The Baum-Connes conjecture, localisation of categories and quantum groups

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Contents

Chapter 1

Noncommutative algebraic topology

1.1 What is noncommutative (algebraic) topology?

We can distinguish three stages of noncommutative algebraic topology:

1. K-theory of C*-algebras.

2. Topological invariants of C*-algebras.

3. Bivariant K-theory - KK-theory.

In this section we will deal with the second point. A topological invariant for C*-algebras is a functor F on the category of C*-algebras and *-morphisms, with certain formal properties. These properties are

(H) **Homotopy invariance**. If $f_0, f_1 \colon A \to B$ are two *-morphisms, then a homotopy between them is a *-homomorphism $f \colon A \to C([0,1], B)$ such that $\mathrm{ev}_t \circ f = f_t$. Homotopy invariance states that if f_0, f_1 are homotopic, then $F(f_0) = F(f_1)$.

(E) **Exactness**. For any C*-algebra extension

$$I \rightarrowtail E \twoheadrightarrow Q \tag{1.1}$$

the sequence

$$F(I) \to F(E) \to F(Q) \tag{1.2}$$

is exact.

Since KK-theory does not have this property we also allow functors that are semi-split exact, that is, a sequence (1.2) is exact only for semi-split extensions. We say that the extension (1.1) is semi-split if it has completely positive contractive section $s \colon Q \to E$. Recall that a map $s \colon Q \to E$ is positive if and only if $x \geq 0$ implies $s(x) \geq 0$. It is completely positive if and only if $M_n(s) \colon M_n(Q) \to M_n(E)$ is positive for all $n \geq 0$. A map $s \colon Q \to E$ is called contractive if $\|s\| \leq 1$.

Theorem 1.1. *The extension $I \rightarrowtail E \twoheadrightarrow Q$ with Q nuclear is semi-split.*

Theorem 1.2 (Stinespring)**.** *If $s \colon Q \to E$ is a completely positive contractive map, then there exists a C*-morphism $\pi \colon Q \to B(\mathcal{H})$, and adjointable contractive isometry $T \colon E \to \mathcal{H}_E$ ($\mathcal{H}_E$ is a Hilbert E-module) such that $s(q) = T^*\pi(q)T$.*

We say that a functor F is split-exact if for every split extension

$$I \rightarrowtail E \overset{s}{\underset{\curvearrowleft}{\twoheadrightarrow}} Q \tag{1.3}$$

The sequence

$$F(I) \longrightarrow F(E) \overset{F(s)}{\underset{\curvearrowleft}{\longrightarrow}} F(Q)$$

is exact, that is $F(E) \simeq F(I) \oplus F(Q)$.

K-theory is homotopy invariant, exact and split-exact.

Proposition 1.3. *Let F be a homotopy invariant and (semi-split) exact functor. Then for any (semi-split) extension $I \rightarrowtail E \twoheadrightarrow Q$ there is a natural long exact sequence*

$$\ldots \to F(S^2Q) \to F(SI) \to F(SE) \to F(SQ) \to F(I) \to F(E) \to F(Q) \tag{1.4}$$

where $SA := C_0((0,1), A)$ is the suspension functor.

(M) **Morita equivalence** or **C*-stability**. The third condition for a topological invariant is Morita equivalence. It is of different nature than homotopy invariance and exactness. It is a special feature of the non-commutative world.

For all C*-algebras A the corner embedding

$$A \to \mathcal{K}(l^2\mathbb{N}) \otimes A$$

induces an isomorphism $F(A) \simeq F(\mathcal{K} \otimes A)$.

We say that two C*-algebras A, B are Morita equivalent if there exists a two sided Hilbert module ${}_A\mathcal{H}_B$ over $A^{op} \otimes B$ such that

$$({}_A\mathcal{H}_B) \otimes_B ({}_B\mathcal{H}^*_A) \simeq {}_AA_A$$
$$({}_B\mathcal{H}^*_A) \otimes_A ({}_A\mathcal{H}_B) \simeq {}_BB_B$$

Theorem 1.4 (Brown–Douglas–Rieffel)**.** *Two separable C*-algebras A, B are Morita equivalent, $A \sim_M B$, if and only if $A \otimes \mathcal{K} \simeq B \otimes \mathcal{K}$.*

Definition 1.5. *A **topological invariant** for C*-algebras is a functor $F\colon C*-\mathbf{Alg} \to \mathbf{Ab}$ which is C*-stable, split exact, semi-split exact and homotopy invariant.*

Theorem 1.6 (Higson)**.** *If $F\colon C*-\mathbf{Alg} \to \mathbf{Ab}$ is C*-stable and split exact then it is homotopy invariant.*

Alse if $F\colon C*-\mathbf{Alg} \to \mathbf{Ab}$ is semi-split exact and homotopy invariant then it is split exact.

Actually, any topological invariant has many more formal properties like Bott periodicity, Pimsner–Voiculescu exact sequence for crossed product by $\mathbb{Z}$, Connes–Thom isomorphism for crossed products by $\mathbb{R}$, Mayer-Vietoris sequences.

Bott periodicity states that $F(S^2A) \simeq F(A)$ with a specified isomorphism. To prove it one can use two extensions

$$\mathcal{K} \rightarrowtail \mathcal{T} \twoheadrightarrow C(\mathrm{U}(1)) \quad \text{(Toeplitz extension)}$$

$$C_0((0,1)) \to C_0((0,1]) \xrightarrow{\text{ev}_1} \mathbb{C} \quad \text{(cone extension)}$$

$$\begin{array}{ccc} \mathcal{K} \rightarrowtail & \mathcal{T} & \twoheadrightarrow C(\mathrm{U}(1)) \\ & \cup\uparrow & \cup\uparrow \\ & \mathcal{T}_0 & \twoheadrightarrow C_0(\mathrm{U}(1) \setminus \{1\}) \end{array}$$

From the long exact sequence in proposition (1.4) we get boundary maps

$$F(S^2A) \to F(\mathcal{K} \otimes A) \simeq F(A)$$

The theorem is that this natural map is invertible for any topological invariant.

Corollary 1.7. *For any topological invariant F, and any split extension*

$$I \rightarrowtail E \twoheadrightarrow Q$$

there is a cyclic six-term exact sequence

$$\begin{array}{ccccc} F(I) & \longrightarrow & F(E) & \longrightarrow & F(Q) \\ \uparrow & & & & \downarrow \\ F(SQ) & \longleftarrow & F(SE) & \longleftarrow & F(SI) \end{array}$$

If F is a topological invariant, A C*-algebra, then $D \mapsto F(A \otimes D)$ is also a topological invariant. Therefore Bott periodicity is equivalent to the fact, that $F(\mathbb{C}) \simeq F(C_0(\mathbb{R}^2))$ for all topological invariants F.

1.1.1 Kasparov KK-theory

The reason why topological invariants have these nice properties is bivariant K-theory (also called KK-theory or Kasparov theory). Both functors $B \mapsto \mathrm{KK}(A, B)$ and $A \mapsto \mathrm{KK}(A, B)$ are topological invariants.

There is a natural product

$$\begin{aligned} \mathrm{KK}(A, B) \otimes \mathrm{KK}(B, C) &\to \mathrm{KK}(A, C) \\ (x, y) &\mapsto x \otimes_B y \end{aligned}$$

This turnes Kasparov theory into a category **KK**.

We can characterize **KK** using the universal property.

Definition 1.8. $C*-$**Alg** $\to$ **KK** *is the universal split exact, C*-stable (homotopy) functor.*

This means that the functor $C*-$**Alg** $\to$ **KK**, which maps a *-homomorphism $A \to B$ into its class in $\mathrm{KK}(A, B)$, is split exact, and C*-stable. Moreover, for any other functor F from (separable) C*-algebras to some additive category **C** there is a unique factorisation through **KK**

$$\begin{array}{ccc} C*-\mathbf{Alg} & \longrightarrow & \mathbf{KK} \\ & {\scriptstyle F}\searrow & \downarrow \\ & & \mathbf{C} \end{array}$$

This abstract point of view explains why KK-theory is so important. It is the universal topological invariant. To be useful, we need existence and a concrete description of KK.

We will describe cycles for A, B. Then homotopies will be cycles in $\mathrm{KK}_0(A, C([0,1], B))$. Next we define $\mathrm{KK}_0(A, B)$ as the set of homotopy classes of cycles. Cycles consist of

- a Hilbert B-module $\mathcal{E}$ that is $\mathbb{Z}/2$-graded, $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$
- a *-homomorphism $\varphi\colon A \to B(\mathcal{E})^{\text{even}}$
- an adjointable operator $F \in B(\mathcal{E})^{\text{odd}}$

such that

- $F = F^*$ (or $(F - F*)\varphi(a) \in \mathcal{K}(\mathcal{E})$ for all $a \in A$)
- $F^2 = 1$ (or $(F^2 - 1)\varphi(a) \in \mathcal{K}(\mathcal{E})$ for all $a \in A$)
- $[F, \varphi(a)] \in \mathcal{K}(\mathcal{E})$ for all $a \in A$.

Addition is the direct sum.

For the odd case we can take

$$\mathrm{KK}_1(A, B) \simeq \mathrm{KK}_0(A, SB) \simeq \mathrm{KK}_0(SA, B)$$

or more concretely drop $\mathbb{Z}/2$-grading in the definition of KK_0.

Kasparov uses Clifford algebras to unify KK_0 and KK_1 and the extend the definition to the real case. We do not treat the real case here but mention the following result

Theorem 1.9. *Let $A^{\mathbb{R}}$ and $B^{\mathbb{R}}$ be real C*-algebras and let $A^{\mathbb{C}} = A^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, $B^{\mathbb{C}} = B^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be their complexifications. Then there is a map*

$$\mathrm{KK}^{\mathbb{R}}(A^{\mathbb{R}}, B^{\mathbb{R}}) \to \mathrm{KK}^{\mathbb{C}}(A^{\mathbb{C}}, B^{\mathbb{C}}), \quad f^{\mathbb{R}} \mapsto f^{\mathbb{C}}.$$

Moreover $f^{\mathbb{R}}$ is invertible if and only if $f^{\mathbb{C}}$ is invertible. In particular $B^{\mathbb{R}} \sim 0$ if and only if $B^{\mathbb{C}} \sim 0$.

