On Stochastic Adaptive Control & its Applications

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1. Motivation: Work in the 1970's
2. Adaptive Control of Continuous Time Stochastic Linear, Semilinear, and Nonlinear Systems
3. Noise in the Systems Modeled by:
   a) Brownian Motion
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4. Difficulties Arising in Solving Stochastic Adaptive Control Problems
5. Computational Aspects of Stochastic Adaptive Control
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What is Adaptive Control?

“In everyday language, '*to adapt*' means to change a behavior to conform to new circumstances. Intuitively, an adaptive controller is thus a controller that can modify its behavior in response to changes in the dynamics of the process and the character of the disturbances.”

-*Astrom & Wittenmark Adaptive Control, 1995*

- Many physical systems experience perturbations or there are unmodeled dynamics in the systems.

- These occurrences can often affectively be modeled by a white noise perturbation.

- Examples show that noise may have a stabilizing or a destabilizing effect.

Significance

- Industrial Models can often be described as controlled systems.
- System's behavior depends on the parameters and the fact that the value of the parameters is unknown makes the system unknown.
- Some crucial information concerning the system is not available to the controller and this information should be learned during the system's performance.
- The described problem is the problem of adaptive control.

Identification and Adaptive Control

Adaptive Control Problem:
Identification and Control

Solution to the Adaptive Control Problem:
Strong consistency of the family of estimates
&
Self-optimality of an adaptive control that uses the family of estimates

The general approach to adaptive control that is described here exhibits a splitting or separation of identification and adaptive control.

Identification:

Estimators used:

- Maximum likelihood
- Least Squares
- Weighted Least Squares

For some cases, the weighted least squares estimator is strongly consistent while the least squares estimator is not.

Important Issues for Identification

- Strong consistency
- Recursivity
- Rate of convergence
- Asymptotic behavior of estimators

Adaptive Control

The adaptive control constructed by the so-called certainty equivalence principle, that is the optimal stationary control, is computed by replacing the unknown parameter values by the current estimates of these values.

Important Issues for Adaptive Control

- Self-tuning property

Asymptotically the adaptive control using the estimate of the unknown parameter is as good as the optimal control if we knew the system (the optimal stationary controls as continuous functions of unknown parameters).

- Self-optimizing property

The family of average costs converges to the optimal average costs.

- Numerical computations for adaptive control

Focus on Identification and Adaptive Control of Continuous-Time Stochastic Systems

- Many models evolve in continuous time.
- It is important for the study of discrete time models when the sampling rates are large and for the analysis of numerical and round-off errors.
- Stochastic calculus provides powerful tools: stochastic integral, Ito's differential, martingales.

Stochastic Adaptive Control Problems as Applications of the Stochastic Control Theory

We use the certainty equivalence control as an adaptive control, so we need the optimal control given explicitly or the nearly optimal control.

Illustrative Example

Stochastic Adaptive Control for an Investment Model with Transaction Fees.

Consider a model where an investor has the choice to invest in two assets: a bond *B* with a fixed rate of growth *r* and a stock *S* whose growth is governed by a Brownian motion with drift μ and variance σ^2. The investor controls his asset by transferring money between the stock and the bond.

Control Problem

Let $U(t)$, $Z(t)$ denote the total amount of money transferred from S to B at time t. ($U(0) = Z(0) = 0$) The processes S and B are described by

$$dB(t) = rB(t)\,dt + \lambda\, dU(t) - dZ(t)$$

$$dS(t) = S(t)[\mu - \sigma^2/2]\,dt + \sigma\, S(t)\, dW(t) + dZ(t) - dU(t)$$

Where $W(t), t \geqslant 0$ is a standard Brownian motion.

Let $Y(\mathrm{t}) = S(t) + B(t)$ be the total wealth of the investor at time t.

Goal: Find a pair of optimal controls (U^*, Z^*) such that the expected rate of growth

$$J_y(U, Z) = E\left[\liminf_{t \to \infty} \left(\ln \frac{Y(t)}{t}\right)\right]$$

is maximized.

Identification Problem:

Find μ and σ based on available observations.

Adaptive Control Problem:

Construct the certainty equivalence adaptive control.

The stochastic adaptive control problem was solved. The solution was computed numerically.

