OPEN PROBLEMS SESSION

ON GEOMETRIC COMPLEXITY OF JULIA SETS II

August 2020

Abstract. Questions presented at the open problems session of the online conference On geometric complexity of Julia sets II by A. Eremenko, G. Levin and A. De Zotti, together with further problems submitted by T. Das.

1. Problems submitted by A. Eremenko

1.1. *On Makienko's conjecture.*

Question 1.1 (Makienko conjecture). *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function, and let J be its Julia set. Suppose that D is a component of $\widehat{\mathbb{C}} \setminus J$ such that $\partial D = J$. Then D is completely invariant for the second iterate of f, that is, $f^{-2}(D) = D$.*

There are several restatements of this conjecture. Suppose that D is invariant, then $f\colon D \to D$ is a ramified covering. Let m be the degree of this covering. Then D is completely invariant if and only if $m = d := \deg f$.

Thus, any counterexample to this conjecture must involve a region D, and a ramified covering $f\colon D \to D$ such that f is m-to-1 in D, while f is d-to-1 on ∂D, with $d > m$. It is not even known whether such thing is possible for continuous functions, even with $m = 1$ and $d = 2$. The simplest related open question is the following:

Question 1.2. *Does there exist a map of the form $z \mapsto z^2 + c$ with a Siegel disk whose boundary coincides with its Julia set?*

One can show, see [CMMR09], that for any counterexample, J must be an *indecomposable continuum*, that is, a continuum that cannot be expressed as the union of any two of its proper subcontinua. A separated question is the following.

Question 1.3. *Can the Julia set of a rational function be an indecomposable continuum?*

Question 1.4 (by F. Przytycki). *Can a periodic point in the boundary of an immediate basin of attraction be nonaccessible from the basin (along a convergent curve)? The answer is known to be "yes" if the basin is completely invariant (Douady et al.).*

1.2. ***On completely invariant domains.***

Question 1.5. *How many completely invariant components can the Fatou set of an entire function have?*

A rational function can have at most two completely invariant components. For a long time, it was believed that I.N. Baker proved in 1970 that a transcendental entire function can have at most one, [Bak70]. However, a mistake in Baker's proof was found in 2017, and a counterexample to *his proof* was found.

Baker was proving a more general statement: let D_1 and D_2 be disjoint simply connected domains. Then there is no transcendental entire function such that both $f^{-1}(D_j)$, $j = 1, 2$, are connected. Surprisingly, this turned out to be wrong: even for a simple entire function like $z \mapsto e^z + z$, one can construct infinitely many disjoint simply connected domains whose preimages are connected, [RGS19].

The corresponding question for meromorphic functions is also open. One can conjecture that in the transcendental entire case, there is at most one completely invariant domain, and at most two for meromorphic functions. Both results are known for functions in the *Speiser class* $\mathcal{S}$ consisting of all entire maps with finitely many singular values. More precisely, it is known that for meromorphic functions in class $\mathcal{S}$, the number of completely invariant domains is two, [BKY92], while for transcendental entire functions in class $\mathcal{S}$, this number is one [RGS19].

Remark 1.6 (by W. Bergweiler)**.** *The conjecture above for meromorphic functions is stronger: if there is one completely invariant domain, it must be the whole Fatou set. This result is known to hold in class $\mathcal{S}$.*

1.3. ***On degenerate Herman rings.***

Question 1.7. *Do there exist degenerate Herman rings? That is, can we have an invariant Jordan curve in the Julia set that is not a circle, not a boundary component of Siegel disc or Herman ring, and so that the restriction of the map to this curve is conjugate to an irrational rotation?*

Note that this curve might only exist in rational or meromorphic dynamics.

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2. Problems submitted by G. Levin

2.1. *On the topology of Herman rings.*

Question 2.1. *Is it possible that closures of two different Herman rings intersect?*

Question 2.2. *Can the complement of the closure of a Herman ring be infinitely connected? In particular, can it contain more than two components? More generally, can there be a "buried" point, i.e., a point in the boundary of a Herman ring which is not in the boundary of any component of the complement to the closure of the ring?*

Note that if the boundaries of Herman rings are Jordan curves, then the answer to both questions is negative.

