Equivariant KK-theory and noncommutative index theory

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Contents

Chapter 1

KK-theory

1.1 C*-algebras

Let G be a locally compact, Hausdorff, second countable (the topology of G has a countable base) group. Examples are:

- Lie groups with $\pi_0(G)$ finite - $\mathrm{SL}(n,\mathbb{R})$,
- p-adic groups - $\mathrm{SL}(n,\mathbb{Q}_p)$,
- adelic groups - $\mathrm{SL}(n,\mathbb{A})$,
- discrete groups - $\mathrm{SL}(n,\mathbb{Z})$.

For a group G we have the reduced C*-algebra of G, denoted by C_r^*G. The problem is to compute its K-theory $\mathrm{K}_j(C_r^*G)$, $j=0,1$.

Conjecture 1 (P. Baum - A. Connes)**.** *For all locally compact, Hausdorff, second countable groups G*

$$\mu\colon\ \mathrm{K}_j^G(\underline{\mathrm{E}}G)\to \mathrm{K}_j(C_r^*G)$$

is an isomorphism for $j=0,1$.

Recall some definitions:

Definition 1.1. A **Banach algebra** is an algebra A over $\mathbb{C}$ with a given norm $\|\cdot\|$

$$\|\cdot\|\colon A\to\{t\in\mathbb{R}\mid t\geq 0\}$$

such that A is complete normed algebra, i.e.

- $\|\lambda a\|=|\lambda|\|a\|$, $\lambda\in\mathbb{C}$, $a\in A$,
- $\|a+b\|\leq\|a\|+\|b\|$, $a,b\in A$,
- $\|ab\|\leq\|a\|\|b\|$, $a,b\in A$,
- $\|a\|=0$ if and only if $a=0$,

and every Cauchy sequence is convergent in A (with respect to the metric $\|a-b\|$).

Definition 1.2. A **C*-algebra** is a Banach algebra $(A,\|\cdot\|)$ with a map $*\colon A\to A$, $a\mapsto a^*$ satisfying

- $(a^*)^* = a$,
- $(a+b)^* = a^* + b^*$,
- $(ab)^* = b^* a^*$,
- $(\lambda a)^* = \bar{\lambda} a^*$, $a, b \in A$, $\lambda \in \mathbb{C}$,
- $\|aa^*\| = \|a\|^2 = \|a^*\|^2$.

A ***-morphism** is an algebra homomorphism $\varphi\colon A \to B$ such that $\varphi(a^*) = (\varphi(a))^*$ for all $a \in A$.

Lemma 1.3. *If $\varphi\colon A \to B$ is a *-homomorphism then $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$.*

Example 1.4. Let X be a locally compact Hausdorff topological space, and $X^+ = X \cup \{p_\infty\}$ its one-point compactification. Define

$$C_0(X) := \{\alpha\colon X^+ \to \mathbb{C} \mid \alpha \text{ is continuous}, \alpha(p_\infty) = 0\},$$

$$\|\alpha\| = \sup_{p \in X} |\alpha(p)|, \quad \alpha^*(p) = \overline{\alpha(p)}.$$

with operations

$$\begin{aligned}(\alpha + \beta)(p) &= \alpha(p) + \beta(p),\\ (\alpha\beta)(p) &= \alpha(p)\beta(p),\\ (\lambda\alpha)(p) &= \lambda\alpha(p),\ \lambda \in \mathbb{C}.\end{aligned}$$

If X is compact, then

$$C_0(X) := C(X) = \{\alpha\colon X \to \mathbb{C} \mid \alpha \text{ is continuous}\},$$

Example 1.5. Let $\mathcal{H}$ be a separable Hilbert space (admits a countable or finite orthonormal basis). Define

$$\mathcal{L}(\mathcal{H}) := \{T\colon \mathcal{H} \to \mathcal{H} \mid T \text{ bounded}\},$$

$$\|T\| = \sup_{u \in \mathcal{H}, \|u\|=1} \|Tu\|, \quad \|u\| = \sqrt{\langle u,\, u\rangle},$$

$$\langle Tu,\, v\rangle = \langle u,\, T^* v\rangle \text{ for all } u, v \in \mathcal{H}.$$

with operations

$$\begin{aligned}(T+S)u &= Tu + Su,\\ (TS)u &= T(Su),\\ (\lambda T)u &= \lambda(Tu), \lambda \in \mathbb{C}.\end{aligned}$$

Example 1.6. If $\mathcal{H}$ is a Hilbert space, then define

$$\begin{aligned}\mathcal{K}(\mathcal{H}) &= \{T \in \mathcal{L}(\mathcal{H}) \mid T \text{ is compact operator}\}\\ &= \overline{\{T \in \mathcal{L}(\mathcal{H}) \mid \dim_{\mathbb{C}} T(\mathcal{H}) < \infty\}}\end{aligned}$$

with the closure in operator norm. Then $\mathcal{K}(\mathcal{H})$ is a sub-C*-algebra of $\mathcal{L}(\mathcal{H})$ and an ideal in $\mathcal{L}(\mathcal{H})$.

Example 1.7. Let G be a locally compact Hausdorff second countable topological group. Fix a left-invariant Haar measure dg for G, that is for all continuous $f\colon G \to \mathbb{C}$ with compact support

$$\int_G f(\gamma g)dg = \int_G f(g)dg$$

for all $\gamma \in G$.

Let L^2G be the following Hilbert space

$$L^2G = \{u\colon G \to \mathbb{C} \mid \int_G |u(g)|^2 dg < \infty\}$$

$$\langle u,\, v\rangle = \int_G \overline{u(g)} v(g) dg, \quad u, v \in L^2G.$$

Let $\mathcal{L}(L^2G)$ be the C*-algebra of all bounded operators $T\colon L^2G \to L^2G$. Let

$$C_cG = \{f\colon G \to \mathbb{C} \mid f \text{ is continuous, and has compact support}\}.$$

Then C_cG is an algebra

$$\begin{aligned} (\lambda f)g &= \lambda(fg), \quad \lambda \in \mathbb{C},\ g \in G \\ (f + h)g &= fg + hg \\ (f * h)g_0 &= \int_G f(g)h(g^{-1}g_0)dg, \quad g_0 \in G. \end{aligned}$$

There is an injection of algebras

$$0 \to C_cG \to \mathcal{L}(L^2G)$$

given by $f \mapsto T_f$, $T_f(u) = f * u$, $u \in L^2G$,

$$(f * u)g_0 = \int_G f(g)u(g^{-1}g_0)dg, \quad g_0 \in G.$$

Define the **reduced C*-algebra** C_r^*G of G as the closure of $C_cG \subset \mathcal{L}(L^2G)$ in the operator norm. C_r^*G is a sub-C*-algebra of $\mathcal{L}(L^2G)$.