The Stochastic Control problem

Example 2: Portfolio Selection and Consumption Model

Let $X(t)$, $t \geq 0$ be the wealth at time t of an individual who invests his wealth in two types of assets: the safe asset with return rate r and the risky asset with average return rate α. The wealth $X(t)$ changes according to the stochastic differential equation

$$dX(t) = r(1-U(t))X(t)dt + U_1(t)X(t)dt(\alpha\, dt + \tau\, dW(t)) - U_2(t)dt$$

$U_1(t)$: fraction of wealth invested in risky assets at time t

$U_2(t)$: consumption rate at time t

r, α, τ constants with $r < \alpha, \tau > 0$

$W(t)$, $t \geq 0$: real-valued Wiener process

The controls are naturally constrained as $0 \le U_1(t) \le 1$ and $U_2(t) \ge 0$

The stochastic control problem is to maximize the expected discounted total utility

$$I(U) = E_y \int_s^{\infty} e^{(-\rho t)} F[U_2(t)]\, dt \;, \quad y = W(s)$$

where $F(u) = u^{\varphi}$ with $0 < \varphi < 1$ is the utility function and $\rho > 0$ is the discount rate.

The Stochastic Adaptive Control Problem

Example 3: Portfolio Selection and Consumption Model

Consider the situation described in example 2. For the adaptive control problem, it is assumed that α is an unknown parameter such that

$$\alpha \in [a_1, a_2] \quad with\ r < a_1$$

The adaptive control procedure in this setting is to define the control at a time t, that is, the portfolio selection and the consumption rate, using the optimal infinite-time control, where the estimate at time t of the unknown parameter is used for the unknown parameter.

Stochastic Adaptive Control Problem was Solved in Cases Where:

- Parameters are constants
- Parameters are functions of time
- Parameters are random
- Parameters are stochastic processes

Weighted Least Squares and Continuous Time Adaptive LQG Control

- Linear Gaussian control problem with ergodic, quadratic cost functional is probably the most well known ergodic control problem.

- It is a basic problem to solve for stochastic adaptive control since the optimal control can be easily computed and the existence and invariant measure follows directly from the stability of the optimal system.

- Problem is solved using only the natural assumptions of controllability and observability.
- Weighted least squares scheme is used to obtain the convergence of the family of estimates (self convergence).
- Scheme is modified by a random regularization to obtain the uniform controllability and observability of the family of estimates.
- A diminishing excitation white noise is used to obtain strong consistency.
- Excitation is sufficient to include the identification of unknown deterministic linear systems.

- The approach eliminates some other assumptions that have previously been used that are unnecessary for the control problem for a known system and are often difficult to verify.

- The approach eliminates the need for random switching or resetting which often occurred in previous work.

Weighted Least Squares Identification

Let ($X(t)$, $t \geq 0$) be the process that satisfies the stochastic differential equation

$$dX(t) = AX(t)\,dt + BU(t)\,dt + D\,dW(t)$$

or

$$dX(t) = \Theta^T \varphi(t)\,dt + D\,dw(t)$$

where

$$\Theta^T = [A, B], \varphi(t) = \begin{bmatrix} X(t) \\ U(t) \end{bmatrix}$$

$$X(0)=X_0, X(t)\in R^n, U(t)\in R^m, (W(t), t\geq 0)$$

Is an R^p – valued standard Wienar process, and ($U(t)$, $t \geq 0$) is a control from a family that is specified.

The random variables are defined on a fixed complete proabbility space (Ω, F, P) and there is a filtration (F_t, $t \geq 0$) defined on this space. It is assumed that A, B are unknown.

A family of weighted least squares (WLS) estimates

$$\hat{\Theta}(t), t \geq 0$$

is given by:

$$d\hat{\Theta}(t) = a(t)P(t)\varphi(t)(dX^T(t) - \varphi(t)\hat{\Theta}(t)dt)$$

$$dP(t) = -a(t)P(t)\varphi(t)\varphi^T(t)P(t)dt$$

Where Θ(0) is arbitrary, P(0) > 0 is arbitrary and

$$a(t) = 1/f(\tau(t))$$

$$\tau(t) = e + \int_0^t |\varphi(s)|^2 ds$$

$$f \in F = \left\{ \begin{matrix} f \; A f : R_t \rightarrow R_t \, , f \; is \; slowly \; increasing \\ \int_c^\infty \quad \frac{dx}{x f(x)} < \infty \; for \; some \; c \geq 0 \end{matrix} \right\}$$

The ergodic cost functional is used

$$J(U) = \limsup_{T \rightarrow \infty} \int_0^T \; [X^T(t) Q_1 X(t) + U^T(t) Q_2 U(t)] dt$$

where $(U(t), t \geq 0)$ is an admissible control, $Q_1 \geq 0$, $Q_2 \geq 0$.