1.1.2 Connection between abstract and concrete description

Take a cycle $X = (\mathcal{E}, \varphi, F)$ for $\mathrm{KK}_1(A, B)$. Form $E_X = \mathcal{K}(\mathcal{E}) + \varphi(A)(\frac{1+F}{2})$. This is a C*-algebra because, modulo $\mathcal{K}(\mathcal{E})$, $P := \frac{1+F}{2}$ is a projection which commutes with $\varphi(A)$. By construction there is an extension

$$\mathcal{K}(\mathcal{E}) \rightarrowtail E_X \twoheadrightarrow A'$$

with $\varphi\colon A \to A'$, $\mathcal{K}(\mathcal{E}) \sim_M I \triangleleft B$. We can assume $\mathcal{E}$ is full and $\varphi(A)$ is injective as a map to $B(\mathcal{E})/\mathcal{K}(\mathcal{E})$. Even $\mathcal{E} = l^2\mathbb{N} \otimes B$ is possible by Kasparov's Stabilisation Theorem

$$\mathcal{E} \oplus (l^2\mathbb{N} \otimes B) \simeq l^2\mathbb{N} \otimes B$$

After simplifying using $\mathcal{K}(l^2\mathbb{N} \otimes B) \simeq \mathcal{K}(l^2\mathbb{N}) \otimes B$ we get a C*-extension

$$\mathcal{K} \otimes B \rightarrowtail E_X \twoheadrightarrow A$$

which is semi-split by $a \mapsto P\varphi(a)P$.

Conversely, this process can be inverted using Stinespring's Theorem, and any semi-split extension

$$\mathcal{K} \otimes B \rightarrowtail E \twoheadrightarrow A$$

gives a class in $\mathrm{KK}_1(A, B)$.

Thus we can describe $\mathrm{KK}_1(A, B)$ as the set of homotopy classes of semi-split extensions of A by $\mathcal{K} \otimes B$. A deep result of Kasparov replaces homotopy invariance by more rigid equivalence relation: unitary equivalence after adding split extensions. Two extensions are unitarily equivalent if there is a commuting diagram

$$\begin{array}{ccccc} \mathcal{K} \otimes B & \rightarrowtail & E_1 & \twoheadrightarrow & A \\ \downarrow{\scriptstyle \mathrm{Ad}(u)} & & \downarrow{\scriptstyle \simeq} & & \| \\ \mathcal{K} \otimes B & \rightarrowtail & E_2 & \twoheadrightarrow & A \end{array}$$

with $u \in \mathcal{K} \otimes B$ unitary.

Corollary 1.10. *For any topological invariant F there is a map*

$$\mathrm{KK}_1(Q, I) \otimes F_k(Q) \to F_{k+1}(I),$$

where $F_k(A) := F(S^k A)$.

Proof. Use the boundary map from proposition (1.3) for the extension associated to a class in $\mathrm{KK}_1(Q, I)$. □

Similar construction works in even case. We take

$$\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-, \quad \varphi = \varphi^+ \oplus \varphi^-, \quad F = \begin{pmatrix} 0 & u* \\ u & 0 \end{pmatrix}$$

with u unitary.

$$\varphi \colon A \to B(\mathcal{E}^+), \quad \mathrm{Ad}(u) \circ \varphi^- \colon A \to B(\mathcal{E}^+)$$
$$\varphi^+(a) - \mathrm{Ad}(u)\varphi^-(a) \in \mathcal{K}(\mathcal{E}^+)$$

for all $a \in A$. From a split extension $\mathcal{K}(\mathcal{E}^+) + \varphi^+(A)$ we get an extension

$$\mathcal{K}(\mathcal{E}^+) \rightarrowtail E \twoheadrightarrow A$$

that splits by φ^+ and $\mathrm{Ad}(u) \circ \varphi^-$.

Let F be a topological invariant, then

$$F(E) \simeq F(B) \oplus F(A),$$

$$F(\varphi^+) - F(\mathrm{Ad}(u) \circ \varphi^-) \colon F(A) \to F(B) \subset F(E).$$

Hence we get a map

$$\mathrm{KK}_0(A, B) \otimes F(A) \to F(B).$$

Consider two extensions

$$C \rightarrowtail E_2 \twoheadrightarrow B, \quad B \rightarrowtail E_1 \twoheadrightarrow A$$

These give a map

$$F(A) \to F(S^{-2}C) \simeq F(C).$$

The miracle of the Kasparov product is that this composite map is described by a class in $\mathrm{KK}_0(A, C)$.

Definition 1.11. *Operator F is **Fredholm** if $\ker(F)$ and $\operatorname{coker}(F)$ have finite dimension.*

The operator F in the definition of Kasparov cycles is something like a Fredholm operator. A cycle in $\mathrm{KK}_0(\mathbb{C},\mathbb{C})$ consists of a Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and an operator $F\colon \mathcal{H}_+ \to \mathcal{H}_-$, $FF^* - \mathrm{id} \in \mathcal{K}$, $F^*F - \mathrm{id} \in \mathcal{K}$, so F is Fredholm.

The index map gives an isomorphism

$$\begin{aligned} \mathrm{Index}\colon\ \mathrm{KK}_0(\mathbb{C},\mathbb{C}) &\xrightarrow{\simeq} \mathbb{Z} \\ \mathrm{Index}(F) &= \dim(\ker F) - \dim(\operatorname{coker} F) \end{aligned}$$

In the odd case we have $\mathrm{KK}_1(\mathbb{C},\mathbb{C}) = 0$.

A pair of *-homomorphisms $f, g\colon A \to B$ with $(f-g)(A) \subset \mathcal{K}$ ideal in B gives a morphism $qA \to \mathcal{K}$.

$$\mathrm{KK}(A,B) = [qA, B \otimes \mathcal{K}] \quad \text{(homotopy classes of *-homomorphisms)}$$

$$\begin{array}{ccc} A & & B \\ \sim\uparrow & & \uparrow\sim \\ qA & \longrightarrow & B \otimes \mathcal{K} \end{array}$$

$$qA \rightarrowtail A * A \underset{\mathrm{id}*\mathrm{id}}{\rightleftarrows} A$$

Here qA is the target of the universal quasi-homomorphism.

1.1.3 Relation with K-theory

KK-theory is very close to K-theory. If some construction gives a map $\mathrm{K}_* A \to \mathrm{K}_* B$ it probably gives a class in $\mathrm{KK}_*(A,B)$.

Theorem 1.12. $\mathrm{KK}_*(\mathbb{C}, A) \simeq \mathrm{K}_*(A)$.

The proof requires the concrete description of KK.

Hence there is a canonical map

$$\gamma\colon\ \mathrm{KK}_*(A,B) \to \mathrm{Hom}(\mathrm{K}_* A, \mathrm{K}_* B).$$

In many cases, this map is injective and has kernel $\mathrm{Ext}^1(K_*A, K_{*+1}B)$.

Take $\alpha \in \mathrm{KK}_1(Q,I)$, $\alpha = [I \rightarrowtail E \twoheadrightarrow Q]$. Assume $\gamma(\alpha) = 0$. There is an exact sequence

$$\begin{array}{ccccc} \mathrm{K}_0(I) & \longrightarrow & \mathrm{K}_0(E) & \longrightarrow & \mathrm{K}_0(Q) \\ {\scriptstyle\gamma(\alpha)}\uparrow & & & & \downarrow{\scriptstyle\gamma(\alpha)} \\ \mathrm{K}_1(Q) & \longleftarrow & \mathrm{K}_1(E) & \longleftarrow & \mathrm{K}_1(I) \end{array}$$

We get an extension of $\mathbb{Z}/2$-graded abelian groups.

$$\mathrm{K}_*(I) \rightarrowtail \mathrm{K}_*(E) \twoheadrightarrow \mathrm{K}_*(Q).$$

This defines a natural map

$$\mathrm{KK}_*(A,B) \supset \ker\gamma \to \mathrm{Ext}^1(\mathrm{K}_{*+1}(A), \mathrm{K}_*(B)).$$

In many cases this map and γ provide the Universal Coefficient Sequence (1.5)

Theorem 1.13. *Let* **B** *be the smallest category of separable C^*-algebras closed under suspensions, semi-split extensions,* KK*-equivalence, tensor products, and containing* $\mathbb{C}$*. Then there exists a natural exact sequence*

$$\mathrm{Ext}^1(K_{*+1}A, K_*B) \rightarrowtail \mathrm{KK}_*(A,B) \to \mathrm{Hom}(\mathrm{K}_*A, \mathrm{K}_*B) \tag{1.5}$$

for $A, B \in \mathbf{B}$

Corollary 1.14. *Let X and Y be locally compact spaces. If* $\mathrm{K}^*(X \setminus \{x\}) \simeq \mathrm{K}^*(Y \setminus \{y\})$ *then* $F(C_0(X \setminus \{x\})) \simeq F(C_0(Y \setminus \{y\}))$ *for any topological invariant for C^*-algebras.*

Proof. Denote $\widetilde{X} := X \setminus \{x\}$, $\widetilde{Y} := Y \setminus \{y\}$.

$$\alpha\colon\ \mathrm{K}^*(X \setminus \{x\}) \simeq \mathrm{K}^*(C_0(X \setminus \{x\})) \xrightarrow{\simeq} \mathrm{K}^*(C_0(Y \setminus \{y\}))$$

By the universal coefficients theorem, α lifts to $\widehat{\alpha} \in \mathrm{KK}_0(C_0(\widetilde{X}), C_0(\widetilde{Y}))$. Because $\mathrm{Ext}^1 \circ \mathrm{Ext}^1 = 0$ we know that $\widehat{\alpha}$ is invertible. Since KK is universal, $F(\widehat{\alpha})$ is invertible for any topological invariant F.