Definition 1.8. A subalgebra A of $\mathcal{L}(\mathcal{H})$ is a C*-algebra of operators if and only if

1. A is closed with respect to the operator norm.
2. If $T \in A$, then the adjoint operator $T^* \in A$.

Theorem 1.9 (I. Gelfand, V. Naimark)**.** *Any C*-algebra is isomorphic, as a C*-algebra, to a C*-algebra of operators.*

Theorem 1.10. *Let A be a commutative C*-algebra. Then A is (canonically) isomorphic to $C_0(X)$ where X is the space of maximal ideals of A.*

Thus a non-commutative C*-algebra can be viewed as a "noncommutative locally compact Hausdorff topological space".

We have an equivalence of the following categories

- Commutative C*-algebras with *-homomorphisms,
- Locally compact Hausdorff topological spaces with morphisms from X to Y being a continuous maps $f\colon X^+ \to Y^+$ with $f(p_\infty) = q_\infty$.

1.2 K-theory

Let A be a C*-algebra with unit 1_A,

$$\mathrm{K}_0(A) = \mathrm{K}_0^{alg}(A) = \text{Grothendieck group of finitely generated (left) projective } A\text{-modules}$$

In the definition of $\mathrm{K}_0(A)$ we can forget about $\|\cdot\|$ and $*$. In the definition of $\mathrm{K}_1(A)$ we cannot forget about that.

Take a topological groups $\mathrm{GL}(n, A)$ and embeddings $\mathrm{GL}(n, A) \hookrightarrow \mathrm{GL}(n+1, A)$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & \dots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & 0 \\ 0 & \dots & 0 & 1_A \end{pmatrix}$$

Then $\mathrm{GL}(A) = \varinjlim_{n\to\infty} \mathrm{GL}(n, A)$ with the direct limit topology. Define the K-theory groups

$$\mathrm{K}_j(A) := \pi_{j-1}(\mathrm{GL}(A)), \quad j = 1, 2, 3, \dots.$$

Bott periodicity states that $\Omega^2 \mathrm{GL}(A) \sim \mathrm{GL}(A)$, so $\mathrm{K}_j(A) \simeq \mathrm{K}_{j+2}(A)$ for $j = 0, 1, 2, \dots$. Thus in fact we have two groups $\mathrm{K}_0(A)$ and $\mathrm{K}_1(A)$.

If A is not unital, then we can adjoin a unit,

$$0 \to A \to \widetilde{A} \to \mathbb{C} \to 0$$

and define

$$\begin{aligned} \mathrm{K}_0(A) &:= \ker(\mathrm{K}_0(\widehat{A}) \to \mathrm{K}_0(\mathbb{C})), \\ \mathrm{K}_1(A) &:= \mathrm{K}_1(\widetilde{A}). \end{aligned}$$

If $\varphi\colon A \to B$ is a *-homomorphism, then there is an induced homomorphism of abelian groups $\mathrm{K}_j(A) \to \mathrm{K}_j(B)$.

Example 1.11. $\mathbb{C}$ is a C*-algebra, $\|\lambda\| = |\lambda|$, $\lambda^* = \bar{\lambda}$.

Theorem 1.12 (Bott).

$$\mathrm{K}_j(\mathbb{C}) = \begin{cases} \mathbb{Z} & j \text{ even} \\ 0 & j \text{ odd} \end{cases}$$

Theorem 1.13 (Bott).

$$\pi_j(\mathrm{GL}(n, \mathbb{C})) = \begin{cases} 0 & j \text{ even} \\ \mathbb{Z} & j \text{ odd} \end{cases}$$

for $j = 0, 1, \dots, 2n-1$.

For a locally compact Hausdorff topological space one defines a topological K-theory with compact supports (Atiyah-Hirzebruch)

$$\mathrm{K}^j(X) := \mathrm{K}_j(C_0(X)).$$

If X is compact Hausdorff then $\mathrm{K}^0(X)$ is the Grothendieck group of complex vector bundles on X.

There is a chern character

$$\mathrm{ch}\colon \mathrm{K}^j(X) \to \bigoplus_l \mathrm{H}_c^{j+2l}(X; \mathbb{Q}), \quad j = 0, 1.$$

Theorem 1.14. *For any locally compact Hausdorff topological space X*

$$\mathrm{ch}\colon \mathrm{K}^j(X) \to \bigoplus_l \mathrm{H}_c^{j+2l}(X;\mathbb{Q})$$

is a rational isomorphism, i.e.

$$\mathrm{ch}\colon \mathrm{K}^j(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \bigoplus_l \mathrm{H}_c^{j+2l}(X;\mathbb{Q})$$

is an isomorphism for $j = 0, 1$.

We can use Čech cohomology, Alexander-Spanier cohomology or representable cohomology (all with compact supports).

1.3 Representations

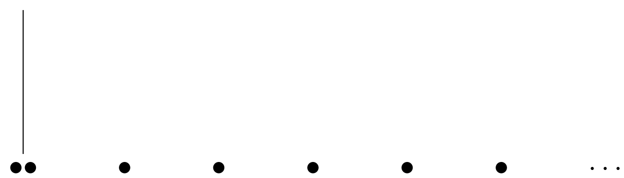
Definition 1.15. A representation of C*-algebra A is a *-homomorphism

$$\varphi\colon A \to \mathcal{L}(\mathcal{H}),$$

where $\mathcal{H}$ is a Hilbert space.

The myth: for a reduced C*-algebra C_r^*G of G there exists a locally compact Hausdorff topological space $\widehat{G}_r$. The space $\widehat{G}_r$ has one point for each distinct (i.e. non-equivalent) irreducible unitary representation of G which is weakly contained in the (left) regular representation of G. $\widehat{G}_r$ is known as the support of the Plancherel measure or the reduced unitary dual of G. The K-theory $\mathrm{K}_*(C_r^*G)$ is the topological K-theory (with compact supports of $\widehat{G}_r$).

Example 1.16. For $G = \mathrm{SL}(2, \mathbb{R})$ we have $\widehat{G}_r$:



1.4 K-homology

Let A be a separable C*-algebra (A has o countable dense subset). We will define generalized elliptic operators over A in the odd and even case.

