Just a little calculation in dynamics

Będlewo, 13-19 August 2017

Abstracts of talks

Elements of thermodynamic formalism for transcendental meromorphic maps

Krzysztof Barański

(Joint work with Bogusława Karpińska and Anna Zdunik)

In this talk I will survey some recent results on the ergodic aspects of the dynamics of some classes of trancendental entire and meromorphic functions on the complex plane. The emphasis will be given to the elements of thermodynamic formalism, such as topological pressure, Bowen's formula and conformal measures.

Almost sure continuity along curves traversing the Mandelbrot set

Michael Benedicks

(Joint work with Jacek Graczyk)

We study continuity properties of dynamical quantities while crossing the Mandelbrot set through typical smooth curves. In particular, we prove that for almost every parameter c_0 in the boundary of the Mandelbrot set M with respect of the harmonic measure and every smooth curve $\gamma : [-1,1] \mapsto \mathbb{C}$ with the property that $c_0 = \gamma(0)$ there exists a set $\mathcal{A}_\gamma$ having 0 as a Lebesgue density point and such that that $\lim_{x\to 0} HDim(J_{\gamma(x)}) = HDim(J_{c_0})$ for the Julia sets J_c.

Regular variations for almost Anosov diffeomorphisms

Henk Bruin

(Joint work with Dalia Terhesiu *(University of Exeter)*)

The current operator renewal-type approach to obtain polynomial mixing rates in various infinite measure-preserving dynamical systems requires that the tails of a certain inducing scheme have regular variation. An almost Anosov diffeomorphism is a diffeomorphism (in our case on the 2-torus) that satisfies the Anosov properties except at a finite set of neutral saddle points. In this invertible setting, the regular variation of the tails has been treated only in very specific settings and/or with unsatisfactory estimates. In this talk I want to present a new methods which works in much greater generality and gives much more precise estimates. The mixing results additionally require the use of an anisotropic Banach space of distribution similar to the one used before by Demers & Liverani and by Liverani & Terhesiu.

Quasisymmetric rigidity of smooth interval mappings

Trevor Clark

(Joint work with Sebastian van Strien)

Quasisymmetric rigidy plays a central role in one dimensional dynamics and it has several important implications. In this talk, I will focus on strategies for proving quasisymmetric rigidity in one-dimensional dynamics using tools from complex analysis, and how they can be applied to smooth mappings. This builds on earlier work of Kozlovski, Shen and van Strien.

Birkhoff averages of perturbations

Neil Dobbs

Birkhoff averages (of an observable along orbits) are objects of interest when investigating statistical behaviour of a dynamical system. If there is a unique physical measure, the Birkhoff averages will converge, for almost every orbit, to the space average (i.e. the integral) of the observable, so the physical measure captures important statistical properties of the dynamical system. However, in the quadratic family, for example, physical measures don't always exist, and even when they do, they don't necessarily depend continuously on the parameter. In joint work with Alexey Korepanov, we examine what happens for finite time Birkhoff averages for nearby parameters.

On the Julia set of the Feigenbaum quadratic polynomial

Artem Dudko

The Feigenbaum quadratic polynomial $p_F(z) = z^2+c_F$ was discovered by Feigenbaum and Coullet-Tresser in 1970s in relation to the universality phenomenon for period-doubling bifurcations. This discovery lead to a rapid development of renormalization theory in complex dynamics. Many striking results related to p_F and other Feigenbaum maps were proven by Lanford, Sullivan, McMullen, Lyubich and others. Recently, Avila and Lyubich constructed examples of Feigenbaum maps with Julia sets of positive area and

Feigenbaum maps with Julia sets of Hausdorff dimension less than two. However, the question about the area and Hausdorff dimension of the Julia set J_F of p_F remained open. Using computer assistance we show that J_F has Hausdorff dimension less than two (and therefore is of zero Lebesgue measure). The talk is based on a joint work with Scott Sutherland.