There are analogies and contrasts between homotopy theory and noncommutative topology. We will summarize them in a table: □

Homotopy theory	**Noncommutative topology**
Spaces	C*-algebras
Stable homotopy category	KK
Stable homotopy groups of spheres	Morphisms from $\mathbb{C}$ to $\mathbb{C}$ in KK
$\pi_*^s(S^0) = \mathrm{Mor}_*(\mathrm{pt}, \mathrm{pt})$	$\mathrm{KK}^*(\mathbb{C}, \mathbb{C}) = \mathbb{Z}[\beta, \beta^{-1}]$, $\deg(\beta) = 2$
	Bott periodicity
Homology $\mathrm{H}_*(-)$	K-theory $\mathrm{K}_*(-)$
Adams spectral sequence	Universal coefficients theorem for KK
Always works but complicated	Not always works, but it is easy when it works
Interesting topology - no analysis	Simple topology - interesting analysis

1.2 Equivariant theory

In equivariant bivariant Kasparov theory additional symmetries create interesting topology, making tools from homotopy theory more relevant.

What equivariant situations are being considered?

- Group actions (of locally compact groups)
- Bundles of C*-algebras $(A_x)_{x \in X}$ over some space X
- Locally compact groupoids
- locally compact quantum group actions (Baaj-Skandalis)
- C*-algebras over non-Hausdorff space (Kirchberg)

In each case, there is an equivariant K-theory with similar properties as the nonequivariant one, with a similar concrete description – add equivariance condition – and an universal property.

Proposition 1.15. *If G is a group, then $\mathrm{KK}^G(\mathbb{C},\mathbb{C})$ is a graded commutative ring, and the exterior product coincides with composition product. Furthermore $\mathrm{KK}^G(\mathbb{C},\mathbb{C})$ acts on $\mathrm{KK}^G(A,B)$ for all $A, B \in \mathbf{C^*-alg}_G$ by exterior product.*

Let $\mathcal{G}$ be a groupoid, and A a C*-algebra. Then we say that $\mathcal{G}$ acts on A, $\mathcal{G} \curvearrowright A$, if A is a bundle over $\mathcal{G}^0$, $\mathcal{G}$ acts fiberwise on this bundle. Continuity of the action is expressed by the existence of a bundle isomorphism $\alpha\colon s^*A \to r^*A$, where r, s are the range and source maps of $\mathcal{G}$.

$$\mathcal{G}^1 \underset{s}{\overset{r}{\rightrightarrows}} \mathcal{G}^0\ , \quad s^*A \xrightarrow{\alpha} r^*A, \quad (s^*A)_y = A_x.$$

$$g\colon x \to y \implies \alpha_g\colon A_x \to A_y \text{ *-isomorphism}$$

We fix some category of C*-algebras with symmetries, equivariant *-homomorphisms. We denote it $\mathbf{C^*-alg}_G$. We study functors F from $\mathbf{C^*-alg}_G$ to an additive category, such that if

$$I \rightarrowtail E \twoheadrightarrow Q \quad (\text{with a section } Q \to E)$$

is a split extension in $\mathbf{C^*-alg}_G$, then

$$F(I) \longrightarrow F(E) \longrightarrow F(Q) \quad (\text{with a section } F(Q) \to F(E))$$

Split exactness is considered for equivariant *-homomorphisms in extensions, and the section is supposed to be also equivariant.

Let A be an object of $\mathbf{C^*-alg}_G$ and $\mathcal{H}$ a G-equivariant full Hilbert module over A. Then F is stable if both maps

$$A \to \mathcal{K}(\mathcal{H} \oplus A) \leftarrow \mathcal{K}(\mathcal{H})$$

coming from inclusions of Hilbert modules $A \hookrightarrow \mathcal{H} \oplus A \hookleftarrow \mathcal{H}$ become isomorphisms after applying F

$$F(A) \to F(\mathcal{K}(\mathcal{H} \otimes A)) \leftarrow F(\mathcal{K}(\mathcal{H}))$$

In the cases mentioned above, KK^G is the universal split-exact stable functor on $\mathbf{C^*-alg}_G$ (separable), that is, any other functor with this properties factors uniquely through KK^G.

$$\begin{array}{ccc} \mathrm{KK}^G(A,B) \times F(A) & \longrightarrow & F(B) \\ \uparrow & \nearrow & \\ \mathrm{Hom}^G(A,B) \times F(A) & & \end{array}$$

1.2.1 Tensor products

The following discussion also shows how the universal property of KK can be used to construct functors between KK-categories and to prove adjointness relations between such functors.

The minimal tensor product of two G-C^*-algebras is again a G-C^*-algebra if G is a groupoid. Here we use the diagonal action of the groupoid. This yields a functor

$$\otimes\colon \mathbf{C^*-alg}_G \times \mathbf{C^*-alg}_G \to \mathbf{C^*-alg}_G, \quad (A,B) \mapsto A \otimes B.$$

For a group(oid) diagonal action of G on $A \otimes B$, if G acts on A, B. This descends to

$$\begin{array}{ccc} \mathbf{C^*-alg} \times \mathbf{C^*-alg}_G & \longrightarrow & \mathbf{C^*-alg}_G \\ \downarrow \quad\quad \downarrow & & \downarrow \\ \mathrm{KK} \times \mathrm{KK}^G & \longrightarrow & \mathrm{KK}^G \end{array}$$

We will provide the concrete description. Let $\beta \in \mathrm{KK}^G*(B_1, B_2)$, $\alpha \in \mathrm{KK}^G(A_1, A_2)$. The tensor product is given by

$$\begin{aligned} \alpha \otimes \beta &= (\alpha \otimes \mathrm{id}_{B_2}) \circ (\mathrm{id}_{A_1} \otimes \beta) \\ &= (\mathrm{id}_{A_2} \otimes \beta) \circ (\alpha \otimes \mathrm{id}_{B_1}). \end{aligned}$$

$$\begin{array}{ccc} A_1 \otimes B_1 & \xrightarrow{\mathrm{id}_{A_1} \otimes \beta} & A_1 \otimes B_2 \\ {\scriptstyle \alpha\otimes \mathrm{id}_{B_1}}\downarrow & & \downarrow{\scriptstyle \alpha\otimes \mathrm{id}_{B_2}} \\ A_2 \otimes B_1 & \xrightarrow[\mathrm{id}_{A_2} \otimes \beta]{} & A_2 \otimes B_2 \end{array}$$

In the abstract approach we fix A and consider functor

$$\mathbf{C}^* - \mathbf{alg}_G \to \mathbf{C}^* - \mathbf{alg}_G \to \mathrm{KK}^G$$

$$B \mapsto A \otimes B \mapsto A \otimes B$$

which is split-exact, stable. The functor $\mathrm{KK}^G \to \mathrm{KK}^G$ exists by the universal property.

In general, if $F_1, F_2 \colon \mathbf{C}^* - \mathbf{alg}_G \to \mathbf{Ab}$ are split exact and stable, and $\Phi \colon F_1 \to F_2$ is a natural transformation, then there exist $\overline{F_1}, \overline{F_2} \colon \mathrm{KK}^G \to \mathbf{Ab}$ and a natural transformation $\overline{\Phi} \colon \overline{F_1} \to \overline{F_2}$ such that the following diagram commutes for $\alpha \in \mathrm{KK}^G(A_1, A_2)$

$$\begin{array}{ccc} \overline{F_1}(A_1) & \xrightarrow{\overline{\Phi}_{A_1}} & \overline{F_2}(A_1) \\ {\scriptstyle \overline{F_1}(\alpha)}\downarrow & & \downarrow{\scriptstyle \overline{F_2}(\alpha)} \\ \overline{F_1}(A_2) & \xrightarrow{\overline{\Phi}_{A_2}} & \overline{F_2}(A_2) \end{array}$$

The diagram above commutes for α, β KK-morphisms provided it commutes for α, β equivariant *-homomorphisms. This is a part of the universal property of KK^G.

If A, B are $\mathcal{G}$-C*-algebras, then $A \otimes B$ gives a tensor product in $\mathrm{KK}^{\mathcal{G}}$. Descent functor $\mathrm{KK}^{\mathcal{G}} \to \mathrm{KK}$ is obtained by taking crossed products on objects and *-homomorphisms.

The functor

$$A \mapsto G \ltimes_r A$$

is split-exact, stable, so it descends to KK^G

$$\mathrm{KK}^G(A, B) \to \mathrm{KK}(G \ltimes_r A, G \ltimes_r B).$$

If $H \leq G$ is a closed subgroup, $H \curvearrowright A$, then $\mathrm{Ind}_H^G A \curvearrowleft G$, where

$$\mathrm{Ind}_H^G A := \{f \in C_0(G, A) \mid f(gh) = (\alpha_h f)(g),\ \|f\| \in C_0(G/H)\}.$$

(On the level of spaces the induction is $\mathrm{Ind}_H^G \colon X \mapsto G \times_H X$). It induces

$$\mathrm{Ind}_H^G \colon \mathrm{KK}^H \to \mathrm{KK}^G.$$

The composition

$$\begin{array}{ccccc} \mathbf{C}^* - \mathbf{alg}_H & \xrightarrow{\mathrm{Ind}_H^G} & \mathbf{C}^* - \mathbf{alg}_G & \xrightarrow{-\ltimes_r G} & \mathbf{C}^* - \mathbf{alg} \\ & \searrow^{-\ltimes_r H} & & & \wr\, {\scriptstyle M.E.} \\ & & & & \mathbf{C}^* - \mathbf{alg} \end{array}, \quad \begin{array}{ccccc} A & \xmapsto{\mathrm{Ind}_H^G} & \mathrm{Ind}_H^G A & \xmapsto{-\ltimes_r G} & G \ltimes_r \mathrm{Ind}_H^G A \\ & \searrow^{-\ltimes_r H} & & & \wr\, {\scriptstyle M.E.} \\ & & & & H \ltimes_r A \end{array}$$

becomes a natural isomorphism in $\mathrm{KK}(H \ltimes_r A, G \ltimes_t \mathrm{Ind}_H^G A)$ for H-equivariant *-homomorphisms or for KK^H-morphisms (equivalent by the universal property of KK^H).