Definition 1.17 (odd case)**.** A **generalized odd elliptic operator over** A is a triple $(\mathcal{H}, \psi, T)$ such that

1. $\mathcal{H}$ is a separable Hilbert space,
2. $\psi\colon A \to \mathcal{L}(\mathcal{H})$ is a *-homomorphisms,
3. $T \in \mathcal{L}(\mathcal{H})$

and

$$T = T^*, \quad \psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)(1 - T^2) \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$.

We will denote the set of such triples by $\mathcal{E}^1(A)$. If $\varphi\colon A \to B$ is a *-homomorphism then there is an induced map

$$\varphi^*\colon \mathcal{E}^1(B) \to \mathcal{E}^1(A), \quad \varphi^*(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi \circ \varphi, T).$$

Example 1.18. $S^1 := \{(t_1, t_2) \in \mathbb{R} \mid t_1^2 + t_2^2 = 1\}$, $A = C(S^1)$, $\psi\colon C(S^1) \to \mathcal{L}(L^2(S^1))$

$$\psi(\alpha)(u) = \alpha(u), \quad \alpha \in C(S^1),\ u \in L^2(S^1),$$

$$(\alpha u)(\lambda) = \alpha(\lambda)u(\lambda), \quad \lambda \in S^1.$$

The Dirac operator D of S^1 is $-i\frac{\partial}{\partial\theta}$. If we take a basis $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ of $L^2(S^1)$, then

$$D(e^{in\theta}) = \left(-i\frac{\partial}{\partial\theta}\right)(e^{in\theta}) = ne^{in\theta}.$$

Set $T = D(I + DD)^{-\frac{1}{2}}$. Then

$$T(e^{in\theta}) = \frac{n}{\sqrt{1+n^2}}e^{in\theta},$$

and $(L^2(S^1), \psi, T) \in \mathcal{E}^1(C(S^1))$.

We will define odd K-homology of A by

$$\mathrm{K}^1(A) := \mathcal{E}^1(A)/\sim (= \mathrm{KK}(A, \mathbb{C})),$$

where the relation $\sim$ is homotopy, which is defined below.

Definition 1.19. Let $\xi = (\mathcal{H}, \psi, T)$, $\eta = (\mathcal{H}', \psi', T')$ be elements of $\mathcal{E}^1(A)$. We say that ξ is **isomorphic** to η, $\xi \simeq \eta$ if there exists a unitary operator $U\colon \mathcal{H} \to \mathcal{H}'$ with commutativity in the diagrams

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{U} & \mathcal{H}' \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T'} \\ \mathcal{H} & \xrightarrow[U]{} & \mathcal{H}' \end{array} \qquad \begin{array}{ccc} \mathcal{H} & \xrightarrow{U} & \mathcal{H}' \\ {\scriptstyle \psi(a)}\downarrow & & \downarrow{\scriptstyle \psi'(a)} \\ \mathcal{H} & \xrightarrow[U]{} & \mathcal{H}' \end{array}$$

for all $a \in A$.

Definition 1.20. We say that $\xi = (\mathcal{H}, \psi, T), \eta = (\mathcal{H}', \psi', T') \in \mathcal{E}^1(A)$ are **strictly homotopic** if there exists a continuous function $[0, 1] \to \mathcal{L}(\mathcal{H})$, $t \mapsto T_t$ such that

1. $T_0 = T$,
2. for all $t \in [0, 1]$, $(\mathcal{H}, \psi, T_t) \in \mathcal{E}^1(A)$,
3. $(\mathcal{H}, \psi, T_1) \simeq (\mathcal{H}', \psi', T')$.

Definition 1.21. We say that a generalized elliptic operator $(\mathcal{H}, \psi, T) \in \mathcal{E}^1(A)$ is **degenerate** if and only if

$$\psi(a)T - T\psi(a) = 0, \quad \psi(a)(I - T^2) = 0, \quad \text{for all } a \in A.$$

Definition 1.22. We say that $\xi = (\mathcal{H}, \psi, T), \eta = (\mathcal{H}', \psi', T') \in \mathcal{E}^1(A)$ are **homotopic**, $\xi \sim \eta$, if and only if there exists degenerate generalized elliptic operators $\widetilde{\xi}$, $\widetilde{\eta}$ with $\xi \oplus \widetilde{\xi}$ strictly homotopic to $\eta \oplus \widetilde{\eta}$.

Definition 1.23. Odd K-homology of a C*-algebra A is defined as the group of homotopy classes of generalized odd elliptic operators,

$$\mathrm{K}^1(A) := \mathcal{E}^1(A)/\sim .$$

It is an abelian group with respect to

$$(\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T')$$

with inverse defined by

$$-(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, -T).$$

If $\varphi\colon A \to B$ is a *-homomorphism, then there is an induced map

$$\varphi^*\colon \mathrm{K}^1(B) \to \mathrm{K}^1(A), \quad \varphi^*(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi \circ \varphi, T).$$

Now we will define even elliptic operators and $\mathrm{K}^0(A)$.

Definition 1.24 (even case)**.** A **generalized even elliptic operator** over A is a triple $(\mathcal{H}, \psi, T)$ such that

1. $\mathcal{H}$ is a separable Hilbert space,
2. $\psi\colon A \to \mathcal{L}(\mathcal{H})$ is a *-homomorphisms,
3. $T \in \mathcal{L}(\mathcal{H})$

and

$$\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)(1 - TT^*) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)(1 - T^*T) \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$.

We will denote the set of such triples by $\mathcal{E}^0(A)$.

Definition 1.25. Even K-homology of a C*-algebra A is defined as the group of homotopy classes of generalized even elliptic operators,

$$\mathrm{K}^0(A) := \mathcal{E}^0(A)/\sim .$$

It is an abelian group with respect to

$$(\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T')$$

with inverse defined by

$$-(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, -T).$$

If $\varphi\colon A \to B$ is a *-homomorphism, then there is an induced map

$$\varphi^*\colon \mathrm{K}^0(B) \to \mathrm{K}^0(A), \quad \varphi^*(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi \circ \varphi, T).$$

1.5 Equivariant K-homology

Let G be a locally compact Hausdorff second countable group, and $\mathcal{H}$ a separable Hilbert space. Denote the set of unitary operators on $\mathcal{H}$ by

$$\mathcal{U}(\mathcal{H}) := \{U \in \mathcal{L}(\mathcal{H}) \mid UU^* = U^*U = I\}$$

Definition 1.26. A **unitary representation** of G is a group homomorphism $\pi\colon G \to \mathcal{U}(\mathcal{H})$ such that for each $v \in \mathcal{H}$ the map $G \to \mathcal{H}$, $g \mapsto \pi(g)v$ is a continuous map from G to $\mathcal{H}$.

