Embeddings of Groups II

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CND Kernel

A kernel on a set X = a symmetric map $\psi : X \times X \to \mathbb{R}_+$ such that $\psi(x,x) = 0$.
For instance, a pseudo-metric.

Definition

A kernel $\psi : X \times X \to \mathbb{R}_+$ is conditionally negative definite (CND) if for every $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ with $\sum_{i=1}^n \lambda_i = 0$:

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j \psi(x_i, x_j) \leq 0 \,.$$

Kernel of type p

Theorem (Schoenberg 1938)

- *If $\psi : X \times X \to \mathbb{R}_+$ is CND and $0 < \alpha \leq 1$ then ψ^α is CND.*
- *In L^p–spaces, $\|x - y\|^p$ is CND.*
- *ψ is CND iff $\exists f : X \to H$ Hilbert such that $\psi(x, y) = \|f(x) - f(y)\|^2$.*

kernel of type p: $\psi(x, y) = \|f(x) - f(y)\|_p^p$ for some $f : X \to L^p(ZS, \mu)$.

Every CND kernel is of type 2.

Comparing types

Theorem (Bretagnolle-Dacunha Castelle- Krivine)

Let $1 \leq p \leq q \leq 2$*.*

1. *The normed space* $(L^q(X,\mu)\,,\,\|\cdot\|_q)$ *can be embedded linearly and isometrically into* $(L^p(X',\mu')\,,\,\|\cdot\|_p)$ *for some measure space* (X',μ')*.*
2. $(L^p(X,\mu)\,,\,\|\cdot\|_p^\alpha)$ *can be embedded isometrically into* $(L^q(X',\mu')\,,\,\|\cdot\|_q)$ *for some measured space* (X',μ') *if and only if* $0 < \alpha \leq \frac{p}{q}$*.*

Corollary

Consider $1 \leq p < q \leq 2$*.*

- *Every kernel of type* p *is of type* q*.*
- *If* ψ *is a kernel of type* q *then* $\psi^{\frac{p}{q}}$ *is of type* p*.*

Strict inclusion

Let $1 \leq p < q \leq 2$.

The inclusion $\{$ kernels of type $p\} \subseteq \{$ kernels of type $q\}$ is strict.

We explain an example for $p = 1$ and $q = 2$.

Theorem (Faraut-Harzallah 1974)

The hyperbolic metrics on the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ and on the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ are CND kernels.

1. the metric on $\mathbb{H}^n_{\mathbb{R}}$ is also a kernel of type 1 ($\mathbb{H}^n_{\mathbb{R}}$ has a structure of measured walls).
2. the metric on $\mathbb{H}^n_{\mathbb{C}}$ cannot be a kernel of type 1. Otherwise:
   - $\mathbb{H}^n_{\mathbb{C}}$ isometrically embeddable into an L^1-space;
   - $\mathbb{H}^n_{\mathbb{C}}$ would have walls $\{h, h^c\}$ with both h and h^c convex.

Keep this in mind until slide 8.

Definition

G locally compact second countable.

property (T) of Kazhdan: Every continuous action of G on a Hilbert space by (affine) isometries has a global fixed point.

Haagerup property (or a-(T)-menability): There exists a continuous action of G on a Hilbert space by (affine) isometries that is proper.

- all isometries of a Hilbert are: $v \mapsto Uv + b$ (Mazur-Ulam).
- continuous action: for every vector v the orbit map $g \mapsto gv$ from G to $\mathcal{H}$ is continuous.
- proper action: for every bounded subset $B \subset \mathcal{H}$ the set $\{g \in G : gB \cap B \neq \emptyset\}$ is relatively compact.

a-(T)-menability: implied by amenability, strong negation of (T).

Relation with CND kernels

Theorem (Delorme-Guichardet, Akermann-Walter)

- *G has property (T) iff every continuous left invariant CND kernel on G is bounded.*
- *G is a-(T)-menable iff there exists a continuous left invariant CND kernel on G that is proper.*

left-invariant means $\psi(gx, gy) = \psi(x, y)$.

proper kernel: if x leaves every compact set then $\psi(1, x) \to \infty$.