For open $H \leq G$
$$\mathrm{KK}^G(\mathrm{Ind}_H^G A, B) \simeq \mathrm{KK}^H(A, \mathrm{Res}_H^G B)$$
the following compositions
$$\mathrm{Ind}_H^G \mathrm{Res}_G^H A \simeq C_0(G/H) \otimes A \hookrightarrow \mathcal{K}(l^2(G/H)) \otimes A \sim_{M.E.} A.$$
$$B \mapsto \mathrm{Res}_G^H \mathrm{Ind}_H^G B$$
are natural for *-homomorphisms, hence KK-morphisms.

1.3 KK as triangulated category

The category KK is additive, but not abelian. However it can be triangulated. This notion is motivated by examples in homological algebra: derived category of an abelian category, homotopy category of chain complexes over an additive category, homotopy category of spaces.

The additional structure in a triangulated category consists of

- **translation/suspension functor**. In $\mathrm{KK}^{\mathcal{G}}$:
$$A[-n] := C_0(\mathbb{R}^n) \otimes A, \quad \text{for } n \geq 0.$$
- **exact triangles**
$$A \to B \to C \to A[1].$$

Merely knowing the KK-theory class of i, p in a C*-algebra extension
$$I \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$$
does not determine the boundary maps. This requires a class in $\mathrm{KK}_1(Q, I)$.

Definition 1.16. *A diagram*
$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$
in KK^G *is called an exact triangle if there are* KK*-equivalences* α, β, γ *such that the following diagram commutes*
$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\ \alpha \downarrow \simeq & & \beta \downarrow \simeq & & \gamma \downarrow \simeq & & \alpha[1] \downarrow \simeq \\ A' & \xrightarrow[{[i]}]{} & B' & \xrightarrow[{[p]}]{} & C' & \xrightarrow[\delta]{} & A'[1] \end{array}$$
where $A' \rightarrowtail B' \twoheadrightarrow C'$ *is a C*-algebra semi-split extension, and* δ *is its class in* $\mathrm{KK}_1(C, A)$.

Proposition 1.17. *With this additional structure* KK_G *is a triangulated category.*

In general the structure of a triangulated category consists of an additive category $\mathcal{T}$, an automorphism $\Sigma \colon \mathcal{T} \to \mathcal{T}$, and a class $\mathcal{E} \subseteq \mathrm{Triangles}(\mathcal{T})$ of exact triangles.

Example 1.18. Homotopy category of chain complexes over A

$$\Sigma(C_n, d_n) = (C_{n-1}, -d_{n-1}), \quad \Sigma(f_n) = f_{n-1}(f_n\text{- chain map})$$

Triangle is exact if it is isomorphic to an exact triangle

$$I \overset{i}{\rightarrowtail} E \overset{\overset{s}{\curvearrowleft}}{\overset{p}{\twoheadrightarrow}} Q,$$

where I, E, Q are chain complexes, i, p are chain maps, s is a morphism in A. Define

$$\delta_s \colon Q \to I[1], \quad \delta_s = d^E \circ s - s \circ d^Q$$

Then

$$I \xrightarrow{i} E \xrightarrow{p} Q \xrightarrow{\delta_s} I[1]$$

is an extension triangle. However the diagram

$$\begin{array}{ccc} E & \xleftarrow{\;s\;} & Q \\ {\scriptstyle d^E}\downarrow & & \downarrow{\scriptstyle d^Q} \\ E[1] & \xleftarrow[s[1]]{} & Q[1] \end{array}$$

is not commutative.

It is easier to work with mapping cone triangles instead of extension triangles. Let $f\colon A \to B$ be a *-homomorphism. Then we define its cone as the algebra

$$C_f := \{(a, b) \in A \oplus C_0([0, 1]) \otimes B \mid f(a) = b(1)\}$$

$$SB \rightarrowtail C_f \twoheadrightarrow A$$

is a C*-algebra semi-split extension.

On the level of spaces, if $f\colon X \to Y$ is a map, then

$$C_f = x \times [0, 1] \amalg Y/(x, 0) \sim (x', 0) \sim (*, t), \ (x, 1) \sim f(x)$$

$K_*(C_f)$ gives a relative K-theory for f. The Puppe exact sequence for F is a long exact sequence

$$\ldots \to F(SC_f) \to F(SA) \to F(SB) \to F(C_f) \to F(A) \xrightarrow{F(f)} F(B)$$

Long exact sequence, say for KK, are often estabilished by first checking exactness of the Puppe sequence, then getting other extensions from that.

Definition 1.19. *A **mapping cone triangle** is a triangle that is isomorphic to*

$$SB \to C_f \to A \xrightarrow{f} B$$

for some f in KK^G.

Theorem 1.20. *A triangle in KK^G is exact (isomorphic to an exact triangle) if and only if it is isomorphic to a mapping cone triangle.*

Proof. Consider extension

$$\begin{array}{ccccccc} SQ & \xrightarrow{\delta} & I & \xrightarrow{i} & E & \xrightarrow{p} & Q \\ \| & & \downarrow{\scriptstyle i} & & \| & & \| \\ SQ & \longrightarrow & C_p & \longrightarrow & E & \longrightarrow & Q \end{array}$$

Exact sequences for KK are estabilished by showing that $I \hookrightarrow C_p$ is a KK-equivalence if the extension is semi-split.

Cuntz-Skandalis: exact triangles are isomorphic to mapping cone triangles.

Conversely, consider a mapping cylinder for a *-homomorphims $f\colon A \to B$, that is

$$Z_f := A \oplus_B B \otimes C([0,1]),$$

and two extensions

$$\begin{array}{ccccccc} SB & \xrightarrow{\delta} & C_f & \xrightarrow{i} & z_f & \xrightarrow{p} & B \\ \| & & \downarrow{\scriptstyle i} & & \uparrow{\scriptstyle j} & & \| \\ SB & \longrightarrow & C_f & \longrightarrow & A & \longrightarrow & B \end{array}$$

where $j\colon A \to Z_f$ is a homotopy equivalence. If the triangle

$$C[-1] \to A \to B \to C$$

is exact, then it is isomorphic to

$$SY \to C_f \to X \xrightarrow{f} Y.$$

Next we get an extension triangle

$$SX \xrightarrow{-Sf} SY \rightarrowtail C_f \twoheadrightarrow X,$$

so the triangle

$$B[-1] \xrightarrow{-w} C[-1] \xrightarrow{u} A \xrightarrow{v} B$$

is exact. □

1.4 Axioms of a triangulated categories

Triangulated category consists of an additive category with suspension automorphism and a class of exact triangles. These are supposed to satisfy the following axioms (TR0-TR4)

(TR0) If a triangle is isomorphic to an exact triangle, then it is exact. Triangles of the form

$$0 \to A \xrightarrow{\text{id}} A \to 0$$

are exact.

(TR1) Any morphism $f\colon A \to B$ can be embedded in an exact triangle

$$\Sigma B \to C \to A \xrightarrow{f} B$$

(we will see that this triangle is unique up to isomorphism and call C a cone for f).

The best proof of this for KK uses extension triangles. Let $f \in \mathrm{KK}_0(A,B) \simeq \mathrm{KK}_1(\Sigma A, B) \simeq \mathrm{Ext}(\Sigma A, B)$. Hence f generates a semi-split extension

$$\underbrace{B \otimes \mathcal{K}}_{\mathcal{K}(\mathcal{H}_B)} \rightarrowtail E \twoheadrightarrow \mathcal{G}A,$$

which yields an extension triangle

$$\begin{array}{ccccccc} \Sigma^2 A & \longrightarrow & \mathcal{K}(\mathcal{H}_B) & \longrightarrow & E & \longrightarrow & \Sigma A \\ \simeq \big\downarrow \text{Bott} & & \simeq \big\downarrow \text{M.E.} & & & & \\ A & \xrightarrow[f]{} & B & & & & \end{array}$$

Now rotate this sequence to bring f to the right place.

(TR2) The triangle

$$\Sigma B \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B$$

is exact if and only if the triangle

$$\Sigma A \xrightarrow{-\Sigma w} \Sigma B \xrightarrow{-u} C \xrightarrow{-v} A$$

is exact. We can get rid of some minus signs by taking

$$\begin{array}{ccccccc} \Sigma A & \xrightarrow{-\Sigma w} & \Sigma B & \xrightarrow{-u} & C & \xrightarrow{-v} & A \\ \text{id}\big\downarrow & & \text{id}\big\downarrow & & -\text{id}\big\downarrow & & -\text{id}\big\downarrow \\ \Sigma A & \xrightarrow[-\Sigma w]{} & \Sigma B & \xrightarrow[u]{} & C & \xrightarrow[v]{} & A \end{array}$$

By applying three times we get that

$$\Sigma^2 B \xrightarrow{-\Sigma u} \Sigma C \xrightarrow{-\Sigma v} \Sigma A \xrightarrow{-\Sigma w} \Sigma B$$

is exact. The reason for a sign is that the suspension of a mapping cone triangle for f is the mapping cone triangle for Σf but this involves a coordinate flip on $\mathbb{R}^2$ on $\Sigma^2 B = C_0(\mathbb{R}^2, B)$, which generates a sign.

Definition 1.21. *A functor F from a triangulated category to an abelian category is called homological if*

$$F(C) \to F(A) \to F(B)$$

is exact for any exact triangle

$$\Sigma B \to C \to A \to B.$$

Example 1.22. If F is a semi-split exact, split exact, C*-stable functor on $\mathbf{C^* - alg}$, then its extension to KK is homological.