Definition 1.27. A **G-C*-algebra** is a C*-algebra A with a given continuous action

$$G \times A \to A$$

by automorphisms.

Example 1.28. Let X be a locally compact G-space. Then G acts on $C_0(X)$ by

$$(g\alpha)(x) = \alpha(g^{-1}x),\ g \in G,\ \alpha \in C_0(X),\ x \in X.$$

This makes $C_0(X)$ a G-C*-algebra.

Let A be a (separable) G-C*-algebra.

Definition 1.29. A **covariant representation** of A is a triple $(\mathcal{H}, \psi, \pi)$ such that

- $\mathcal{H}$ is a separable Hilbert space,
- $\psi\colon A \to \mathcal{L}(\mathcal{H})$ is a *-homomorphism,
- $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of G,
- and
$$\psi(ga) = \pi(g)\psi(a)\pi(g^{-1})$$
for all $g \in G$, $a \in A$.

Definition 1.30. Equivariant odd K-homology $\mathrm{K}^1_G(A)$ of a G-C*-algebra A is the group of homotopy classes of quadriples $(\mathcal{H}, \psi, T, \pi)$, where $(\mathcal{H}, \psi, \pi)$ is a covariant representation of A, and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$T = T^*, \pi(g)T - T\pi(g) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)(1 - T^2) \in \mathcal{K}(\mathcal{H})$$

for all $g \in G$, $a \in A$.

$$\mathrm{K}^1_G(A) = \{(\mathcal{H}, \psi, \pi, T)\}/\sim$$

Example 1.31. Let $G = \mathbb{Z}$, $X = \mathbb{R}$, $A = C_0(\mathbb{R})$. Consider the action by translations

$$\mathbb{Z} \times \mathbb{R} \to \mathbb{R}, \quad (n, t) \mapsto n + t.$$

Let $\mathcal{H} = L^2(\mathbb{R})$. Define $\psi\colon A \to \mathcal{L}(\mathcal{H})$ by

$$\psi(\alpha)u = \alpha u, \quad \alpha u(t) = \alpha(t)u(t), \quad \alpha \in C_0(\mathbb{R}),\ u \in L^2(\mathbb{R}),\ t \in \mathbb{R}.$$

The representation $\pi\colon \mathbb{Z} \to \mathcal{U}(L^2(\mathbb{R}))$ is defined by

$$(\pi(n)u)(t) := u(t - n).$$

As an operator on $L^2(\mathbb{R})$ we take $-i\frac{d}{dx}$. It is not a bounded operator on $L^2(\mathbb{R})$, but we can"normalize" it to obtain a bounded operator T. Since $-i\frac{d}{dx}$ is self-adjoint ther is functional calculus, and T can be taken to be the function $\frac{x}{\sqrt{1+x^2}}$ applied to $-i\frac{d}{dx}$,

$$T := \left(\frac{x}{\sqrt{1+x^2}}\right)(-i\frac{d}{dx}).$$

Equivalently, T can be constructed using Fourier transform. Let $\mathcal{M}_x$ be the operator of "multiplication by x"

$$(\mathcal{M}_x f)(x) = xf(x).$$

Fourier transform converts $-i\frac{d}{dx}$ to $\mathcal{M}_x$ i.e. there is a commutativity in the diagram

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\ {\scriptstyle -i\frac{d}{dx}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_x} \\ L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \end{array}$$

where $\mathcal{F}$ denotes the Fourier transform. Let $\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ be the operator of "multiplication by $\frac{x}{\sqrt{1+x^2}}$". Then

$$\left(\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}} f\right)(x) = \frac{x}{\sqrt{1+x^2}} f(x),$$

and $\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ is a bounded operator

$$\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}} : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Now, T is the unique bounded operator $T\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ such that there is commutativity in the diagram

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}} \\ L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \end{array}$$

Then

$$(L^2(\mathbb{R}), \psi, \pi, T) \in \mathcal{E}^1_{\mathbb{Z}}(\mathbb{R}).$$

Definition 1.32. Equivariant even K-homology $\mathrm{K}^0_G(A)$ of a G-C*-algebra A is the group of homotopy classes of quadriples $(\mathcal{H}, \psi, T, \pi)$, where $(\mathcal{H}, \psi, \pi)$ is a covariant representation of A, and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\pi(g)T - T\pi(g) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)(1-T^*T) \in \mathcal{K}(\mathcal{H}), \quad \psi(a)(1-TT^*) \in \mathcal{K}(\mathcal{H})$$

for all $g \in G$, $a \in A$.

$$\mathrm{K}^0_G(A) = \{(\mathcal{H}, \psi, \pi, T)\}/\sim$$

If A, B are G-C*-algebras, and $\varphi\colon A \to B$ is a G-equivariant *-homomorphism, then $\varphi^*\colon \mathcal{E}^j_G(B) \to \mathcal{E}^j_G(A)$ for $j = 0, 1$ is given by

$$\varphi^*(\mathcal{H}, \psi, \pi, T) \mapsto (\mathcal{H}, \psi \circ \varphi, \pi, T).$$

Addition in $\mathrm{K}^j_G(A)$ is direct sum

$$(\mathcal{H}, \psi, \pi, T) + (\mathcal{H}', \psi', \pi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', \pi \oplus \pi', T \oplus T'),$$

and the inverse is

$$-(\mathcal{H}, \psi, \pi, T) = (\mathcal{H}, \psi, \pi, -T).$$

1.6 Hilbert modules

Let A be a C*-algebra. Recall that an element $a \in A$ is positive (notation: $a \geq 0$) if and only if there exists $b \in A$ such that $b^*b = a$.

Definition 1.33. A **pre-Hilbert A-module** is a right A-module $\mathcal{H}$ with a given A-valuead inner product $\langle -, - \rangle$ such that

$$\begin{aligned} \langle u, v_1 + v_2 \rangle &= \langle u, v_1 \rangle + \langle u, v_2 \rangle \\ \langle u, va \rangle &= \langle u, v \rangle a \\ \langle u, v \rangle &= \langle v, u \rangle^* \\ \langle u, u \rangle &\geq 0 \quad \forall u \in A \\ \langle u, u \rangle &= 0 \equiv u = 0 \end{aligned}$$

for $u, v_1, v_1, v \in \mathcal{H}$, $a \in A$.