Exponential equidistribution of standard pairs for piecewise expanding maps of metric spaces
Peyman Eslami

For a large class of piecewise expanding maps of metric spaces we show the equidistribution of standard pairs at an exponential rate. As a corollary such systems have a unique absolutely continuous invariant measure with respect to which the system is mixing. We allow for unbounded, non-compact spaces, countably many branches and do not assume big images or the existence of a Markov structure. We show how to control the complexity growth of the dynamical partition of the map in order to avoid counterexamples. Our method gives explicit estimates on the exponential rate of equidistribution.

Iterated function systems of logistic maps: synchronization and intermittency
Masoumeh Gharaei

We discuss iterated function systems generated by finitely many logistic maps, with a focus on synchronization and intermittency. We provide sufficient conditions for synchronization, involving negative Lyapunov exponents and minimal dynamics. A number of results that clarify the scope of these conditions are also discussed. We analyze a mechanism for intermittency that involves the full map $x \mapsto 4x(1-x)$ as one of the generators of the iterated function system. The talk is based on a joint work [1] with Ale Jan Homburg (University of Amsterdam) and Neda Abbasi (Shahid Beheshti University).

References

[1] N. Abbasi, M. Gharaei, A.J. Homburg. *Iterated function systems of logistic maps: synchronization and intermittency.* preprint 2017

Two dimensional interval exchange transformations: Piecewise Isometries
Arek Goetz

We offer a glimpse of how rich the dynamics of piecewise isometries is, especially as seen from computer graphics. We survey some recurrence/rescaling results and techniques for a pizza map. A pizza map is an invertible map of the plane that acts as first a permutation of cones that partition the plane followed by a translation. Investigation has been joint mainly with Peter Ashwin from Exeter, UK and Anthony Quas from Victoria.

Periodic Islands for 2-dim Maps
Paweł Góra

We present a number of examples of periodic islands for a two dimensional map

$$G(x,y) = (y, f(\alpha y + (1-\alpha)x)),$$

where $f(t) = 4t(1-t)$ is the logistic map, and α has some specific values.
No theoretical results are presented, we only show computer generated images.

A Few Small Substitution Calculations
Mike Keane

In this lecture, I would like to discuss two recent joint articles submitted and published this year showing, in the spirit of this conference, a few small calculations. In the first article we show, given a constant length primitive substitution, that there are finitely many "other" injective primitive substitutions of constant length with topologically conjugate dynamic systems, and an effective algorithm for making a list of these substitutions. In the second article, it is shown that the Fibonacci substitution, surprisingly, has infinitely many such injective primitive isomorphic copies; we still do not have any generality here for more general classes. The few small calculations will be given by restricting our attention to two well-known constant length substitution, Toeplitz and Morse, and to the Fibonacci substitution. Here are the references:

Article 1. Topological conjugacy of constant length substitution dynamical systems. Indagat.Math 28(2017), 91-107 (joint work with Ethan Coven and Michel Dekking).

Article 2. On the conjugacy class of the Fibonacci dynamical system. Theoretical Computer Science 668(2017), 59-69 (joint work with Michel Dekking).

Monotonicity of entropy and positively oriented transversality for families of interval maps
Genadi Levin
(Joint work with Weixiao Shen and Sebastian van Strien)

We present a general approach to show that critical relations of locally holomorphic maps on the complex plane unfold transversally in a "positively oriented" way. In one-parameter interval families this property implies the monotonicity of kneading sequence and topological entropy. We mainly illustrate this approach on a wide class of one-parameter families of interval maps, for example maps with flat critical points, piecewise linear maps, maps with discontinuities but also for families of maps with complex analytic extensions such as certain polynomial-like maps.

Stable Hall's ray for some generalized Lagrange spectra
Luca Marchese

The classical Lagrange spectrum is a subset of the positive real line, corresponding to a filtration of the set of badly approximable real numbers. Elements in the spectrum can be expressed also as maximal asymptotic excursion of bounded geodesic in the modular surface, which is the quotient of the upper half plane by $\mathrm{SL}(2,\mathbb{Z})$. Well-known features of such spectrum are its discrete lower part, whose elements are called Markoff numbers, and its upper part, which is an entire half-line and is know as Hall's ray.