If the answer to both questions is negative, then the union of closures of Herman rings has the property that any continuous on this union and holomorphic inside function is uniformly approximated by rational functions, with poles outside. That would remove a condition in a characterisation of fixed points of the so-called Ruelle-Thurston push-forward operator in a rather general situation. In turn, it implies some transversality result for rational functions with summable critical points in natural local spaces, see [Lev20].

Similar questions can also be asked for Siegel discs, noting that for polynomials, by the Maximum modulus principle, bounded components of the complement to the closure of a Siegel disc (if any) must be components of the Fatou set and, on the other hand, their boundaries are in the boundary of the basin of infinity. In particular, there are no "buried" points for polynomial Siegel discs.

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3. Problems submitted by A. De Zotti

3.1. *On dimension paradoxes for irrational indifferent fixed points.*

We consider rational maps with a Siegel disk Δ and its associated critical orbit. Let Λ be the closure of the forward orbit of the critical orbit. We assume that the rotation number α of the fixed point is of high type. Then, see [Che17],

(1) if it is Herman, Λ is a Jordan curve and boudanry of the Seigel disk Δ;
(2) if α is Bryuno but not Herman, Λ is a one sided hairy Jordan curve (Cantor bouquet supported on a Jordan curve);
(3) if α is not Bryuno, then we get a Cantor bounquet.

Moreover, we show in [CDY20] that if the rotation number α is not Herman, then the Hausdorff dimension $\mathrm{HD}(\Lambda) = 2$.

Let E be the set of endpoints of the one sided hairy Jordan curve/Cantor bouquet. Then, there are some rotation numbers (in both classes"Bryuno and not Herman" and "not Bryuno") for which $\mathrm{HD}(\Lambda \backslash (E \cup \overline{\Delta})) = 1$, see [CDY20].

This is a dimension paradox: objects made of curves have dimension 2, but the dimension of the endpoints only is 2 while the whole of the curves (without the endpoints) has only Hausdorff dimension 1. This phenomenon was first discovered by B. Karpińska for exponential maps [Kar99b, Kar99a]. The proof in [CDY20] is very similar to the original proof on exponential maps. It works with the assumption that the rotation number has an"exponential type tower".

This type of combinatorics guarantees that renormalization has a similar behaviour to iterating the exponential maps.

Question 3.1. *Does the paradox hold in general for high type non Herman rotation numbers? Or are there counterexamples in simple families of rational/polynomial maps? Does this depend on the combinatorics?*

Remark 3.2 (by F. Przytycki)**.** *A related problem is whether* $\mathrm{HD}(J) > 1$ *for all rational functions with connected Julia set not being an analytic curve.*

References

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4. Problems submitted by T. Das

4.1. **On correspondences after Lomonaco–Bullett.** [BL20].

Based on the brief discussion after Luna Lomonaco's wonderful talk, it seems reasonable to conjecture (or, perhaps state as a moral :-) that the boundary of Lomonaco–Bullett's M_Γ has Hausdorff dimension 2. This may be titled the "Lomonaco–Rempe Problem", after the discussants!

Further natural questions/problems in this theme:
– Have you developed a thermodynamic formalism for your correspondences?
– For example, is there an analogue of Ruelle's formula for $c \mapsto HD(z^2 + c)$?
– Can you get some nice stochastic laws? E.g., as in the talks of Anna Zdunik and Fabrizio Bianchi.

4.2. **An inequality in the dimension theory of meromorphic dynamics.** [DFSU18].

Consider the following two cases:

Case 1. $\mathcal{D} = \widehat{\mathbb{C}}$, and $f : \mathcal{D} \to \widehat{\mathbb{C}}$ is a rational function of degree at least two;
Case 2. $\mathcal{D} = \mathbb{C}$, and $f : \mathcal{D} \to \widehat{\mathbb{C}}$ is a transcendental meromorphic function;

and commonly refer to them as meromorphic dynamical systems. We recall that the *Fatou set* of f consists of all those points $z \in \mathcal{D}$ that admit some neighborhood $U_z \subset \mathcal{D}$ such that the iterates of f are well-defined on U_z and their restrictions to U_z form a normal family. The *Julia set* $J = J_f$ of f is defined to be the complement of the Fatou set of f in $\widehat{\mathbb{C}}$. The Julia set J_f is a nonempty perfect subset of $\widehat{\mathbb{C}}$, enjoying the following invariance properties:

$$f(J_f \cap \mathcal{D}) \subset J_f \quad \text{and} \quad f^{-1}(J_f) = J_f \cap \mathcal{D}.$$

Note that in Case 2, the Julia set contains ∞, but in Case 1, this may or may not be true.