Definition 1.34. A **Hilbert A-module** is a pre-Hilbert A-module $\mathcal{H}$ which is complete in the norm

$$\|u\| = \|\langle u, u \rangle\|^{\frac{1}{2}}$$

Example 1.35. A Hilbert $\mathbb{C}$-module is a Hilbert space (viewed as a right $\mathbb{C}$-module).

If $\mathcal{H}$ is a Hilbert A-module, and A has unit 1_A, then $\mathcal{H}$ is a $\mathbb{C}$-vector space with

$$u\lambda = u(\lambda 1_A), \quad \lambda \in \mathbb{C}.$$

Moreover, even if A does not have a unit, then by using approximate identity in A, it is a $\mathbb{C}$-vector space.

Example 1.36. Let A be C*-algebra. We define a Hilbert A-module structure on $\mathcal{H} = A^n$ by

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n),$$

$$(a_1, \ldots, a_n)a = (a_1 a, \ldots, a_n a),$$

$$\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = a_1^* b_1 + a_2^* b_2 + \ldots a_n^* b_n.$$

Example 1.37. Let

$$\mathcal{H} = \{(a_1, a_2, \ldots) \mid \sum_{j=1}^{\infty} a_j^* a_j \text{ is norm-convergent in } A\}$$

with the operations

$$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots),$$

$$(a_1, a_2, \ldots)a = (a_1 a, a_2 a, \ldots),$$

$$\langle (a_1, a_2, \ldots), (b_1, b_2, \ldots) \rangle = \sum_{j=1}^{\infty} a_j^* b_j.$$

Then $\mathcal{H}$ is a Hilbert A-module.

Example 1.38. Let G be a locally compact Hausdorff second countable topological group. Fix a left-invariant Haar measure dg for G. Let A be a G-C*-algebra. Denote

$$L^2(G, A) := \{f\colon G \to A \mid \int_G g^{-1} f(g)^* f(g) dg \text{ is norm-convergent in } A\}.$$

Then $L^2(G, A)$ is a Hilbert A-module with operations

$$\begin{aligned}(f + h)g &= f(g) + h(g),\\ (fa)(g) &= f(g)[ga],\\ \langle f,\, h\rangle &= \int_G g^{-1} f(g)^* h(g) dg.\end{aligned}$$

Definition 1.39. An A-module map $T\colon \mathcal{H} \to \mathcal{H}$ is **adjointable** if there exists an A-module map $T^*\colon \mathcal{H} \to \mathcal{H}$ with

$$\langle Tu,\, v\rangle = \langle u,\, T^* v\rangle$$

for all $u, v \in \mathcal{H}$.

If T^* exists, then it is unique, and $\sup_{\|u\|=1} \|Tu\| < \infty$. Set

$$\mathcal{L}(\mathcal{H}) := \{T\colon A \to A \mid \|T \text{ is adjointable}\}.$$

Then $\mathcal{L}(\mathcal{H})$ is a C*-algebra with operations

$$\begin{aligned}(T + S)u &= Tu + Su\\ (ST)(u) &= S(Tu)\\ (T\lambda)u &= (Tu)\lambda\\ \|T\| &= \sup_{\|u\|=1} \|Tu\|\end{aligned}$$

for $u \in \mathcal{H},\ \lambda \in \mathbb{C}$.

1.7 Reduced crossed product

Let A be a G-C*-algebra. Denote

$$C_c(G, A) = \{f\colon G \to A \mid f \text{ is continuous and has compact support}\}$$

Then $C_c(G, A)$ is an algebra with operations

$$\begin{aligned}(f + h)(g) &= f(g) + h(g)\\ (f\lambda)(g) &= f(g)\lambda\\ (f * h)(g_0) &= \int_G f(g)[gh(g^{-1} g_0)] dg\end{aligned}$$

for $g, g_0 \in G,\ \lambda \in \mathbb{C}$. The operation $*$ is the **twisted convolution**. There is an injection of algebras $C_c(G, A) \to \mathcal{L}(L^2(G, A))$.

$$f \mapsto T_f, \quad T_f(u) = f * u$$

$$(f * u)(g_0) = \int_G f(g)(gu(g^{-1} g_0)) dg.$$

Definition 1.40. The **reduced crossed product** C*-algebra $C_r(G, A)$ is the completion of $C_c(G, A)$ in $\mathcal{L}(L^2(G, A))$ with respect to the norm $\|f\| = \|T_f|$.

Example 1.41. Let G be a finite group and A a G-C*-algebra. Assume that each $g \in G$ has mass 1. Then

$$C_r^*(G, A) = \{\sum_{\gamma\in\Gamma} a_\gamma[\gamma] \mid a_\gamma \in A\}$$

with the following operations

$$\left(\sum_{\gamma\in\Gamma} a_\gamma[\gamma]\right) + \left(\sum_{\gamma\in\Gamma} b_\gamma[\gamma]\right) = \sum_{\gamma\in\Gamma}(a_\gamma + b_\gamma)[\gamma]$$

$$(a_\gamma[\gamma])(b_\beta[\beta]) = a_\alpha(ab_\beta)[\alpha\beta]$$

$$\left(\sum_{\gamma\in\Gamma} a_\gamma[\gamma]\right)^* = \sum_{\gamma\in\Gamma}(\gamma^{-1}a_\gamma^*)[\gamma^{-1}]$$

$$\left(\sum_{\gamma\in\Gamma} a_\gamma[\gamma]\right)\lambda = \sum_{\gamma\in G}(a_\lambda\lambda)[\gamma]$$

for $\gamma \in G$, $\lambda \in \mathbb{C}$.

Let X be a locally compact G-space. Then $C_0(X)$ is a G-C*-algebra with

$$(gf)(x) = f(g^{-1}x), \quad , f \in C_0(X),\ g \in G,\ x \in X.$$

We will denote $C_r^*(G, C_0(X))$ by $C_r^*(G, X)$. We ask about the K-theory of this C*-algebra. If G is compact, then $\mathrm{K}_j(C_r^*(G, X))$ is the Atiyah-Segal group $\mathrm{K}_G^j(X)$, $j = 0, 1$. Hence for G non-compact $\mathrm{K}_j(C_r^*(G, X))$ is the natural extension of the Atiyah-Segal theory to the case when G is non-compact.