Proposition 1.23. *If F is homological, then any exact triangle yields a long exact sequence*

$$\dots F_n(C) \to F_n(A) \to F_n(B) \to F_{n-1}(C) \to \dots$$

where $F_n(A) := F(\Sigma^n A)$, $n \in \mathbb{Z}$.

Proof. Use axiom (TR2). □

(TR3) Consider a commuting diagram with exact rows

$$\begin{array}{ccccccc}
\Sigma B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & B \\
{\scriptstyle \Sigma\beta}\downarrow & & \vdots\,{\scriptstyle \exists\gamma} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
\Sigma B' & \longrightarrow & C' & \longrightarrow & A' & \longrightarrow & B'
\end{array}$$

There exists $\gamma\colon C \to C'$ making the diagram commutative (but it is not unique).

We will proof (TR3) for KK. We may assume that rows are mapping cone triangles

$$\begin{array}{ccccccc}
\Sigma B & \longrightarrow & C_f & \longrightarrow & A & \xrightarrow{f} & B \\
{\scriptstyle \Sigma\beta}\downarrow & & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\
\Sigma B' & \longrightarrow & C'_{f'} & \longrightarrow & A' & \xrightarrow[f']{} & B'
\end{array}$$

We know that α is a KK-cycle for $A \to A'$, β is a KK-cycle for $B \to B'$, and there exists a homotopy H from $\beta \circ f$ to $f' \circ \alpha$ (because the classes $[\beta \circ f] = [f' \circ \alpha]$ in KK).

Denote

$$\begin{aligned}
\alpha &= (\mathcal{H}^{\alpha}_{A}, \varphi^{\alpha}, F^{\alpha} \in B(\mathcal{H}^{\alpha})),\\
\beta &= (\mathcal{H}^{\beta}_{B}, \varphi^{\beta}, F^{\beta} \in B(\mathcal{H}^{\beta})),\\
H &= (\mathcal{H}^{H}_{C([0,1],B')}, \varphi^{H}, F^{H} \in B(\mathcal{H}^{H})),
\end{aligned}$$

such that

$$\begin{aligned}
H|_0 &= \beta \circ f = (\mathcal{H}^{\beta}, \varphi^{\beta} \circ f, F^{\beta}),\\
H|_1 &= f' \circ \alpha = (\mathcal{H}^{\alpha} \otimes_{f'} B', \varphi^{\alpha} \otimes \mathrm{id}_{B'}, F^{\alpha} \otimes \mathrm{id}_{B'}).
\end{aligned}$$

Then

$$\mathcal{H}^{\beta} \otimes C([0, \tfrac{1}{2}]) \oplus_{\mathcal{H}^{\beta} \text{ at } \frac{1}{2}} \mathcal{H}^{H} \oplus_{\mathcal{H}^{\alpha} \otimes_{f'} B'} \mathcal{H}^{\alpha}$$

is a mapping cone of f'. Now $\varphi^{\beta} \otimes C([0, \frac{1}{2}])$, φ^{H}, φ^{α} glue to $\varphi^{\gamma}\colon A \to B(\mathcal{H}^{\gamma})$. Similarly for F.

Many results use only axioms (TR0)-(TR3). The last one, (TR4) will be given at the end. Before that we will prove

Proposition 1.24. *Let D be an object of a category $\mathcal{T}$. Then the functor $A \to \mathcal{T}(D, A)$ is homological. Dually $A \mapsto \mathcal{T}(A, B)$ is cohomological for every object B in $\mathcal{T}$.*

Proof. Let

$$\Sigma B \to C \to A \to B$$

be an exact triangle in $\mathcal{T}$. We have to verify the exactness of

$$\mathcal{T}(D, C) \to \mathcal{T}(D, A) \to \mathcal{T}(D, B).$$

We use the fact that in an exact triangle, the composition $C \to A \to B$ is zero. Hence

$$\underbrace{\mathcal{T}(D, C) \longrightarrow \mathcal{T}(D, A) \longrightarrow \mathcal{T}(D, B)}_{0}$$

Now we use (TR3) to complete diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & D & = & D & \longrightarrow & 0 \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \hat{f}} & & \downarrow & & \downarrow{\scriptstyle 0} \\ \Sigma B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & B \end{array}$$

with $\hat{f}\colon D \to C$. $\square$

Example 1.25. $\mathrm{KK}^G(-, D)$ is homological, and $\mathrm{KK}^G(D, -)$ is cohomological.

Lemma 1.26 (Five lemma)**.** *Consider morphism of exact triangles*

$$\begin{array}{ccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ {\scriptstyle \Sigma\beta}\downarrow & & {\scriptstyle \gamma}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle \beta}\downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

If two of α, β, γ are invertible, then so is the third.

Proof. Assume α, β are invertible. Then $\mathcal{T}(D, \alpha)$, $\mathcal{T}(D, \beta)$, and $\mathcal{T}(D, \Sigma\alpha)$, $\mathcal{T}(D, \Sigma\beta)$ are invertible. We can use exact sequences from the proposition (1.24) and write a diagram

$$\begin{array}{ccccccccc} \mathcal{T}(D, \Sigma A) & \longrightarrow & \mathcal{T}(D, \Sigma\beta) & \longrightarrow & \mathcal{T}(D, C) & \longrightarrow & \mathcal{T}(D, A) & \longrightarrow & \mathcal{T}(D, B) \\ {\scriptstyle \mathcal{T}(D,\Sigma\alpha)}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \mathcal{T}(D,\Sigma\beta)}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \mathcal{T}(D,\gamma)}\downarrow & & {\scriptstyle \mathcal{T}(D,\alpha)}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \mathcal{T}(D,\beta)}\downarrow{\scriptstyle \simeq} \\ \mathcal{T}(D, \mathcal{G}A') & \longrightarrow & \mathcal{T}(D, \Sigma B') & \longrightarrow & \mathcal{T}(D, C') & \longrightarrow & \mathcal{T}(D, A') & \longrightarrow & \mathcal{T}(D, B') \end{array}$$

Rows are exact chain complexes, so the five lemma yields $\mathcal{T}(D, \gamma)$ invertible. $\square$

Proposition 1.27. *Let $f\colon A \to B$ be a morphism. There is up to isomorphism a unique exact triangle*

$$\Sigma B \to C \to A \xrightarrow{f} B$$

Proof. Existence comes from (TR1). From the (TR3) we get γ in the following diagram

$$\begin{array}{ccccccc} \Sigma B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & B \\ \| & & \downarrow{\scriptstyle \gamma} & & \| & & \| \\ \Sigma B & \longrightarrow & C' & \longrightarrow & A & \longrightarrow & B \end{array}$$

From the five lemma (1.26) we get that γ is invertible, which gives uniqueness. $\square$

Lemma 1.28. *Let*

$$\Sigma B \xrightarrow{u} C \xrightarrow{v} A \xrightarrow{w} B$$

be an exact triangle. Then

1. *$B = 0$ if and only if v is invertible*

2. *$u = 0$ if and only if $C \to A \to B$ is a split extension ($A \simeq C \oplus B$)*

Proof. 1. If v is invertible, then

$$0 \to C \xrightarrow{v} A \to 0$$

is an exact triangle by (TR0) and

$$\begin{array}{ccccccc} 0 & \longrightarrow & C & \xrightarrow{v} & A & \longrightarrow & 0 \\ \| & & \simeq \downarrow & & \| & & \| \\ 0 & \longrightarrow & A & = & A & \longrightarrow & 0 \end{array}$$

For the converse we use long exact sequence for $\mathcal{T}(D, -)$. We have $\mathcal{T}(D, B) = 0$ if and only if $\mathcal{T}(D, v)$ invertible. Then we use the Yoneda lemma.

2. If $A \to B$ is split epimorphism, then $B \to \Sigma^{-1}C$ vanishes because $A \to B \to \Sigma^{-1}C$ vanishes.

   Assume $u = 0$. We use exactness of

$$\mathcal{T}(B, A) \to \mathcal{T}(B, B) \to \mathcal{T}(B, \Sigma^{-1}C)$$

   to get $s\colon B \to A$

$$s \mapsto \mathrm{id}_B \mapsto 0$$

   which gives a section for $w\colon A \to B$, $w \circ s = \mathrm{id}_B$.

   Exactness of

$$\ldots \xrightarrow{0} \mathcal{T}(D, C) \to \mathcal{T}(D, A) \to \mathcal{T}(D, B) \xrightarrow{0} \ldots$$

implies that $\mathcal{T}(D, v)$ and $\mathcal{T}(D, s)$ give isomorphism

$$\mathcal{T}(D, C) \oplus \mathcal{T}(D, B) \to \mathcal{T}(D, A)$$

for all D, so (s, v) give isomorphism $C \oplus B \xrightarrow{\simeq} A$. Given B, C embed $B \oplus C \to B$ in an exact triangle

$$\Sigma B \to D \to B \oplus C \to B$$

Since $B \oplus C \xrightarrow{w} B$ is an epimorphism we have $u = 0$. From the long exact sequence

$$\ldots \xrightarrow{0} \mathcal{T}(X, D) \to \mathcal{T}(X, B \oplus C) \to \mathcal{T}(X, B) \xrightarrow{0} \ldots$$

we get $\mathcal{T}(X, D) \simeq \mathcal{T}(X, C)$ for all $X \in \mathcal{T}$, so $D \simeq C$. □

Proposition 1.29. *If*

$$\Sigma B_i \to C_i \to A_i \to B_i$$

are exact triangles for all $i \in I$, and direct sums exist, then

$$\bigoplus_{i \in I} \Sigma B_i \to \bigoplus_{i \in I} C_i \to \bigoplus_{i \in I} A_i \to \bigoplus_{i \in I} B_i$$

is exact. The same holds for products.