We assume that (A, B) is controllable and that $(A, Q_1^{1/2})$ is observable.

Controllability

Controllability & Observability are highly desirable is a system, say one given by:

$$x_{k+1} = Ax_k + Bu_k \qquad\qquad y_k = Hx_k$$

The above system is controllable when you can steer the system from an arbitrary initial point to an arbitrary final point.

(A,B) is controllable $\leftrightarrow$ rank $[B\ AB\ \ldots\ A^{n-1}B] = n$

Observability

The above system is observable when an arbitrary initial point can be determined as a function of the observable data points.

(H,A) is observable $\leftrightarrow$ $$rank \begin{bmatrix} H \\ HA \\ \vdots \\ HA^{n-1} \end{bmatrix} = n$$

(A,B) is controllable $\leftrightarrow$ (B^T, A^T) is observable.

Adaptive Control: The diminishing excited lagged certainty equivalence control is used.

Identification: To obtain the strong consistency for the family of estimates, a diminishing excitation is added to the adaptive control.

The complete solution to the adaptive control problem with the most natural assumptions.

Solution to the Adaptive Control Problem for Stochastic Continuous Time Linear and some Non-linear Systems has been obtained.

Stochastic Control Theory recently focuses on Identification and Control of stochastic systems with a noise modeled by a Fractional Brownian Motion.

In the recent paper (T. Duncan, B. Pasik-Duncan), an adaptive control problem for a scalar linear stochastic control system perturbed by a fractional Brownian motion with the Hurst parameter H in (1/2, 1) is solved. A necessary ingredient of a self-optimizing adaptive control is the corresponding optimal control for the known system. It seems that the optimal control problem has only been solved for a scalar system. In the solution of the adaptive control problem, a strongly consistent family of estimators of the unknown parameter are given and a certainty equivalence control is shown to be self-optimizing in an $L^2(P)$ sense. It seems that this paper is the initial work on the adaptive control of such systems.

Standard Fractional Brownian Motion

$(B(t), t \geq 0)$ is a standard fractional Brownian motion with $H \in (0, 1)$ if it is a Gaussian process with continuous sample paths that satisfies

$$E[B(t)] = 0$$

$$E[B(s)B(t)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$$

for all $s, t \in \Re_{+}$.

Standard Fractional Brownian Motion

Three properties of FBM:

1. Self- similarity

   if $a > 0$, then $(B(at), t \geq 0)$ and $(a^H B(t), t \geq 0)$ have the same probability law,

2. Long-range dependence for $H \in (1/2, 1)$

$$r_H(n) = E[(B(1) - B(0))(B(n+1) - B(n))]$$

$$\sum_{n=1}^{\infty} r_H(n) = \infty$$

3. p^{th} variation is nonzero and finite only for $p = 1/H$.

Since a (standard) fractional Brownian motion B with the Hurst parameter $H \neq ½$ is not a semimaritngale, the stochastic calculus for a Brownian motion, or more generally for a continuous square integrable martingale, is not applicable. However, a stochastic calculus for a fractional Brownian motion particularly for $H \in (½, 1)$ has been developed which preserves some of the properties for the (Itô) stochastic calculus for Brownian motion.

The linear-quadratic control problem is reviewed. Let ($X(t)$, $t \geq 0$) be the real-valued process that satisfies the stochastic differential equation

$$dX(t) = \alpha_0 X(t)\,dt + bU(t)\,dt + dB(t)$$

$$X(t) = X_0 \qquad (1)$$

where X_0 is a constant, ($B(t)$, $t \geq 0$) is a standard fractional Brownian motion with the Hurst parameter $H \in (½, 1)$, $\alpha_0 \in [a_1, a_2]$ where $a_2 < 0$, $b \in \Re \setminus \{0\}$.

For $t \geq 0$, let F_t be the P-completion of the sub-σ algebra $\sigma(B(u), 0 \leq u \leq t)$. The family of sub-σ algebras $(\mathsf{F}_t, t \geq 0)$ is called the filtration associated with $(B(t), t \geq 0)$. Let $(U(t), t \geq 0)$ be a process adapted to $(\mathsf{F}_t\ t \geq 0)$. It is known that he filtration generated by $(X(t), t \geq 0)$ is the same as the filtration generated by $(B(t), t \geq 0)$. The process U is adapted to the filtration $(\mathsf{F}_t, t \geq 0)$ such that (1) has one and only one solution.