The radial (or conical) Julia set of f, denoted $J_{rad}(f)$, or just J_{rad}, consists of all those points $z \in J_f$ for which there exists $\delta > 0$ such that for infinitely many $n \in \mathbb{N}$, the map f^n admits an analytic local inverse branch $f_z^{-n} : B_{\text{sph}}(f^n(z), \delta) \to \mathcal{D}$ sending $f^n(z)$ to z. Here $B_{\text{sph}}(x, y)$ refers to a ball with center x and radius y w.r.t. spherical metric.

Theorem 4.1 ([DFSU18]). *If $f : \mathcal{D} \to \widehat{\mathbb{C}}$ is a meromorphic dynamical system, then*

$$\dim_{\text{dyn}}(J_f) = \dim_{\text{hyp}}(J_f) = \dim_{\text{H}}(J_{rad}(f)) = \dim_{\text{IFS}}(J_f) \le \dim_{\text{conf}}(J_f). \tag{1}$$

Here $\dim_{\text{H}}$ *refers to Hausdorff dimension, and the common number appearing in* (1) *was named the* dynamically accessible dimension *of the Julia set.*

Question 4.2. *Does equality hold in* (1) *for all transcendental functions f?*

Apart from the Hausdorff dimension of the radial Julia set of f, the four remaining numerical quantities associated with a meromorphic dynamical system in the theorem above are defined (see [DFSU18] for more details) as follows:

(1) The conformal dimension of J_f denoted $\dim_{\text{conf}}(J_f)$ is the infimum of the exponents of all locally finite conformal measures supported on J_f.

(2) The dynamical dimension of J_f denoted $\dim_{\text{dyn}}(J_f)$ is the supremum of the Hausdorff dimensions of all f-invariant ergodic probability measures μ on J_f with positive Lyapunov exponent[1] that satisfy $\text{Supp}(\mu) \subset \mathcal{D}$.

(3) The hyperbolic dimension of J_f denoted $\dim_{\text{hyp}}(J_f)$ is the supremum of the Hausdorff dimensions of all hyperbolic subsets of J_f.

(4) The IFS dimension of J_f denoted $\dim_{\text{IFS}}(J_f)$ is the supremum of the Hausdorff dimensions of all limit sets of finite inverse branch IFSes[2] of f.

I'd like to highlight a challenging question that Lasse Rempe reminded the community of in [RG14], viz. does there exist a rational function f such that

$$\dim_{\text{hyp}}(J_f) < \dim_{\text{H}}(J_f) < 2?$$

[1] The Lyapunov exponent of a measure μ is the number $\chi_\mu := \int \log |f'| \, d\mu$.

[2] An inverse branch IFS of f is a conformal IFS whose elements are local analytic inverse branches of (positive) iterates of f.

4.3. Transcendental rigidity/flexibility. [DFSU18].

A meromorphic function $f : \mathcal{D} \to \widehat{\mathbb{C}}$ is reducible if some relatively open subset of J_f is contained in a real-analytic curve; and f is irreducible if it is not reducible. It was shown by Bergweiler, Eremenko, and van Strien that a rational function is irreducible if and only if its limit set is not contained in any generalized circle (i.e. either a circle or the union of $\{\infty\}$ with a line).

Question 4.3 ([DFSU18])**.** *Does there exist a transcendental function whose limit set is contained in a curve, but not in a generalized circle?*

Can you construct a counterexample? Perhaps using quasiconformal surgery?

4.4. Bowen–Sullivan rigidity. [DSU17a].

Bowen proved that the limit set of a convex-cocompact quasi-Fuchsian group[3] is either a generalized circle or has Hausdorff dimension strictly greater than 1. Sullivan responded by showing that if the Julia set of a hyperbolic rational map is a Jordan curve, then it is either a generalized circle or has Hausdorff dimension > 1.