Definition 1.30. *A square*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \beta \downarrow & & \downarrow \beta' \\ X' & \xrightarrow{\alpha'} & Y' \end{array}$$

*is called **homotopy Cartesian** with differential* $\gamma\colon \Sigma Y' \to X$ *if*

$$\Sigma Y' \xrightarrow{\gamma} X \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} Y \oplus X' \xrightarrow{\beta', -\alpha'} Y'$$

is exact.

Given α, β in the definition we get α', β', γ' unique up to isomorphism by embedding $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ in an exact triangle (homotopy pushout). Dually, given α', β' there are α, β, γ unique up to isomorphism (homotopy pullback).

Definition 1.31. *Let* $(A_n, \alpha_n^{n+1}\colon A_n \to A_{n+1})_{n\in\mathbb{N}}$ *be an inductive system in a triangulated category. We define its **homotopy colimit*** $\underrightarrow{\operatorname{holim}}(A_n, \alpha_n^{n+1}\colon A_n \to A_{n+1})_{n\in\mathbb{N}}$ *as the desuspended cone of the map*

$$\bigoplus_{n\in\mathbb{N}} A_n \xrightarrow{\mathrm{id}-S} \bigoplus_{n\in\mathbb{N}} A_n$$

$$S|_{A_n} = \alpha_n^{n+1}\colon A_n \to A_{n+1}$$

It is unique up to isomorphism but not functorial.

$$\bigoplus_{n\in\mathbb{N}} A_n \xrightarrow{\mathrm{id}-S} \bigoplus_{n\in\mathbb{N}} A_n \to \underrightarrow{\operatorname{holim}}(A_n, \alpha_n^{n+1}) \to \bigoplus_{n\in\mathbb{N}} \Sigma^{-1} A_n$$

Proposition 1.32. *Let* $F\colon \mathcal{T} \to \mathbf{Ab}$ *be homological and commuting with* $\oplus$*, then*

$$F(\underrightarrow{\operatorname{holim}}\, A_n) = \underrightarrow{\lim}\, F(A_n)$$

If $\widetilde{F}\colon \mathcal{T} \to \mathbf{Ab}^{op}$ *is contravariant cohomological and* $\widetilde{F}(\bigoplus A_n) = \prod \widetilde{F}(A_n)$*, then there is an exact sequence*

$$\underleftarrow{\lim}^1 \widetilde{F}(A_n) \rightarrowtail \widetilde{F}(\underrightarrow{\operatorname{holim}}\, A_n) \twoheadrightarrow \underleftarrow{\lim}\, \widetilde{F}(A_n)$$

Proof. Apply F to the exact triangle defining $\underrightarrow{\operatorname{holim}}$

$$\bigoplus F_n(A_m) \xrightarrow{\mathrm{id}-S} \bigoplus F_n(A_m) \to F_n(\underrightarrow{\operatorname{holim}}\, A_n) \to \bigoplus F_{n-1}(A_m) \rightarrowtail \bigoplus F_{n-1}(A_m) \to \dots$$

$$\operatorname{coker}(\mathrm{id} - S) = \underrightarrow{\lim}\, F_n(A_m), \qquad \ker(\mathrm{id} - S) = 0.$$

□

Fact 1.33. *If* $A \to B \to C \to D \to E$ *is exact, then*

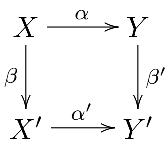
$$\operatorname{coker}(A \to B) \rightarrowtail C \to \ker(D \to E)$$

is an extension.

Example 1.34. Let $e\colon A \to A$ be an idempotent morphism. Then $\underrightarrow{\operatorname{holim}}(A, e\colon A \to A)$, $A \xrightarrow{e} A \xrightarrow{e} A \xrightarrow{e} \dots$ is a range object for e and $A \simeq eA \oplus (1-e)A$.

There are two questions concerning C*-algebras:

1. Let



be a pullback diagram of C*-algebras, so that

$$X = \{(x', y) \in X' \times Y | \alpha'(x') = \beta'(y)\}.$$

When is this image in KK homotopy Cartesian?

2. Let (A_n, α_n) be an inductive system of C*-algebras. Is $\varinjlim(A_n, \alpha_n)$ also a homotopy colimit?

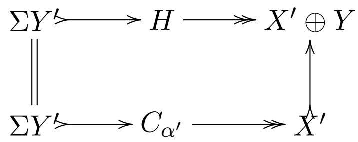
Ad 1. Compare X to the homotopy pullback

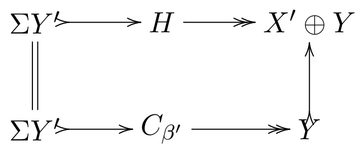
$$H = \{(x', y', y) \in X' \times C(I, Y) \times Y \mid \alpha'(x') = y'(0), \beta'(y) = y'(1)\}$$

H is a part of an extension

$$\Sigma Y' \rightarrowtail H \twoheadrightarrow X' \oplus Y$$

which is semisplit. Its class in $\mathrm{KK}_1(X' \oplus Y, \Sigma Y) \simeq \mathrm{KK}_0(X' \oplus Y, Y')$ is $(\beta', -\alpha')$, so H is a homotopy pullback.





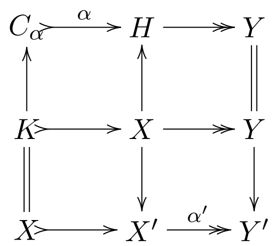
Definition 1.35. *The pullback square is admissible if $X \to H$ is a* KK*-equivalence.*

Proposition 1.36. *If α' is a semisplit epimorphism then so is α, and the pullback square is admissible. Thus we get a long exact sequence*

$$\ldots \to F_n(X) \to F_n(X') \oplus F_n(Y) \to F_n(Y') \to \ldots$$

for any semisplit-exact C-stable homotopy functor.*

Proof. If α' is semisplit epimorphism, then α is a semisplit epimorphism.



$$
\begin{array}{ccccc}
C_{\alpha} & \rightarrowtail & Z_{\alpha'} & \twoheadrightarrow & Y' \\
\uparrow & & \uparrow & & \| \\
X & \rightarrowtail & X' & \overset{\alpha'}{\twoheadrightarrow} & Y'
\end{array}
$$

The map $X' \to Z_{\alpha'}$ is a homotopy equivalence, and $K \to C_{\alpha'}$ is a KK-equivalence because the extension $K \rightarrowtail X' \twoheadrightarrow Y'$ is semisplit. Now use five lemma in KK to get that $X \to H$ is a KK-equivalence. $\square$

Ad 2. If all A_n are nuclear, then $\varinjlim(A_n, \alpha_n)$ is a homotopy colimit.

There is a fourth axiom of triangulated categories which is about exactness properties of cones of maps.

(TR4)

$$
\begin{array}{ccccccc}
X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & Z & \xrightarrow{\alpha_3} & \Sigma X \\
\| & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \delta_1} & & \| \\
X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & \Sigma X \\
 & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \Sigma\alpha_1} \\
 & & W & = & W & \xrightarrow{\beta_3} & \Sigma Y \\
 & & \downarrow{\scriptstyle \beta_3} & & \downarrow{\scriptstyle \delta_3} & & \\
 & & \Sigma Y & \xrightarrow{\Sigma\alpha_2} & \Sigma Z & &
\end{array}
$$

Given solid arrows so that $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, $(\gamma_1, \gamma_2, \gamma_3)$ are exact triangles, we can find exact triangle $(\delta_1, \delta_2, \delta_3)$ making the diagram commute.

We should warn the reader that the arrows are reversed here compared to the previous convention.

There are equivalent versions of the axiom (TR4):

(TR4') Every pair of maps

$$
\begin{array}{ccc}
X & \xrightarrow{\alpha} & Y \\
\downarrow & & \\
X' & &
\end{array}
$$

can be completed to a morphism of exact triangles

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \| & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & \Sigma X'
\end{array}
$$

such that the first square is homotopy Cartesian with differential $Y' \to \Sigma X'$.

(TR4") Given a homotopy Cartesian square

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & Y'
\end{array}
$$

and differential $\delta\colon Y' \to \Sigma X$, it can be completed to a morphism of exact triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \| & \overset{\delta}{\nearrow} & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & \Sigma X' \end{array}$$

Proposition 1.37. *The axioms (TR4), (TR4'), (TR4") are equivalent.*

1.5 Localisation of triangulated categories

Roughly speaking localisation enlarges a ring (or a category) by adding inversions of certain ring elements (or morphisms). However strange things can happen here due to non-commutativity. Actually in all examples we are going to study the localisation is just a quotient of the original category.

The motivating example is the derived category of an abelian category, which is defined as a localisation of its homotopy category of chain complexes. For any additive category $\mathbf{A}$, the homotopy category of chain complexes in $\mathbf{A}$ is a triangulated category. The suspension is a shift here.

Mapping cones for chain maps behave as in homotopy theory. If $f\colon K \to L$ is a chain map, then

$$K \xrightarrow{f} L \to C_f \to K[1]$$

is a mapping cone triangle. For C*-algebras the contravariance of the functor $\mathbf{Spaces} \to \mathbf{C^*-alg}$, $X \mapsto C(X)$ causes confusion about direction of arrows.

If $F\colon \mathbf{A} \to \mathbf{A}'$ is additive functor, then the induced functor

$$\mathrm{Ho}(F)\colon \mathrm{Ho}(\mathbf{A}) \to \mathrm{Ho}(\mathbf{A}')$$

is exact - preserves suspensions and exact triangles.

Example 1.38. Let $\Sigma\colon \mathcal{T} \to \mathcal{T}$ be a suspension functor, and

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$$

an exact triangle. The triangle

$$A[1] \xrightarrow{-u[1]} B[1] \xrightarrow{-v[1]} C \xrightarrow{-w[1]} A[2]$$

could be non-exact. To correct it we use an isomorphism

$$\Sigma(A[1]) \xrightarrow{-\mathrm{id}} (\Sigma A)[1]$$

Passage to the derived category introduces homological algebra. The quasi-isomorphisms class, that is maps that induce an invertible maps on homology, is the class of morphisms which should be inverted in derived category.

