τ be the default time, $\mathbf{F}$ the reference filtration and $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$. Assume that

$$G_t^\theta = P(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty f_t(u) du$$

We also assume that $G_t(t)$ is continuous. Then the process

$$M_t = H_t - \int_0^t \lambda_s^{\mathbf{G}} ds$$

where

$$\lambda_t^{\mathbb{G}} = \mathbb{1}_{t<\tau} \lambda_t^{\mathbb{F}} = \mathbb{1}_{t<\tau} \frac{f_t(t)}{G_t(t)} = \mathbb{1}_{t<\tau} \frac{f_t(t)}{\int_t^\infty f_t(u) du}.$$

is a $\mathbf{G}$-martingale.

Assume that $\widetilde{S} = (\widetilde{S}_t, t \le T)$ is an $\mathbb{R}^{n+2}$ valued process constructed on $(\Omega, \mathcal{A}, \mathbb{P})$, S^0 denoting the saving accounts, and $\mathbb{G}$ is the natural filtration generated by $\widetilde{S}$.

We emphasize that $\mathbb{P}$ is a probability measure defined on $\mathcal{A}$.

We denote by $\Theta^{\mathbb{G}}_{\mathbb{P}}(\widetilde{S})$ the set of $\mathbb{G}$-e.m.ms, i.e., the set of probability measures $\mathbb{Q}$ defined on $\mathcal{A}$, equivalent to $\mathbb{P}$ on $\mathcal{A}$, such that the discounted process $(\widetilde{S}_t/S^0_t, t \le T)$ is a $(\mathbb{G}, \mathbb{Q})$-local martingale.

In what follows, we assume that $S^0 \equiv 1$.

Assume that $\mathbf{F}$ is the natural filtration of the $\mathbb{R}^{n+1}$-valued process S and that this market is complete. Let $\mathbb{P}^*$ be an e.m.m. (the restriction of $\mathbb{P}^*$ to $\mathbf{F}$ is unique) For every $X \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$, there exists two $\mathbb{G}$-predictable process β and γ such that

$$dX_t = \gamma_t d\widehat{S}_t + \beta_t dM_t.$$

There exists a probability $\mathbb{Q} \in \Theta^{\mathbb{G}}_{\mathbb{P}^*}(S)$ such that immersion property holds under $\mathbb{Q}$

If the market generated by S is incomplete, we assume that the market chooses an e.m.m. $\mathbb{P}^*$. We assume that a default sensitive asset S^{n+2} is traded.

There exists a unique $\mathbb{G}$-e.m.m. $\mathbb{Q} \in \Theta_{\mathbb{P}^*}^{\mathbb{G}}\left(\widetilde{S}\right)$, that preserves $\mathcal{F}_T$, i.e.,

$$\mathbb{E}^{\mathbb{Q}}\left(X_T\right) = \mathbb{E}^*\left(X_T\right),$$

for any $X_T \in L^2\left(\mathcal{F}_T\right)$.

Valuation of Single-Name CDSs

Let us now examine the valuation of single-name CDSs under the assumption that the interest rate equals zero.

We consider the CDS

- with the constant spread κ,
- which delivers $\delta(\tau)$ at time τ if $\tau < T$, where δ_1 is a

deterministic function.

The value of the CDS is

$$
\begin{aligned}
S_t &= \mathbb{1}_{t<\tau}\mathbb{E}(\delta(\tau)\mathbb{1}_{t<\tau\le T} - \kappa((T\wedge\tau)-t)|\mathcal{G}_t) \\
&= \mathbb{1}_{t<\tau}\frac{1}{G_t(t)}\int_t^T (\delta(u)f_t(u) - G_t(u)\kappa)\,du
\end{aligned}
$$

Recall that

$$G_t = G_t^t = G_0^t + \int_0^t g_s(s) dW_s + \int_0^t f_s(s)\eta(ds)$$

The dynamics of a CDS can be obtained in a closed form

$$\begin{aligned} dS_t(\kappa) &= r(t)S_t(\kappa)\,dt - S_{t-}(\kappa)\,dM_t + (\kappa - \frac{f_t(t)}{G_t(t)}\delta(t))(1-H_t)dt \\ & +(1-H_t)\sigma_t(T)\left(dW_t - \frac{g_t(t)}{G_t(t)}\,dt\right). \end{aligned}$$

If immersion property holds, then $g_t(t) = 0$, otherwise

$$W_{t\wedge\tau} - \int_0^{t\wedge\tau} \frac{g_s(s)}{G_s(s)} ds$$

is a **G**-martingale

Several Defaults

For any $t_1, t_2, t \geq 0$, we assume that the density process $(f_t(t_1, t_2), t \geq 0)$ of (τ_1, τ_2) exists, i.e.

$$G_t(t_1, t_2) = \mathbb{P}(\tau_1 > t_1, \tau_2 > t_2 | \mathcal{F}_t) = \int_{t_1}^{\infty} du_1 \int_{t_2}^{\infty} du_2 \, f_t(u_1, u_2).$$

Let

$$G_t(t_1, t_2) = G_0(t_1, t_2) + \int_0^t g_s(t_1, t_2) dW_s$$

The process

$$M_t^1 = H_t^1 - \int_0^{t\wedge\tau_1\wedge\tau_2} \widetilde{\lambda}_u^1\, du - \int_{t\wedge\tau_1\wedge\tau_2}^{t\wedge\tau_1} \lambda_u^{1|2}(\tau_2)du,$$

is a **G**-martingale. Here

$$\widetilde{\lambda}_t^i = -\frac{\partial_i G_t(t,t)}{G_t(t,t)}, i = 1, 2 \quad \lambda_t^{1|2}(s) = -\frac{f_t(t,s)}{\partial_2 G_t(t,s)}$$

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We consider the CDS

- with the constant spread κ,
- which delivers $\delta(\tau_1)$ at time τ_1 if $\tau_1 < T$, where δ_1 is a

deterministic function.