In this talk we consider the same problem for any non co-compact Fuchsian group G with finite co-volume in $\mathrm{SL}(2,\mathbb{R})$. We prove the existence of Hall ray for the corresponding Lagrange spectrum. We also prove that the Hall ray persists replacing the penetration in to cusp by a Lipschitz perturbation of it. This is a joint work with M. Artigiani and C. Ulcigrai.

If time will allow it, we will also discuss complementary results on spectra whose lower part is not discrete, based on a joint work with P. Hubert, S. Lelièvre and C. Ulcigrai.

Diophantine type and dynamics of interval exchange maps and of translation flows

Stefano Marmi

The notion of diophantine type of an irrational number has equivalent characterizations in terms of rigidity of irrational rotations and linear flows on tori. Recently Roth type and higher type diophantine conditions have been introduced for translation flows on higher genus surfaces and interval exchange maps. Some rigidity results have also been proved, as well as some estimates on recurrence and hitting times, providing an extension of the theory beyond the torus case which is still incomplete but nevertheless quite broad. (Based on joint works with Moussa and Yoccoz and with Kim and Marchese).

On problem 17 in Bowen's notebook

Carlos Matheus

In his notebook, Bowen left a list of 157 mathematical problem. Among them, the problem 17 asks for "symbolic dynamics for billiards".

In this talk, we discuss a joint work with Yuri Lima extending Sarig's theory of symbolic models for smooth surface diffeomorphisms to the case of surface maps with discontinuities, including Sinai and Bunimovich billiards.

Lozi-like maps

Michał Misiurewicz

(Joint work with Sonja Štimac)

We define a broad class of piecewise smooth plane homeomorphisms which have properties similar to the properties of Lozi maps, including the existence of a hyperbolic attractor. We call those maps *Lozi-like*. The basic structure of such a map is determined by the set of *kneading sequences*, or each of the two equivalent objects: the *folding pattern* and the *folding tree*.

Convergence to the Nash equilibrium for the Best Response Dynamics in the non-zero sum bimatrix games

Cezary Olszowiec
(Joint work with S. van Strien and D. Turaev)

We consider the differential inclusion known as Best Response Dynamics for a bimatrix game (A, B), with A, B - the payoff matrices for two agents. It is known that if (A, B) is equivalent to a zero-sum game and the Nash equilibrium is an internal point of the phase space, then it is also stable in the sense of Lyapunov. Hofbauer's conjecture states that the internal Nash equilibrium can be stable only if (A, B) is equivalent to a zero-sum game. We claim that Hofbauer's conjecture is false and so we will present our approach to this problem.

Lyapunov spectrum for step one $SL(2,\mathbb{R})$ matrix cocycles

Michał Rams
(Joint work with Lorenzo Díaz and Katrin Gelfert)

Let $A_1, \dots, A_k$ be $SL(2,\mathbb{R})$ matrices. Given a sequence $\omega \in \{1,\dots,k\}^{\mathbb{N}}$, define the Lyapunov exponent

$$\chi(\omega) = \lim_{n\to\infty} \frac{1}{n} \log ||A_{\omega_n} \cdot \ldots \cdot A_{\omega_1}||$$

(if such a limit exists). Let

$$L(\alpha) = \{\omega; \chi(\omega) = \alpha\}.$$

The function $\alpha \to f(\alpha) = h_{\text{top}}(L(\alpha))$ is called Lyapunov spectrum of the matrix cocycle $(A_1, \dots, A_k)$.

We give a description of the Lyapunov spectrum for generic $SL(2,\mathbb{R})$ matrix cocycles. Of more interesting properties: it is a concave function, a Legendre transform of some properly defined pressure function, it has a maximum $\log k$ which is achieved only at one point, for an elliptic cocycle generically we have $f(0) \in (0, \log k)$.