Consequence for actions on L^p

Corollary (Delorme-Guichardet, Akermann-Walter)

1. *If G has property (T) then for every $p \in (0, 2]$, every continuous action by isometries of G on a subset of a space $L^p(X, \mu)$ has bounded orbits (fixed point, for $p > 1$).*
2. *G has the Haagerup property if there exists $p \in (0, 2]$, and a continuous proper action by isometries of G on a subset of some $L^p(X, \mu)$.*

Corollary

Isom($\mathbb{H}^n_{\mathbb{R}}$) and Isom($\mathbb{H}^n_{\mathbb{C}}$), all their subgroups have the Haagerup property.

Actions on median spaces

Theorem (Chatterji-Druțu-Haglund)

Let G be a locally compact second countable group.

1. *G has (T) iff any continuous action by isometries on a median space has bounded orbits.*
2. *G has Haagerup iff it admits a proper continuous action by isometries on a median space.*

median metric = kernel of type 1, so direct implication in 1, converse in 2 follow from Delorme-Guichardet, Akermann-Walter.

For G countable the direct implication in 1 was given geometric proofs by Niblo-Reeves, Roller, Nica.

Actions on spaces with measured walls

Theorem (Chatterji-Druțu-Haglund)

1. *G has (T) iff any continuous action by automorphisms on a space with measured walls has bounded orbits.*
2. *G has Haagerup iff it admits a proper continuous action by automorphisms on a space with measured walls.*

For G countable, direct implication in 1, converse in 2 proved by Cherix-Martin-Valette.

key remark: action by automorphisms on a space with measured walls $\Rightarrow$ action by affine isometries on an L^p, $p > 0$:

- take $\mathcal{H}$= set of halves h s.t. $\{h, h^c\}$ wall;
- $\mathcal{H} \to \mathcal{W}$ double cover $\Rightarrow$ can define a measure $\mu_{\mathcal{H}}$ on $\mathcal{H}$;
- action on $L^p(\mathcal{H}, \mu_{\mathcal{H}})$ defined by $g \cdot f = f \circ g + \chi_{\mathcal{H}_{gx}} - \chi_{\mathcal{H}_x}$, where x arbitrary point in X, $\mathcal{H}_x = \{h \in \mathcal{H} \,;\, x \in h\}$.

Actions on L^p–spaces

Fix $p > 0$. property FL^p: Every continuous action by affine isometries on a space $L^p(X, \mu)$ has bounded orbits (equivalently, for $p > 1$, it has a fixed point).

a-FL^p-menability: There exists a proper affine isometric continuous action on some space $L^p(X, \mu)$.

for $p \in (0, 2]$, $FL^p \Leftrightarrow (T)$ and a-FL^p-menability $\Leftrightarrow$ a-(T)-menability.

for $p \geq 2$, FL^p is stronger, a-FL^p-menability is weaker:

- Assume G has FL^p.
- Take an action of G on a space with measured walls.
- Consider the action on $L^p(\mathcal{H}, \mu_{\mathcal{H}})$, $g \cdot f = f \circ g + \chi_{\mathcal{H}_{gx}} - \chi_{\mathcal{H}_x}$.
- FL^p implies bounded orbit for $f \equiv 0$.
- $\|\chi_{\mathcal{H}_{gx}} - \chi_{\mathcal{H}_x}\|_p = \mu(\mathcal{W}(gx|x))^{\frac{1}{p}}$ uniformly bounded.
- apply the first part of the Theorem on the previous slide: G has then property (T).

Fixed point properties

Pansu, Cornulier-Tesera-Valette: The group of isometries of $\mathbb{H}^n_{\mathbf{H}}$ (the quaternionic hyperbolic space)

- acts properly on an L^p for $p > 4n + 2$;
- has property (T).

For G group with property (T) define

$$\wp(G) = \{p \in (0, \infty); \; ; G \text{ has } FL^p\} .$$

$\wp(G)$ is an open set $\Leftrightarrow$ $\wp(G)^c$ is closed:
If $p_n \to p$ then a sequence of actions on spaces L^{p_n} without fixed point has an ultralimit an action on an L^p without fixed point.