Consider the optimal control problem where the state X satisfies (1) and the ergodic (or average cost per unit time) cost function J is

$$J(U) = \limsup_{T\to\infty} \int_0^T (qX^2(t)+rU^2(t))\,dt$$

where $q > 0$ and $r > 0$ are constants. The family U of admissible controls is all (F_t) adapted processes such that (1) has one and only one solution.

To introduce some notation, recall the well-known solution with $H = ½$, that is $(B(t), t \geq 0)$ is a standard Brownian motion. An optimal control is U^* given by

$$U^*(t) = \frac{-b}{r}\rho_0 X^*(t)$$

where $(X^*(t), t \geq 0)$ is the solution of (1) with the control U^*, ρ_0 is the unique positive solution of the scalar algebraic Riccati equation

$$\frac{b^2}{r}\rho^2 - 2a\rho - q = 0$$

So

$$\rho_0 = \frac{r}{b^2}\left[\alpha_0 + \delta_0\right]$$

$$\delta_0 = \sqrt{\alpha_0^2 + \frac{b^2}{r}q}$$

Furthermore,

$$J(U^*) = \rho_0 \qquad a.s.$$

The following result is given in Kleptsyna et al. And solves the analogous problem for $H \in (½, 1)$.

<u>Theorem 1</u>

Let $(U^*(t),\ t \geq 0)$ be the control given by

$$U^*(t) = \frac{-b}{r}\rho_0[X^*(t) + V^*(t)]$$

$$V^*(t) = \int_o^t \delta_0 V^*(s)\,ds + \int_o^t [\bar{k}(t,s) - 1](dX^*(s) - \alpha_0 X^*(s) - bU^*(s))\,ds$$

$$V^*(t) = \int_t^\infty e^{-\delta_0(s-t)}\,dB(s|t)$$

Where ($X^*(t)$, $t \geq 0$) is the solution of (1) with the admissible control ($U^*(t)$, $t \geq 0$) and

$$\bar{k}(t,s) = -c_H^{-1} s^{1/2-H} \frac{d}{ds} \int_s^t (r-s)^{1/2-H} \gamma(r,r)\, dr$$

$$\gamma(t,s) = \delta e^{st} \int_t^{\infty} e^{\delta_0} K_H(\tau,s)\, d\tau$$

$$K_H(t,s) = H(2\text{H}-1) \int_s^{\tau} r^{H-1/2} (r-s)^{H-3/2}\, dr$$

$$B(s|t)=E[B(s)|F_t]=B(t)+\int_0^t u_{1/2-H}(I_t^{1/2-H}(I_s^{H-1/2}1_{(t,s)}))\,dB$$

$$B(s|t)=B(t)+\int_0^t u_{1/2-H}(I_S^{H-1/2}u_{H-1/2}1_{(t,s)})\,dW$$

Where c_H is a constant that only depends on H, $u_a(s) = s^a$ for $s \geq 0$, $I^{H-½}$ is the fractional integral, $I^{½-H}$ is the fractional derivative and ($W(t)$, $t \geq 0$) is a standard Brownian motion (Wiener process) associated with ($B(t)$, $t \geq 0$).

Then the control U^* is optimal in U and the optimal cost is

$$J(U^*) = \lambda \qquad a.s.$$

where

$$\lambda = \frac{q\Gamma(2H+1)}{2\delta_0^{2H}}\left[1 + \frac{\delta_0 + \alpha_0}{\delta_0 - \alpha_0}\sin \pi H\right]$$

If α_0 is unknown, then it is important to find a family of strongly consistent estimators of the unknown parameter α_0 in (1). A method is used by Duncan and P.D. That is called pseudo-least squares because it uses the least squares estimate for α_0 assuming H = ½, that is, B is a standard Brownian motion in (1). It is shown that the family of estimators ($\hat{\alpha}(t)$, $t \geq 0$) is strongly consistent for $H \in \hat{\alpha}$ (½, 1) where

$$\hat{\alpha}(t) = \alpha_0 + \frac{\int_0^t X^0(s)\,dB(s)}{\int_0^t (X^0(s))^2\,ds}$$

$$dX^0(t) = \alpha_0 X^0(t)\,dt + dB(t)$$

$$X^0(0) = X_0$$

This family of estimators can be obtained from (1) by removing the control term. The family of estimators $\hat{\alpha}$ is modified here using the fact that $\alpha_0 \in [a_1, a_2]$ as

$$\alpha(t) = \hat{\alpha}(t) 1_{[a_1, a_2]}(\hat{\alpha}(t)) + a_1 1_{(-\infty, a_1)}(\hat{\alpha}(t)) + a_2 1_{(a_2, \infty)}(\hat{\alpha}(t))$$

for $t \geq 0$. $\hat{\alpha}(0)$ is chosen arbitrarily in $[a_1, a_2]$.