Zero temperature phase transitions of geometric Gibbs states

Juan Rivera-Letelier
(Joint work with Daniel Coronel)

For one-dimensional maps we study zero temperature phase transitions of geometric Gibbs states. For quadratic-like maps, we show a related phenomenon of sensitive dependence: There are analytic families of quadratic-like maps for which an arbitrarily small perturbation of the parameter can have a definite effect on the low-temperature geometric Gibbs states. In contrast with what is known in statistical mechanics, this phenomenon is robust: There is an open set of analytic 2-parameter families of quadratic-like maps that exhibit sensitive dependence of geometric Gibbs states. The key ingredient is a geometric version of the Peierls condition for contour models ensuring that the low-temperature Gibbs states are concentrated near the critical orbit.

Renormalisation of a cone exchange transformation

Ana Rodrigues
(Joint work with P. Peres)

Piecewise Isometries (PWIs) are higher dimensional generalizations of *interval exchange transformations* (IETs) that have highly complex and little understood behaviour. In this talk I will introduce a new family of piecewise isometries and renormalize its first return map to a subset of its partition. This implies in particular the existence of infinitely many periodic islands for this map as well as non ergodicity close to the origin.

Rotation theory and mixed-mode oscillations in a hybrid neuron model

Justyna Signerska-Rynkowska

Integrate-and-fire (IF) models combine excitable nature of nerve cells membranes with the discrete nature of spike emission and thus belong to the so-called hybrid dynamical systems, coupling continuous dynamics given by ordinary differential equations with the reset mechanism.

Qualitative dynamics of (non-linear) IF models can be effectively analysed by iterations of the so-called *adaptation map*, with fixed points of this map corresponding to regular spiking and periodic orbits to bursting. However, so far such analysis has been only performed in the specific case where the vector field of the subthreshold system has no singular points and the induced adaptation map is continuous.

Interestingly, as will be shown in this talk, in the presence of subthreshold singularities these systems are able to exhibit mixed-mode (bursting) oscillations (MM(B)Os), where consecutive spikes or bursts are alternated by small subthreshold oscillations, as observed in different cell types and brain areas. In this case the emerging adaptation map is no longer continuous and moreover shows divergence of the derivative in the neighbourhood of discontinuity points. Notwithstanding, such a map, cut to the proper invariant interval, can be seen as a discontinuous circle map, falling into the framework of either the non-overlapping lifts ([2]) or the so-called "old heavy maps" ([1]), with the univocal bidirectional link between the rotation number of the trajectory and the signature of the generated MM(B)Os.

The talk is based on a joint work ([3]) with J. Rubin (Univ. Pittsburgh), J. Touboul (College de France & INRIA) and A. Vidal (Univ. Evry & INRIA).

The research has been supported by Polish National Science Centre grant 2014/15/B/ST1/01710 and partly by US National Science Foundation awards DMS 1312508 and 1612913.

References

[1] M. Misiurewicz. Rotation intervals for a class of maps of the real line into itself. Ergodic Theory Dynam. Systems 6 (1986), 117–132.
[2] F. Rhodes, Ch.L. Thompson. Rotation numbers for monotone functions on the circle. J. London Math. Soc. 34 (1986), 360–368.
[3] J. Rubin, J. Signerska-Rynkowska, J. Touboul, A. Vidal. Wild oscillations in a nonlinear neuron model with resets: (II) Mixed-mode oscillations. (DCDS-B - accepted).

Hausdorff dimension for some non-Markovian repellers which were inspired by fractal image compression

Károly Simon

(Joint work with Balázs Bárány *(Budapest and Jerusalem)* and Michał Rams *(IMPAN)*)

The motivation of our research was to answer a fractal image compression related question asked by Michael Barnsley.