Question

Other features of $\wp(G)$?

Uniform embeddings

Theorem (Guoliang Yu)

A group with a uniform embedding in a Hilbert space satisfies the Novikov conjecture and the coarse Baum-Connes conjecture.

A uniform embedding in a Hilbert: $\varphi : G \to H$ such that

$$\rho(\mathrm{dist}_G(x,y)) \leq \|\varphi(x)-\varphi(y)\| \leq C\mathrm{dist}_G(x,y)\,, \tag{1}$$

with $\lim_{x\to\infty} \rho(x) = \infty$ and $C > 0$.

Question

Maybe all f.g. groups admit a uniform embedding in a Hilbert space ?

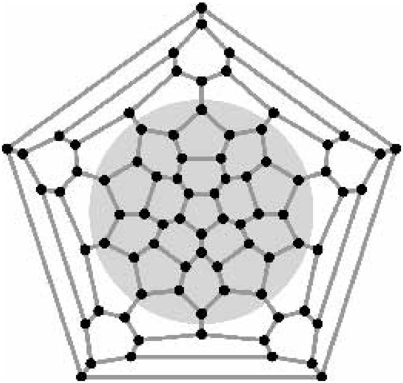
Expanders

Definition

A (d, λ)–expander is a finite graph Γ:

- of valence d in every vertex;
- such that for every set S containing at most half of the vertices, the set $E(S, S^c)$ of edges with exactly one endpoint in S has at least $\lambda \cdot \mathrm{card} S$ elements.

A Ramanujan graph



Expanders and embeddings

Theorem (obstruction to uniform embedding)

Let $\mathcal{G}_n$ be an infinite sequence of (d, λ)–expanders.
The space $\bigvee_{n\in\mathbb{N}} \mathcal{G}_n$ cannot be embedded uniformly in a Hilbert space.

Question

How to construct expanders ?

Constructions of expanders

- Consider G with property (T) and a finite generating set S (e.g. $G = SL(n, \mathbb{Z}), n \geq 3$).
- Consider G_N a sequence of finite index subgroups (e.g. $G_N = \{A \in SL(n, \mathbb{Z})\; ;\; A = \mathrm{Id}_n \text{ modulo } N\}$.
- The Cayley graphs of G/G_N for generating sets $\pi_N(S)$ are (d, λ)–expanders.

  Kassabov: The sequence of permutation groups S_n yields expanders (for good choices of generators).

Other examples of expanders

V. Lafforgue: Uniform lattices G in $SL(3, \mathbb{Q}_p)$ satisfy a stronger version of property (T).
Take G_N finite index subgroups, $\mathcal{G}_N$ the Cayley graphs of G/G_N.

The space $\bigvee_{n\in\mathbb{N}} \mathcal{G}_N$ cannot be embedded uniformly into any uniformly convex Banach space.

Question

Is the same true for an arbitrary family of expanders ?

uniformly convex: $\forall R > 0, \delta > 0$ there exists $\varepsilon = \varepsilon(R, \delta) > 0$ such that $\|x\| = \|y\| = R\,,\ \|x - y\| \geq \delta \Rightarrow \left\|\frac{1}{2}(x + y)\right\| \leq R - \varepsilon$.

Brown-Guentner: any countable graph of bounded degree can be uniformly embedded into the Hilbertian sum $\oplus_{p\in\mathbb{N}} l^p(\mathbb{N})$ (reflexive, strictly convex, not uniformly convex).

Gromov's example

Theorem (Gromov, Arzhantseva-Delzant)

The exist f.g. groups with a family of expanders quasi-isometrically (uniformly) embedded in a Cayley graph.

- A family of expanders with particular properties is needed (with increasing girth).
  Cayley graphs of G/G_N, G non-free, do not work.
- Proof uses random groups.
- The group is a direct limit of hyperbolic quotients.