We say that the G-space is G-compact if and only if the quotient space X/G is compact. If X is a proper G-compact G-space, then an equivariant $\mathbb{C}$-vector bundle E on X determines an element $[E] \in \mathrm{K}_0(C_r^*(G, X))$.

Theorem 1.42 (W. Lück, B. Oliver)**.** *If* Γ *is a (countable) discrete group and* X *is a proper* Γ*-compact* Γ*-space, then* $\mathrm{K}_0(C_R^*(\Gamma, X))$ *is the Grothendieck group of* Γ*-equivariant* $\mathbb{C}$*-vector bundles on* X*.*

1.8 Topological K-theory of Γ

Consider pairs (M, E) such that M is a C^∞ manifold without boundary, with a given smooth proper co-compact action of Γ and a given Γ-equivariant Spin^c-structure, and E is a Γ-equivariant vector bundle on M. We introduce an equivalence relation on such pairs, which is generated by three elementary steps

- Bordism
- Direct sum - disjoint union
- Vector bundle modification

Then we define **topological K-theory of** Γ as

$$\mathrm{K}_0^{top}(\Gamma) \oplus \mathrm{K}_1^{top}(\Gamma) = \{(M, E)\}/\sim .$$

Addition will be disjoint sum

$$(M, E) + (M', E') = (M \cup M', E \cup E').$$

The main result of this section is:

Theorem 1.43 (P. Baum, N. Higson, T. Schick)**.** *The map*

$$\tau\colon\ \mathrm{K}_j^{top}(\Gamma) \to \mathrm{K}_j^{\Gamma}(\underline{\mathrm{E}}\Gamma)$$

is an isomorphism for $j = 0, 1$.

We will describe the equivalence relation $\sim$ in details. We say that (M, E) is **isomorphic** to (M', E') if and only if there exist a Γ-equivariant diffeomorphism $\psi\colon M \to M'$ preserving the Γ-equivariant Spin^c-structures on M, M' with $\psi^* E' \simeq E$. The equivalence relation is generated by three elementary steps:

- **Bordism:** we say that (M_0, E_0) is bordant to (M_1, E_1) if and only if there exists (W, E) such that

  1. W is a C^∞ manifold with boundary, with a given smooth proper co-compact action of Γ
  2. W has a given Γ-equivariant Spin^c-structure
  3. E is a Γ-equivariant vector bundle on W
  4. $(\partial W, E|_{\partial W}) \simeq (M_0, E_0) \cup (-M_1, E_1)$.

- **Direct sum - disjoint union:** if E, E' are Γ-equivariant vector bundles on M, then

  $$(M, E) \cup (M, E') \sim (M, E \oplus E').$$

- **Vector bundle modification:** let F be a Γ-equivariant Spin^c vector bundle on M. Assume that for every fiber F_p we have $\dim_{\mathbb{R}}(F_p) = 0 \mod 2$. Take a one-dimensional Γ-equivariant trivial bundle $\mathbf{1} = M \times \mathbb{R}$, $\gamma(p, t) = (\gamma p, t)$. Let $S(F \oplus \mathbf{1})$ be the unit sphere bundle of $F \oplus \mathbf{1}$. $F \oplus \mathbf{1}$ is a Γ-equivariant Spin^c vector bundle with odd dimensional fibers. Let Σ be the spinor bundle for $F \oplus 1$

  $$\pi\colon\ \mathbb{C}\mathrm{l}(F_p \oplus \mathbb{R}) \otimes \Sigma_p \to \Sigma_p.$$

  Decompose $\pi^*\Sigma = \beta_+ \oplus \beta_-$. Then

  $$(M, E) \sim (S(F \oplus 1), \beta_+ \otimes \pi^* E).$$

1.9 KK-theory

Let A be a C*-algebra, $\mathcal{H}$ a Hilbert module, $u, v \in \mathcal{L}(\mathcal{H})$. Denote

$$\theta_{u,v} \in \mathcal{L}(\mathcal{H}), \quad \theta_{u,v}(\xi) = u\langle v, \xi\rangle, \quad \theta^*_{u,v} = \theta_{v,u}.$$

The $\theta_{u,v}$ are the **rank one** operators on $\mathcal{H}$. A **finite rank** operator on $\mathcal{H}$ is any $T \in \mathcal{L}(\mathcal{H})$ such that T is a finite sum of $\theta_{u,v}$.

$$T = \theta_{u_1,v_1} + \theta_{u_2,v_2} + \ldots + \theta_{u_n,v_n}.$$

The compact operators $\mathcal{K}(\mathcal{H})$ are defined as the norm closure in $\mathcal{L}(\mathcal{H})$ of the space of finite rank operators. It is an ideal in $\mathcal{L}(\mathcal{H})$.

We say that $\mathcal{H}$ is **countably generated** if in $\mathcal{H}$ there is a countable (or finite) set such that the A-module generated by this set is dense in $\mathcal{H}$.

Let A, B be C*-algebras, $\varphi\colon A \to B$ a *-homomorphism, and $\mathcal{H}$ a Hilbert A-module. We will define $\mathcal{H} \otimes_A B$ which will be a Hilbert B-module. First form the algebraic tensor product $\mathcal{H} \odot_A B$. It is a right B-module

$$(h \otimes b)b' = h \otimes bb', \ h \in \mathcal{H}, \ b, b' \in B.$$

Now define B-valued inner product $\langle -, - \rangle$ on $\mathcal{H} \odot_A B$ by

$$\langle h \otimes b, h' \otimes b' \rangle = b^* \varphi(\langle h, h' \rangle) b'.$$

Set

$$\mathcal{N} := \{\xi \in \mathcal{H} \odot_A B \mid \langle \xi, \xi \rangle = 0\}.$$

It is a B-submodule of $\mathcal{H} \odot_A B$, and $\mathcal{H} \odot_A B/\mathcal{N}$ is a pre-Hilbert B-module.

Definition 1.44. $\mathcal{H} \otimes_A B$ is the completion of $\mathcal{H} \odot_A B/\mathcal{N}$.

Let A, B be separable C*-algebras, $\mathcal{E}^1(A, B) = \{(\mathcal{H}, \psi, T)\}$, where $\mathcal{H}$ is a countably generated Hilbert B-module, $\psi\colon A \to \mathcal{L}(\mathcal{H})$ is a *-homomorphism, $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\begin{aligned} T &= T* \\ \psi(a)(I - T^2) &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)T - T\psi(a) &\in \mathcal{K}(\mathcal{H}) \end{aligned}$$

for all $a \in A$.