For the optimal control ($U^*(t)$, $t \geq 0$), the corresponding solution ($X^*(t)$, $t \geq 0$) can be expressed as

$$X^*(t) = e^{-\delta_0 t} X_0 + \int_0^t e^{-\delta_0 (t-s)} [-(\alpha_0 + \delta_0) V^*(s) ds + dB(s)]$$

where

$$dX^*(t) = \alpha_0 X^*(t) dt - \frac{b^2}{r} \rho_0 [X^*(t) + V^*(t)] dt + dB(t)$$

$$. = -\delta_0 X^*(t) dt - (\alpha_0 + \delta_0) V^*(t) dt + dB(t)$$

An adaptive control (U^(t), $t \geq 0$), is obtained from the certainty equivalence principle, that is, at time t, the estimate α(t) is assumed to be the correct value of the parameter. Thus the stochastic equation for the system (1) with the control U^ is

$$dX\wedge(t)=(\alpha_0-\alpha(t)-\delta(t))X\wedge(t)\,dt-\frac{b\rho(t)}{r}V\wedge(t)\,dt+dB(t)$$

$$dX\wedge(t)=(-\alpha_0-\alpha(t)-\delta(t))X\wedge dt-(\alpha(t)+\delta(t))V\wedge(t)\,dt+dB(t)$$

$$X\wedge(0)=X_0$$

Where

$$\delta(t)=\sqrt{\alpha^2(t)+\frac{b^2}{r}q}$$

$$U\wedge(t)=\frac{-b\rho(t)}{r}[X\wedge(t)+V\wedge(t)]$$

$$\rho(t)=\frac{r}{b^2}[\alpha(t)+\delta(t)]$$

And

$$V^\wedge(t) = \int_0^t \tilde{\delta}(s) V^\wedge(s)\, ds + \int_0^t [\tilde{k}(t,s) - 1][dX^\wedge(s) - \alpha(s) X^\wedge(s) ds - bU^\wedge(s) ds]$$

$$:= \int_0^t \tilde{\delta}(s) V^\wedge(s)\, ds + \int_0^t [\tilde{k}(t,s) - 1][dB(s) + (\alpha_0 - \alpha(t)) X^\wedge(s) ds]$$

$$\tilde{\delta}(t) = \delta(t) + \alpha(t) - \alpha_0$$

and $\tilde{k}$ denotes the use of $\tilde{\delta}$ instead of δ_0 in $\bar{k}$. Note that $\delta(t) \geq -\alpha(t) + c$ for some $c > 0$ and all $t \geq 0$ so that

$$\alpha_0 - \alpha(t) - \delta(t) < -c$$

The solution of the stochastic equation is

$$X\wedge(t)=e^{-\int_0^t \tilde{\delta}} X_0+\int_0^t e^{-\int_0^t \hat{\delta}}\left[-(\alpha(s)+\delta(s))V\wedge(s)\,ds+dB(s)\right]$$

The following result states that the adaptive control ($U\wedge(t)$, $t \geq 0$) is self-optimizing in $L^2(P)$, that is, the family of average costs converge in $L^2(P)$ to the optimal average cost.

Theorem 2

Let ($\alpha(t)$, $t \geq 0$) be the family of estimators of α_0, let ($U\hat{}(t)$, $t \geq 0$) be the associated adaptive control, and let ($X\hat{}(t)$, $t \geq 0$) be the solution with the control $U\hat{}$. Then

$$\lim_{t\to\infty} \frac{1}{t} E \int_0^t \left| U^*(s) - U\hat{}(s) \right|^2 ds = 0$$

$$\lim_{t\to\infty} \frac{1}{t} E \int_0^t \left| X^*(s) - X\hat{}(s) \right|^2 ds = 0$$

$$\lim_{t\to\infty} \frac{1}{t} E \int_0^t \left(q(X\hat{}(s))^2 + r(U^*(s))^2 \right) ds = \lambda$$

where λ is given above.