Sullivan's result was improved by Hamilton, who proved: if the Julia set of a rational map is a Jordan curve, then it is either a generalized circle or has Hausdorff dimension > 1. For polynomials, the assumption that the Julia set is a Jordan curve can be weakened to merely assume that the Julia set is connected; see [Urb91, p.168], where this is proven by combining the results of Douady–Hubbard, Przytycki, and Zdunik.

Theorem 4.4 (Theorem 2.3 in [DSU17a])**.** *Let $T : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational function, and let $\mathbf{K}$ denote the Julia set of T, and $\mathbf{L}$ denote the radial Julia set of T. Let $\delta = \dim_{\mathrm{H}}(\mathbf{L})$ denote the Hausdorff dimension of the radial Julia set, and $k = \dim_{\mathrm{T}}(\mathbf{K})$ denote the topological dimension of the Julia set. Assume that the non-radial points in the Julia set have zero δ-Hausdorff measure*[4]*, i.e.*

$$\mathcal{H}^{\delta}(\mathbf{K} \setminus \mathbf{L}) = 0 \tag{2}$$

(which holds, for example, if $\dim_{\mathrm{H}}(\mathbf{L}) < \dim_{\mathrm{H}}(\mathbf{K})$). Then the following dichotomy holds: either $\delta > k$, or $\mathbf{K}$ is either a generalized circle or a segment of a generalized circle (if $k = 1$) or the entire Riemann sphere (if $k = 2$).

Question 4.5. *Is Theorem 4.4 still true if the hypothesis (2) is removed?*

We also highlight an old question of Chris Bishop [Bis01, p.207], who asked:

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[3]A convex-cocompact Kleinian group $G \leq \mathrm{Mob}(\widehat{\mathbb{C}})$ is quasi-Fuchsian if it is conjugate to some cocompact Fuchsian group (i.e. a uniform lattice) in $\mathrm{Mob}(S^1)$ by some quasiconformal homeomorphism of $\widehat{\mathbb{C}}$.

[4]The hypothesis (2) is satisfied for several classes of conformal dynamical systems, since it is often the case that the appropriate analogues of $\mathbf{K} \setminus \mathbf{L}$ have Hausdorff dimension zero. In particular, this is true for geometrically finite Kleinian groups (e.g. [DSU17b, Theorem 12.4.5]), finitely generated conformal IFSes (trivially since $\mathbf{K} = \mathbf{L}$), topological Collet–Eckmann rational maps [PRL07, p.139, para.3], rational maps with no recurrent critical points [Urb94, Theorem 6.1], and certain more general classes of rational maps [RLS14, Corollary 6.3].

> ...the following Bowen-type problem is still open as far as I know: if J is a connected Julia set of a rational map, then is it true that J is either an analytic arc or has Hausdorff dimension strictly larger than 1.

If the hypothesis of the Julia set being connected is strengthened to assume that the Julia set is a Jordan curve, then the question is solved by Hamilton's theorem stated above, see [Ham95, Theorem 1]. In the other direction, if "rational function" is replaced by "transcendental meromorphic function" or "transcendental entire function", then the answer is known to be negative [Ham96, Bis18].

Question 4.6. *Are there rational maps f with*

$$\dim_{\mathrm{H}}(\mathbf{L}) \leq \dim_{\mathrm{H}}(\mathbf{K} \setminus \mathbf{L})?$$

If this is false, then the answer to Question 4.5 is yes.

4.5. On random Julia sets after Lech–Zdunik [arXiv 2004.06955].

After Krzysztof Lech's excellent short talk[5] on Lech–Zdunik, Total disconnectedness of Julia sets of random quadratic polynomials, arXiv 2004.06955, it appears that Feliks and I had the same line of questions about what can be said if one sampled parameters from a ball centered at 1/4 (rather than one centered at the origin). One also wonders what may be said for sampling from more generally placed balls; as well as for sampling from random balls (i.e. considering Julia sets that are spatially and temporally random).

References

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[5] This occurred after the problem session!

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