Example 1.39. The following map is a quasi-isomorphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 \\ & & & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \end{array}$$

Definition 1.40. *The localisation of a category* $\mathbf{C}$ *in a family of morphisms* S *is a category* $\mathbf{C}[S]$ *together with a functor* $F\colon \mathbf{C} \to \mathbf{C}[S^{-1}]$ *such that*

1. $F(s)$ *is invertible for all* $s \in S$
2. F *is universal among functors with this property, that is if* $G\colon \mathbf{C} \to \mathbf{C}'$ *is another functor with* $G(s)$ *invertible for all* $s \in S$*, then there is a unique factorisation*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{C}[S^{-1}] \\ & {\scriptstyle G}\searrow \quad \swarrow{\scriptstyle \exists!} & \\ & \mathbf{C} & \end{array}$$

In good cases there are some "commutation relations". We can introduce also a calculus of fractions. The pair

$$\begin{array}{ccc} & \bullet & \\ {\scriptstyle s,\ \simeq}\swarrow & & \searrow{\scriptstyle f} \\ A & & B \end{array} \quad \text{can be rewritten as} \quad \begin{array}{ccc} & \bullet & \\ {\scriptstyle g}\nearrow & & \searrow{\scriptstyle f} \\ A & & B \end{array}$$

In good cases:

- For all $f \in \mathbf{C}$, $s \in S$ there exist g, t such that $tf = gs \implies fs^{-1} = t^{-1}g$
- $S \circ S \subseteq S$ - compositoin of morphisms in S is in S.
- $s \cdot t \in S \implies t \in S$ - cancelation law.

In triangulated categories it is easier to specify which objects should become zero. Indeed for an exact triangle

$$A \xrightarrow{f} B \to C \to A[1]$$

if G is an exact functor, then $G(f)$ invertible implies $G(C) \simeq 0$.

Definition 1.41. *A class* $\mathcal{N}$ *of objects in a triangulated category* $\mathcal{T}$ *is calles* ***thick*** *if it satisfies the following conditions*

1. $0 \in \mathcal{N}$,
2. *If* $A \oplus B \in \mathcal{N}$ *then* $A, B \in \mathcal{N}$,
3. *If the triangle* $A \to B \to C \to A[1]$ *is exact, and* $A, B \in \mathcal{N}$*, then* $C \in \mathcal{N}$.

Notice that the object kernel $\{A \in \mathcal{T} \mid G(A) \simeq 0\}$ of an exact functor satisfies this.

Definition 1.42. *Given a thick subcategory* $\mathcal{N} \in \mathcal{T}$ *an* $\mathcal{N}$***-equivalence*** *is a morphism in* $\mathcal{T}$ *which cone belongs to* $\mathcal{N}$.

Denote

$$\mathcal{T}/\mathcal{N} := \mathcal{T}[(\mathcal{N} - \text{equivalences})^{-1}]$$

Theorem 1.43. *Given a thick subcategory* $\mathcal{N}$ *in a (small) triangulated category* $\mathcal{T}$*, the* $\mathcal{N}$*-equivalences have a calculus of fractions,* $\mathcal{T}/\mathcal{N}$ *is again a trianguleated category, and* $\mathcal{T} \to \mathcal{T}/\mathcal{N}$ *is an exact functor.*

Definition 1.44. ***Left orthogonal complement*** *of a class of objects $\mathcal{N}$ in $\mathcal{T}$*

$$\mathcal{N}^{\perp} := \{P \in \mathcal{T} \mid \mathcal{T}(P, N) = 0 \ \forall\, N \in \mathcal{N}\}$$

Definition 1.45. *Two thick classes of objects $\mathcal{P}, \mathcal{N}$ in $\mathcal{T}$ are called **complementary** if*

- $\mathcal{P} \subseteq \mathcal{N}^{\vdash}$
- *For all $A \in \mathcal{T}$ there is an exact triangle*

$$P \to A \to N \to P[1], \quad P \in \mathcal{P},\ N \in \mathcal{N}.$$

Theorem 1.46. *Let $(\mathcal{P}, \mathcal{N})$ be complementary. Then*

1. $\mathcal{P} = \mathcal{N}^{\vdash}$, $\mathcal{N} = \mathcal{P}^{\dashv}$
2. *the exact triangle $P \to A \to N \to P[1]$ with $P \in \mathcal{P}$, $N \in \mathcal{N}$ is unique up to canonical isomorphism and functorial in $\mathcal{A}$*
3. *the functors $\mathcal{T} \to \mathcal{P}$, $A \mapsto P$, $\mathcal{T} \to \mathcal{N}$, $A \mapsto N$ are exact.*
4. *$\mathcal{P} \to \mathcal{T}$ to$\mathcal{T}/\mathcal{N}$ and $\mathcal{N} \to \mathcal{T} \to \mathcal{T}/\mathcal{P}$ are equivalences of categories.*

Example 1.47. Take $\mathrm{Ho}(\mathcal{A})$, $\mathcal{A}$ abelian, $\mathcal{N} = \{\text{exact complexes}\}$. If $P \in \mathcal{A}$ is projective, then homotopy classes of chain maps $P \to C_\bullet$ (there is an inclusion $\mathcal{A} \hookrightarrow \mathrm{Ho}(\mathcal{A})$) are in bijection with maps $P \to \mathrm{Ho}(C_\bullet)$.

$$\begin{array}{ccccc} C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{d_0} & C_{-1} \\ & & \uparrow{\scriptstyle f} & & \\ 0 & \longrightarrow & P & \longrightarrow & 0 \end{array}$$

Notice that $\mathcal{N}^{\vdash}$ is always thick and closed under direct sums. Subcategories with both properties are called localising.

Example 1.48. Let P_0, P_1 be projective in $\mathcal{A}$, and $f\colon P_1 \to P_0$. Then its cone

$$C_f := (\ldots \to 0 \to \underbrace{P_1}_{0} \xrightarrow{f} \underbrace{P_0}_{-1} \to 0 \to \ldots)$$

Theorem 1.49 (Boekstadt-Neemann)**.** *Suppose that $\mathcal{A}$ is abelian category with enough projectives and countable direct sums. Let $\mathcal{N} \subseteq \mathrm{Ho}(\mathcal{A})$ be the full subcategory of exact chain complexes, and let $\mathcal{P}$ be the localising subcategory generated by the projective objects of $\mathcal{A} \hookrightarrow \mathrm{Ho}(\mathcal{A})$. Then $(\mathcal{P}, \mathcal{N})$ are complementary.*

The functor $P\colon \mathrm{Ho}(\mathcal{A}) \to \mathcal{P}$ replaces a module by a projective resolution of the module

$$P(M) = (\ldots \to \underbrace{P_2}_{3} \to \underbrace{P_1}_{2} \to \underbrace{P_0}_{1} \to \underbrace{M}_{0} \to 0 \to \ldots)$$

Example 1.50. Let $\mathcal{T} = \mathrm{KK}$, $\mathcal{N} = \{A \in \mathrm{KK} \mid \mathrm{K}_*(A) = 0\}$. Then $\mathbb{C} \in \mathcal{N}^{\vdash}$ because $\mathrm{KK}_*(\mathbb{C}, A) = \mathrm{K}_*(A) = 0$ for $A \in \mathcal{N}$. Let $\mathcal{B}$ be the localising subcategory generated by $\mathbb{C}$.

Theorem 1.51. *$(\mathcal{B}, \mathcal{N})$ are complementary.*

$P\colon \mathrm{KK} \to \mathcal{B}$ replaces a separable C*-algebra by one in the bootstrap class with the same K-theory.

Let $(\mathcal{P}, \mathcal{N})$ be complementary subcategories. Then

1. $\mathcal{P} = \mathcal{N}^{\vdash}$. From the assumption $\mathcal{P} \subseteq \mathcal{N}^{\vdash}$. Take $A \in \mathcal{N}^{\vdash}$ and embed it into an exact triangle

$$\underbrace{P}_{\in\mathcal{P}} \to A \xrightarrow{0} \underbrace{N}_{\mathcal{N}} \to P[1]$$

   There is a splitting $A \to P$, so A is a direct summand of P, hence $A \in \mathcal{P}$, because $\mathcal{P}$ is thick.

2. Let $A, A' \in \mathcal{T}$, $f\colon A \to A'$. Then there is a map of exact triangles

$$\begin{array}{ccccccc} P & \longrightarrow & A & \longrightarrow & N & \longrightarrow & P[1] \\ & & \downarrow{\scriptstyle f} & & & & \\ P' & \longrightarrow & A' & \longrightarrow & N' & \longrightarrow & P'[1] \end{array}$$

   with $P, P' \in \mathcal{P}$, $N, N' \in \mathcal{N}$.

   We use long exact sequence

$$\ldots \to \underbrace{\mathcal{T}(P, N')}_{=0} \to \mathcal{T}(P, P') \xrightarrow{\simeq} \mathcal{T}_0(P, A') \to \underbrace{\mathcal{T}(P, N')}_{=0} \to \ldots$$

   to get $P \xrightarrow{P_f} P'$ in the diagram

$$\begin{array}{ccccccc} P & \longrightarrow & A & \longrightarrow & N & \longrightarrow & P[1] \\ \downarrow{\scriptstyle P_f} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \Sigma P_f} \\ P' & \longrightarrow & A' & \longrightarrow & N' & \longrightarrow & P'[1] \end{array}$$

   Then use (TR3) to extend (f, P_f) to a morphism of exact triangles by $N \xrightarrow{N_f} N'$, which is unique making the diagram

$$\begin{array}{ccc} A & \longrightarrow & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle N_f} \\ A' & \longrightarrow & N' \end{array}$$

   commute.

3. $\mathcal{P}$, $\mathcal{N}$ are exact.