We say that $(\mathcal{H}_0, \psi_0, T_0)$, $(\mathcal{H}_1, \psi_1, T_1) \in \mathcal{E}^1(A, B)$ are **isomorphic** if there exists an isomorphism of Hilbert B-modules $\Phi\colon \mathcal{H}_0 \to \mathcal{H}_1$ with

$$\Phi\psi_0(a) = \psi_1(a)\Phi, \ \text{for all } a \in A, \ \Phi T_0 = T_1 \Phi$$

Let A, B, D be separable C*-algebras, $\varphi\colon B \to D$ a *-homomorphism. There is an induced map

$$\varphi_*\colon \mathcal{E}^1(A, B) \to \mathcal{E}^1(A, D),$$

$$\varphi_*(\mathcal{H}, \psi, T) = (\mathcal{H} \otimes_B D, \psi \otimes_B I, T \otimes_B I),$$

where I is the identity operator of D.

Consider two maps $\rho_0, \rho_1 \colon C([0,1],B) \to B$, $\rho_0(f) = f(0)$, $\rho_1(f) = f(1)$. We say that $(\mathcal{H}_0, \psi_0, T_0), (\mathcal{H}_1, \psi_1, T_1) \in \mathcal{E}^1(A,B)$ are **homotopic** if there exists $(\mathcal{H}, \psi, T) \in \mathcal{E}^1(A, C([0,1],B))$ with $(\rho_j)_*(\mathcal{H}, \psi, T) \simeq (\mathcal{H}_j, \psi_j, T_j)$.

For the even case, consider $\mathcal{E}^0(A,B) = \{(\mathcal{H}, \psi, T)\}$, where $\mathcal{H}$ is a countably generated Hilbert B-module, $\psi \colon A \to \mathcal{L}(\mathcal{H})$ is a *-homomorphism, and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H})$$
$$\psi(a)(I - T^*T) \in \mathcal{K}(\mathcal{H})$$
$$\psi(a)(I - TT^*) \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$.

Definition 1.45. We define the **KK-theory** of A, B as

$$\mathrm{KK}^0(A,B) := \mathcal{E}^0(A,B)/\sim$$
$$\mathrm{KK}^1(A,B) := \mathcal{E}^1(A,B)/\sim$$

where the relation $\sim$ is homotopy. $\mathrm{KK}^j(A,B)$ is an abelian group

$$(\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T')$$
$$-(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, T^*).$$

1.10 Equivariant KK-theory

Let A be a G-C*-algebra.

Definition 1.46. A **G-Hilbert A-module** is a Hilbert A-module $\mathcal{H}$ with a given continuous action $G \times \mathcal{H} \to \mathcal{H}$, $(g, v) \mapsto gv$ such that

$$g(u+v) = gu + gv$$
$$g(ua) = (gu)(ga)$$
$$\langle gu,\, gv \rangle = g\langle u,\, v\rangle$$

for $u, v \in \mathcal{H}$, $g \in G$, $a \in A$. Continuity here means that for each $u \in \mathcal{H}$, $g \mapsto gu$ is a continuous map $G \to \mathcal{H}$.

For each $g \in G$, denote by L_g the map $L_g \colon \mathcal{H} \to \mathcal{H}$, $L_g(v) = gv$. Note that L_g might not be in $\mathcal{L}(\mathcal{H})$. But if $T \in \mathcal{L}(\mathcal{H})$, then $L_g T L_g^{-1} \in \mathcal{L}(\mathcal{H})$. Thus $\mathcal{L}(\mathcal{H})$ is a G-C*-algebra with $gT = L_g T L_g^{-1}$.

Example 1.47. If A isa G-C*-algebra, n positive integer. Then A^n is a G-Hilbert A-module with $g(a_1, a_2, \ldots, a_n) = (ga_1, ga_2, \ldots, a_n)$.

Let A, B be separable G-C*-algebras, $\mathcal{E}^1(A,B) = \{(\mathcal{H}, \psi, T)\}$, where $\mathcal{H}$ is a G-Hilbert B-module (countably generated), $\psi \colon A \to \mathcal{L}(B)$ is a *-homomorphism with

$$\psi(ga) = g\psi(a),\ g \in G, a \in A,$$

and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\begin{aligned} T &= T^* \\ gT - T &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)T - T\psi(a) &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)(I - T^2) &\in \mathcal{K}(\mathcal{H}) \end{aligned}$$

for all $g \in G$, $a \in A$.

In the even case we take $\mathcal{E}^0(A,B) = \{(\mathcal{H},\psi,T)\}$, where $\mathcal{H}$ is a G-Hilbert B-module (countably generated), $\psi\colon A \to \mathcal{L}(B)$ is a *-homomorphism with

$$\psi(ga) = g\psi(a),\ g \in G, a \in A,$$

and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\begin{aligned} gT - T &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)T - T\psi(a) &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)(I - T^*T) &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)(I - TT^*) &\in \mathcal{K}(\mathcal{H}) \end{aligned}$$

for all $g \in G$, $a \in A$.

Definition 1.48. We define the **equivariant KK-theory** of A, B as

$$\begin{aligned} \mathrm{KK}^0_G(A,B) &:= \mathcal{E}^0(A,B)/\sim \\ \mathrm{KK}^1_G(A,B) &:= \mathcal{E}^1(A,B)/\sim \end{aligned}$$

where the relation $\sim$ is homotopy. $\mathrm{KK}^j_G(A,B)$ is an abelian group

$$\begin{aligned} (\mathcal{H},\psi,T) + (\mathcal{H}',\psi',T') &= (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T') \\ -(\mathcal{H},\psi,T) &= (\mathcal{H},\psi,T^*). \end{aligned}$$

1.11 K-theory of the reduced group C*-algebra

If a compact group G acts on $\mathbb{C}$ by a C*-automorphisms, then it must act trivially, since $\mathbb{C}$ has no nontrivial *-automorphisms. We will prove the following:

Theorem 1.49. *For a compact group G there is an isomorphism*

$$\mathrm{K}_0(C^*_r(G)) \simeq \mathrm{R}(G).$$

The key element in the proof is the Peter Weyl theorem:

Theorem 1.50 (Peter Weyl)**.** *If G is a compact, Hausdorff, second countable unitary representation of G, then every irreducible unitary representation of G is finite dimensional.*

Proof. Let $\rho\colon G \to \mathrm{U}(\mathcal{H})$ bea an irreducible representation on a separable Hilbert space $\mathcal{H}$. Choose a projection p on $\mathcal{H}$, $p \neq 0$, $p = p^*$ with finitely dimensional range. Let

$$T := \int_G \rho(g)p\rho(g)^* dg,$$

where dg is a Haar measure. Then

- T commutes with $\rho(g)$ for all $g \in G$,
- $T = T^*$, $T \geq 0$, $T \neq 0$,
- T is compact operator, $T \in \mathcal{K}(\mathcal{H})$.