The Hilbert compression of a group G: the supremum of all $\alpha \geq 0$ such that there exists $\varphi : G \to H$ satisfying

$$[\mathrm{dist}_G(x,y)]^\alpha \leq \|\varphi(x) - \varphi(y)\| \leq C\mathrm{dist}_G(x,y)\,, \tag{2}$$

Theorem (Guentner-Kaminker)

If $\alpha > \frac{1}{2}$ then G has property A of Guoliang Yu (a "non-equivariant version of amenability")

Theorem (Naor-Peres)

If G is amenable then its Hilbert compression is at most $\frac{1}{2\beta^*(G)}$, where $\beta^*(G)$ is the speed of divergence of random walks.

$\beta^*(G)$: supremum of β s.t. $\exists S$ set of generators, $c > 0$ s.t. if W_t canonical simple random walk on $\mathrm{Cayley}(G,S)$ starting at 1, for every $t \in \mathbb{N}$ the expectation $E(\mathrm{dist}_S(W_t, 1)) \geq ct^\beta$.

Examples

1. Hyperbolic groups have compression 1, the best possible $\rho(x)$ is $\frac{x}{\log x}$ (Bonk-Schramm, Buyalo-Schroeder, Bourgain).
2. finite dimensional $CAT(0)$ cube complexes have compression 1.
3. Uniform lattices in Lie groups have compression 1 (R. Tessera)
4. The Thompson group F has compression $\frac{1}{2}$ (Arzhantseva-Guba-Sapir)
5. If $\mathbb{Z}_{(1)} = \mathbb{Z}$ and $\mathbb{Z}_{(k+1)} = \mathbb{Z}_{(k)} \wr \mathbb{Z}$ then $\mathbb{Z}_{(k)}$ has compression $\frac{1}{2-2^{1-k}}$ (Naor-Peres).
6. If G hyperbolic relative to $H_1, \dots, H_n$ then the compression of G is the infimum of the compressions of the H_i (D. Hume).
7. There exist solvable groups with compression 0 (T. Austin).

Theorem (Arzhantseva-Druțu-Sapir)

For every $\alpha \in [0, 1]$ there exists a finitely generated group G with Hilbert compression α (and with uniformly convex Banach space compression α). Moreover G has asymptotic dimension 2, hence property A.

Step 1: Let $\mathcal{G}_n$ be a sequence of (d, λ)–expanders. For every α there exists k_n increasing sequence in $\mathbb{N}$ such that $\bigvee_{n\in\mathbb{N}} k_n\mathcal{G}_n$ has Hilbert compression α.

$k_n\mathcal{G}_n$: assume all edges have length k_n instead of 1.

Step 2: Take G a group with property (T) generated by involutions $\sigma_1, ..., \sigma_r$ (e.g. Lafforgue's examples).

Take G_n finite index subgroups, $M_n = G/G_n$ generated by $\sigma_{1,n}, ..., \sigma_{r,n}$.

Let $k_n - k_{n-1} = 2a_n + 1$. Let M be the free product of all M_n (not f.g.)
G generated by M and $t_1, ..., t_r$ s.t. for all $i \in \{1, ..., r\}$, $n \in \mathbb{N}$,

$$\sigma_{i,n+1} = t_i^{a_i} \sigma_{i,n} t_i^{-a_i} .$$

Another description: take r copies $H_1, ..., H_r$ of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}$:

- generator of $\mathbb{Z}/2\mathbb{Z}$ denoted by σ_i,
- generator of $\mathbb{Z}$ denoted by t_i.
- we denote $t_i^k \sigma_i t_i^{-k}$ in H_i by $\sigma_i^{(k)}$.

G is the fundamental group of the graph of groups:

- with vertex groups M and $H_1, ..., H_r$;
- edge groups $M \cap H_i$ free products $*_{n \geq 1} (\mathbb{Z}/2\mathbb{Z})_n$
- $(\mathbb{Z}/2\mathbb{Z})_n$ identified with $\langle \sigma_{i,n} \rangle < M_n$ in M, with $\langle \sigma_i^{(a_n)} \rangle$ in H_i.

$\mathrm{Cayley}(G, \sigma_{1,1}, ..., \sigma_{r,1}, t_1, ..., t_n)$ contains $k_n \mathcal{G}_n$ for every n.

Hence compression is at most α.

For lower bound: a construction of an embedding using the embedding of each expander $\mathcal{G}_n$ in the set of vertices of a simplex with length of the edge equal to k_n.