The value $S(\kappa_1)$ of this CDS, computed in the filtration $\mathbf{G}$, i.e., taking care on the information on the second default contained in that filtration, is computed in two successive steps.

On the set $t < \tau_{(1)} = \tau_1 \wedge \tau_2$, the ex-dividend price of the CDS equals

$$S_t(\kappa) = \widetilde{S}_t(\kappa) = \frac{1}{G_t(t,t)} \left(- \int_t^T \delta(u) \partial_1 G_t(u,t)\, du - \kappa \int_t^T G_t(u,t)\, du \right).$$

On the set $t < \tau_{(1)} = \tau_1 \wedge \tau_2$, the ex-dividend price of the CDS equals

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On the event $\{\tau_2 \leq t < \tau_1\}$, we have that

$$S_t^1(\kappa) = \frac{1}{\partial_2 G_t(t,\tau_2)} \left(- \int_t^T \delta(u) f_t(u,\tau_2)\, du - \kappa \int_t^T \partial_2 G_t(u,\tau_2)\, du \right).$$

Price Dynamics of Single-Name CDSs

By applying the Itô-Wentzell theorem, we get

$$\begin{aligned} G_t(u,t) &= G_0(u,0) + \int_0^t g_s(u,s)\,dW_s + \int_0^t \partial_2 G_s(u,s)\,ds \\ G_t(t,t) &= G_0(0,0) + \int_0^t g_s(s,s)\,dW_s + \int_0^t (\partial_1 G_s(s,s) + \partial_2 G_s(s,s))\,ds. \end{aligned}$$

If immersion property holds between **F** and **G**, the dynamics of the process $\widetilde{S}(\kappa)$ are

$$d\widetilde{S}_t(\kappa) = \Big(- \widetilde{\lambda}_t^1 \delta(t) + \kappa + \widetilde{\lambda}_t \widetilde{S}_t(\kappa) - \widetilde{\lambda}_t^2 S_{t|2}(\kappa) \Big) dt + \sigma_t(T)\, dW_t$$

where

$$\begin{aligned} \sigma_t(T) &= -\frac{1}{G_t(t,t)} \left(\int_t^T \left(\delta(u)\, \partial_1 g_t(u,t) + \kappa g_t(u,t) \right) du \right) \\ S_{t|2}(\kappa) &= \frac{1}{\partial_2 G_t(t,t)} \left(- \int_t^T \delta(u) f_t(u,t)\, du - \kappa \int_t^T \partial_2 G_t(u,t)\, du \right). \end{aligned}$$

The cumulative price (with $B_t = e^{rt}$ the price of the savings account)

$$S_t^c(\kappa) = S_t(\kappa) + B_t \int_{]0,t]} B_u^{-1}\, dD_u$$

where

$$D_t = D_t(\kappa, \delta, T, \tau_1) = \delta(\tau_1) 1\!\!1_{\{\tau_1 \leq t\}} - \kappa(t \wedge (T \wedge \tau_1))$$

satisfies, on $[0, T \wedge \tau_{(1)}]$,

$$dS_t^c(\kappa) \quad = \quad (\delta(t) - \widetilde{S}_t(\kappa))\, d\widehat{M}_t^1 + (S_{t|2}(\kappa) - \widetilde{S}_t(\kappa))\, d\widehat{M}_t^2 + \sigma_t(T) dW_t \,.$$

On $\tau_1 > t > \tau_2$

$$dS_t = \sigma_t^{1|2}(T)dW_t + (\delta(t)\lambda_t^{1|2}(\tau_2) - \kappa + \widehat{S}_t\lambda_t^{1|2}(\tau_2))dt$$

where

$$\begin{aligned}
\sigma_t^{1|2}(T) &= -\int_t^T \delta(u)\partial_1\partial_2 g_t(u,\tau_2)du - \kappa\int_t^T \partial_2 g_t(u,\tau_2)du \\
\lambda^{1|2}(t,s) &= -\frac{f_t(t,s)}{\partial_2 G_t(t,s)}
\end{aligned}$$

In the general setting, the dynamics of the process $\widetilde{S}^1(\kappa)$ (the predefault-price) are

$$\begin{aligned} d\widetilde{S}_t(\kappa) \quad = \quad & \frac{1}{G_t(t,t)}\Big[\delta(t)\partial_1 G_t(t,t) + \kappa G_t(t,t) \\ & -\big(\partial_1 G_t(t,t) + \partial_2 G_t(t,t)\big)\widetilde{S}_t(\kappa) \\ & - \int_t^T \big(\delta(u) f_t(u,t) + \kappa \partial_2 G_t(u,t)\big)\, du\Big] dt \\ & + \sigma_t(T)\,(dW_t - \frac{g_t(t,t)}{G_t(t,t)} dt) \end{aligned}$$

with

$$\sigma_t(T) = -\frac{1}{G_t(t,t)}\left(\int_t^T \big(\delta(u)\,\partial_1 g_t(u,t) + \kappa g_t(u,t)\big)\, du + g_t(t,t)\widetilde{S}_t(\kappa)\right).$$

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