The structure theorem for compact selfadjoint positive operators gives

$$\mathrm{sp}(T) := \{a_n \in \mathbb{R} \mid a_n \to 0\},$$

where each a_n is an eigenvalue with finitely dimensional eigenspace. In particular any compact selfadjoint operator has finite dimensional eigenspace. For T this eigenspace has to be preserved by the group action, so ρ has to be finitely dimensional if it is irreducible. □

Proof. (of Theorem 1.49) Notice that for compact group $C^*_r(G) = C^*(G)$ (there is only one C*-algebra for a compact group). Irreducible unitary representations of G (up to equivalence) form a countable set. There is a C*-isomorphism

$$C^*(G) \simeq \bigoplus_{\sigma \in \mathrm{Irrep}(G)} A_\sigma,$$

where each A_σ is a finitelry dimensional C*-algebra, which is isomorphic to $M_n(\mathbb{C})$, $n = \dim \sigma$. Hence

$$\mathrm{K}_j(C^*(G)) \simeq \bigoplus_{\sigma \in \mathrm{Irrep}(G)} \underbrace{\mathrm{K}_j(A_\sigma)}_{\mathrm{K}_j(\mathbb{C})} \simeq \begin{cases} \mathrm{R}(G) & \text{for } j = 0, \\ 0 & \text{for } j = 1. \end{cases}$$

□

For a compact group G we have the map

$$\mu\colon \mathrm{K}^G_j(\underline{E}G)) \to \mathrm{K}_j(C^*_r(G)).$$

The elements of $\mathrm{K}^G_j(\underline{E}G)$ can be viewed as generalized elliptic operators on $\underline{E}G$. The map μ assigns to auch a generalized elliptic operator its index

$$\mu(\mathcal{H}, \psi, T, \pi) = \ker T - \mathrm{coker}\, T.$$

1.12 $\mathrm{KK}^0_G(\mathbb{C}, \mathbb{C})$

If G is a compact group then $\underline{E}G = \mathrm{pt}$ and $\mathrm{K}_0(C^*_r(G)) = \mathrm{R}(G)$ - the representation ring of G. We obtain $\mathrm{R}(G)$ as a Grothendieck group of the category of finite dimensional (complex) representations of G. It is a free abelian group with one generator for each distinct (i.e. nonequivalent) irreducible representation of G.

Theorem 1.51. *For a compact group G there is an isomorphism*

$$\mathrm{K}_G(\mathbb{C}, \mathbb{C}) \simeq \mathrm{R}(G).$$

Proof. Given $(\mathcal{H}, \psi, T, \pi) \in \mathcal{E}^0_G(\mathbb{C})$ within the equivalence relation on $\mathcal{E}^0_G(\mathbb{C})$ we may assume that

$$T\pi(g) - \pi(g)T = 0, \tag{1.1}$$

because we can average T over the compact group G

$$T' := \int_G \pi(g)T\pi(g)^* dg = 0,$$

$$\begin{aligned} T - T' &= T - \int_G \pi(g)T\pi(g)^* dg \\ &= \int_G (T - \pi(g)T\pi(g)^*)dg \in \mathcal{K}(\mathcal{H}), \end{aligned}$$

because $\int_G T dg = T$ since we normalize Haar measure.

Furthermore we can assume that

$$\psi(\lambda) = \lambda \mathrm{Id}. \tag{1.2}$$

Indeed, $\psi\colon \mathbb{C} \to B(\mathcal{H})$ is a *-homomorphism, and $\psi(1)$ is a selfadjoint projection. For all $\lambda \in \mathbb{C}$

$$\psi(\lambda) = \lambda\psi(1), \quad p := \psi(1).$$

$\mathcal{H}$ splits into $p\mathcal{H} \oplus (1-p)\mathcal{H}$, and

$$Tp - pT \in \mathcal{K}(\mathcal{H}), T(1-p) - (1-p)T \in \mathcal{K}(\mathcal{H}).$$

Compare T to $pTp \oplus (1-p)T(1-p)$, to see that on $(1-p)\mathcal{H}$ ψ is 0.

The only nontrivial condition on $(\mathcal{H}, \psi, T, \pi)$ is

$$\begin{aligned} I - T^*T &\in \mathcal{K}(\mathcal{H}), \\ I - TT^* &\in \mathcal{K}(\mathcal{H}). \end{aligned}$$

These conditions imply that T is Fredholm, that is

$$\begin{aligned} \dim_{\mathbb{C}}(\ker T) &< \infty, \\ \dim_{\mathbb{C}}(\operatorname{coker} T) &< \infty. \end{aligned}$$

The spaces $\ker T$ and $\operatorname{coker} T$ are finite dimensional representations of G. We have

$$\mu(\mathcal{H}, \psi, T, \pi) = \ker T - \operatorname{coker} T \in \mathrm{R}(G).$$

First we will prove the surjection of $\mathrm{K}_G(\mathbb{C}, \mathbb{C}) \to \mathrm{R}(G)$. Let $V \in \mathrm{R}(G)$ be finitely dimensional irreducible unitary representation. Consider countable direct sum $\bigoplus V$ and $\bigoplus \pi$. Let T be a shift

$$(v_1, v_2, \ldots) \mapsto (v_2, v_3, \ldots).$$

Then $\ker T = V$ (first copy), and $\operatorname{coker} T = 0$.

The homomorphism $\mathrm{K}_G(\mathbb{C}, \mathbb{C}) \to \mathrm{R}(G)$ is well defined and injective. Indeed, consider irreducible representation $V \in \mathrm{R}(G)$. There is a canonical decomposition into isotypical components

$$V = n_1 V_1 \oplus n_2 V_2 \oplus \ldots n_n V_k.$$

Then T will preserve this decomposition because it commutes with the group action. If T_t, $t \in [0,1]$ is a homotopy of operators, then also each T_t commutes with $\pi(g)$, $g \in G$. We can stick to $(\mathcal{H}, \psi, \pi, T)$ with the equivalence relation consisting only of homotopy and isomorphism.

When T is unitary then $(\mathcal{H}, \psi, \pi, T)$ is degenerate. □