Namely, in a one dimensional image, the brightness, point by point, is represented by a function $f : I \to \mathbb{R}$, where $I := [0, 1]$. We approximate f by a function $\widetilde{f}$ whose graph is the repeller of a certain $F : I \times \mathbb{R} \to I \times \mathbb{R}$, where F is defined as follows:

Given an $N \in \mathbb{N}$ and a partition $\{I_i\}_{i=1}^{N}$ of I. Moreover, for every I_i we are given $F_i : I_i \times \mathbb{R} \to I \times \mathbb{R}$

$$F_i : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_i \\ e_i \end{pmatrix},$$

where $a_i, b_i, c_i, d_i, e_i \in \mathbb{R}$ and $|a_1| > 1$, $|d_i| > 1$. We define

$$F(x, y) := F_i(x, y) \text{ if } x \in I_i.$$

Consider

$$R := \left\{(x, y) \in I \times \mathbb{R} : \{F^n(x, y)\}_{n=1}^{\infty} \text{ is bounded}\right\}.$$

Then R is the graph of a function $\widetilde{f} : I \to \mathbb{R}$. Recall that this is the function with which we approximate f. The problem asked by Michael Barnsley was to find

$$\dim_{\mathrm{H}}(\mathrm{graph}(\widetilde{f})) =?$$

To solve this problem we combine techniques from one-dimensional dynamics and fractal geometry.

Local Minima in Traning of Neural Networks

Grzegorz Świrszcz

(Joint work with Wojciech Marian Czarnecki and Razvan Pascanu)

There has been a lot of recent interest in trying to characterize the error surface of deep models. This stems from a long standing question. Given that deep networks are highly nonlinear systems optimized by local gradient methods, why do they not seem to be affected by bad local minima? It is widely believed that training of deep models using gradient methods works so well because the error surface either has no local minima, or if they exist they need to be close in value to the global minimum. It is known that such results hold under very strong assumptions which are not satisfied by real models. In this paper we present examples showing that for such theorem to be true additional assumptions on the data, initialization schemes and/or the model classes have to be made. We look at the particular case of finite size datasets. We demonstrate that in this scenario one can construct counter-examples (datasets or initialization schemes) when the network does become susceptible to bad local minima over the weight space.

Heterogeneous Networks

Sebastian van Strien

(Joint work with Matteo Tanzi and Tiago Pereira)

The aim of this work is to rigorously study dynamics of Heterogeneously Coupled Maps (HCM). Such systems are determined by a network with heterogeneous degrees. Some nodes, called hubs, are very well connected and most nodes interact with few others. The local dynamics on each node is chaotic, coupled with other nodes according to the network structure. Such systems are very hard to describe in full. Nevertheless we are able to describe the system over exponentially large time scales (in terms of the total number of nodes of the network). In particular, we show that the dynamics of hub nodes can be very well approximated by a low-dimensional system.

In this talk I will describe these results and also give an outline of the proof - which is based on techniques coming from ergodic theory.

Attractors of Piecewise Translations

Denis Volk

Piecewise Translations is a class of dynamical systems which arises from some applications in computer science, machine learning, and electrical engineering. In dimension 1, it can also be viewed as a non-invertible generalization of Interval Exchange Transformations. These dynamical systems still possess some features of Interval Exchanges but the total volume is no longer preserved and allowed to decay.

Every Piecewise Translation has a well-defined attracting subset which is the locus of our interest. In particular, people are interested in how fast the dynamics lock onto the attractor, geometry of the attractor, and its ergodic properties. I will recall some classical and modern results in $\dim = 1$, discuss generalizations to $\dim > 1$, and illustrate some open questions.

Thin annuli property and exponential distribution of return times

Anna Zdunik

(Joint work with Łukasz Pawelec and Mariusz Urbański)

We introduce the concept of Weakly Markov dynamical systems. We study for these systems the asymptotic behaviour of the distribution of first return times to shrinking balls. We prove for almost all balls its convergence to the exponential law. We obtain this for sets of radii with relative Lebesgue measure converging very fast to one.

We show that several classes of smooth dynamical systems are Weakly Markov: expanding repellers, Axiom A diffeomorphisms, and holomorphic endomorphisms of complex projective spaces (none of them are assumed to be conformal), and conformal ones such as conformal iterated function systems, conformal graph directed Markov systems, conformal expanding repellers, conformal IFSs, rational functions on $\mathbb{C}$ and many more.

For the conformal dynamical systems we in fact prove much more, namely that the convergence to the exponential law is along all radii. We obtain the convergence along all radii by proving the Full Thin Annuli Property, the property interesting on its own.