   From (TR1) there is X in the exact triangle

$$P_A \to P_B \to X \to P_A[1]$$

From (TR3) we can find $X \xrightarrow{f} C$ in the diagram

$$\begin{array}{ccccccc}
P_A & \longrightarrow & P_B & \longrightarrow & X & \longrightarrow & P_A[1] \\
\downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle f} & & \downarrow \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
N_A & \longrightarrow & N_B & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & N_A[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P_A[1] & \longrightarrow & P_B[1] & \longrightarrow & X[1] & \longrightarrow & P_A[1]
\end{array}$$

Thus $X = P_C$ and $\mathrm{Cone}(f) = N_C$ and f must be the canonical map $P_C \to C$.

$\mathcal{T}_*(Q, \pi_A)$ and $\mathcal{T}_*(Q, \pi_B)$ are invertible because $N_A \in \mathcal{N}$, $N_B \in \mathcal{N}$. Now we use the five lemma for

$$\begin{array}{ccccccccc}
\mathcal{T}(Q, P_A) & \longrightarrow & \mathcal{T}(Q, P_B) & \longrightarrow & \mathcal{T}(Q, X) & \longrightarrow & \mathcal{T}(Q, P_A[1]) & \longrightarrow & \mathcal{T}(Q, P_B[1]) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
\mathcal{T}(Q, A) & \longrightarrow & \mathcal{T}(Q, B) & \longrightarrow & \mathcal{T}(Q, C) & \longrightarrow & \mathcal{T}(Q, A[1]) & \longrightarrow & \mathcal{T}(Q, B[1])
\end{array}$$

There is an isomorphism $P_A[1] \simeq P_{A[1]}$.

For an exact triangle

$$P \xrightarrow{u} A \xrightarrow{v} N \xrightarrow{w} P[1]$$

the triangle

$$P[1] \xrightarrow{u} A[1] \xrightarrow{v} N[1] \xrightarrow{-w} P[2]$$

is exact.

We have seen along the way that $\mathcal{T}(Q, P_A) \simeq \mathcal{T}(Q, A)$ for all $Q \in \mathcal{P}$, which means that the functor $P \colon \mathcal{T} \to \mathcal{P}$ is right adjoint to the embedding $\mathcal{P} \hookrightarrow \mathcal{T}$.

Define $\mathcal{T}'$ as the category with the same objects as $\mathcal{T}$ and $\mathcal{T}'(A, B) := \mathcal{T}(P_A, P_B)$. Let $F \colon \mathcal{T} \to \mathcal{T}'$ be the functor that is the identity on objects and P on morphisms. This satisfies the universal property of $\mathcal{T}[(\mathcal{N} - \text{equivalences})^{-1}]$. Notice that $P_A \simeq A$ if $A \in \mathcal{P}$. Also $P_A \to A$ is an $\mathcal{N}$-equivalence.

If the triangle

$$A \xrightarrow{P_u} B \xrightarrow{P_v} C \xrightarrow{P_w} A[1]$$

is exact in $\mathcal{T}'$, then the triangle

$$P_A \xrightarrow{P_u} P_B \xrightarrow{P_v} P_C \xrightarrow{P_w} P_A[1]$$

is exact in $\mathcal{T}$.

P maps $\mathcal{N}$-equivalences to isomorphisms because $P(A) = 0$ for $A \in \mathcal{N}$. If G maps $\mathcal{N}$-equivalences to isomorphisms we get

$$\begin{array}{ccc}
G(P_A) & \longrightarrow & G(P_B) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
G(A) & \dashrightarrow & G(B)
\end{array}$$

so $\mathcal{T}'(A,B)$ gives a map $G(A) \to G(B)$.

Let $\mathcal{T}$ be triangulated and monoidal, and let $\mathcal{P}, \mathcal{N}$ be thick subcategories with $\mathcal{P} \otimes \mathcal{T} \subseteq \mathcal{P}$, $\mathcal{N} \otimes \mathcal{P} \subseteq \mathcal{N}$. If there is an exact triangle

$$P \to \mathbf{1} \to N \to P[1],$$

where $\mathbf{1}$ is the tensor unit, $P \in \mathcal{P}$, $N \in \mathcal{N}$, and $\mathcal{P} \subseteq \mathcal{N}^{\vdash}$, then $(\mathcal{P}, \mathcal{N})$ are complementary. Also for an arbitrary A the triangle

$$P \otimes A \to \mathbf{1} \otimes A \to N \otimes A \to P \otimes A[1],$$

is exact.

We expect that $\mathrm{KK}^{\mathcal{G}}$ has a (symmetric) monoidal structure also if $\mathcal{G}$ is a quantum group.

Example 1.52. Let G be finite group, A, B algebras with G-coaction (grading). Then $A \otimes B$ carries a diagonal coaction

$$(A \otimes B)_g = \bigoplus_{h \in G} A_h \otimes B_{h^{-1}g}$$

We want to equip $A \otimes B$ with a multiplication that is equivariant for the canonical coaction of G on $A \otimes B$. The usual product does not work, because if $a \in A_h$, $b \in B_g$, then $a \cdot b = b \cdot a \in (A \otimes B)_{hg}$ but we need $b \cdot a \in (A \otimes B)_{gh}$. We must therefore impose a commutation relation that is non-trivial. We define

$$b_g \cdot a_h := \alpha_g(a_h) \cdot b_g, \quad \text{for } a_h \in A_h,\ b_g \in B_g,$$

where $\alpha_g \colon A \to A$ for $g \in G$ is some linear map. Associativity dictates that $\alpha_g(a_1 \cdot a_2) = \alpha_g(a_1)\alpha_g(a_2)$, and $\alpha_{g_1}\alpha_{g_2} = \alpha_{g_1 g_2}$. It is natural to require also $\alpha_1 = \mathrm{id}_A$, so that α is an action of G on A by algebra automorphisms. Finally covariance dictates that $\alpha_g(A_h) \subseteq A_{ghg^{-1}}$ for all $g, h \in G$.

The extra structure α should always exist on a stabilisation $E_A := \mathrm{End}(A \otimes \mathbb{C}[G])$ with the coaction of G induced by the tensor product coaction on $A \otimes \mathbb{C}[G]$. $A_h \otimes |\delta_g\rangle\langle\delta_l|$ maps $(A \otimes \mathbb{C}[G])_x$ to $A_{xl^{-1}h} \otimes \mathbb{C}[G]_g \subseteq (A \otimes \mathbb{C}[G])_{xl^{-1}hg}$, hence

$$(E_A)_g = \sum_{x,y,z \in G,\ x^{-1}yz = g} A_y \otimes |\delta_z\rangle\langle\delta_x|$$

Let G act on $A \otimes \mathbb{C}[G]$ by the regular representation. This induces an action $\alpha \colon G \times E_A \to E_A$ by conjugation. We check that if $x^{-1}yz = h$, then

$$\alpha_g(A_y \otimes |\delta_z\rangle\langle\delta_x|) = A_y \otimes |\delta_{zg^{-1}}\rangle\langle\delta_{xg^{-1}}| \in (E_A)_{gx^{-1}yzg^{-1}} = (E_A)_{ghg^{-1}}$$

Thus $E_A \otimes B$ carries a canonical algebra structure.

Even in homological algebra, in $\mathrm{Ho}(R - \mathbf{Mod})$ it is not obvious that the exact chain complexes are part of a complementary pair.

$$\mathrm{Der}(R - \mathbf{Mod}) := \mathrm{Ho}(R - \mathbf{Mod})/(\text{exact chain complexes})$$

Recall $(\mathcal{L}, \mathcal{N})$ is complementary if

- $\mathrm{Hom}(\mathcal{L}, \mathcal{N}) = 0$

- For all $A \in \mathcal{T}$ there exist an exact triangle

$$L \to A \to N \to L[1]$$

With $L \in \mathcal{L}$, $N \in \mathcal{N}$.

We will explain a general method for doing homological algebra in a triangulated categories that also, eventually solves this problem.

Assume we want to understand a triangulated category $\mathcal{T}$. As a probe to explore it, we use some homological functor $F : \mathcal{T} \to \mathcal{A}$, where $\mathcal{A}$ is some abelian category.

Examples 1.53.

- $\mathcal{T} = \mathrm{Ho}(\mathcal{A})$, $\mathcal{A}$ an abelian category, and F is a homology functor $\mathrm{Ho}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$.
- $\mathcal{T} = \mathrm{KK}$, $F = \mathrm{K}_*\colon\ \mathrm{KK} \to \mathbf{Ab}^{\mathbb{Z}/2}$.
- $\mathcal{T} = \mathrm{KK}^{(C,\Delta)}$, where (C,Δ) is a quantum group, $F = \mathrm{K}_*\colon\ \mathrm{KK} \to \mathbf{Ab}^{\mathbb{Z}/2}$.

In the examples above, the target category has its own translation (suspension) automorphism, and F intertwines these translation automorphisms, we call F stable if this happens.

Actually, all the relevant information about F is contained in its morphism-kernel

$$(\ker F)(A,B) := \{\varphi\colon A \to B \mid F(\varphi) = 0\}$$

This is a finer invariant than the object kernel. $\ker F$ is called a homological ideal. Using homological ideal we can carry over various notions from homological algebra to our category $\mathcal{T}$.

Definition 1.54. *Let (C_n, d_n) be a chain complex in $\mathcal{T}$. We call it $\ker F$-exact in degree n if*

$$F(C_{n+1}) \to F(C_n) \to F(C_{n-1})$$

is exact at $F(C_n)$

Here F is exact, but it depends only on $\ker F$, so we call it $\ker F$-exact.

Definition 1.55. *An object $A \in \mathcal{T}$ is $\ker F$-projective if the functor $\mathcal{T}(A,-)$ maps $\ker F$-exact chain complexes in $\mathcal{T}$ to exact chain complexes.*

Denote $\mathcal{J} := \ker F$.

Lemma 1.56. *The following statements are equivalent*

1. *an object $A \in \mathcal{T}$ is $\mathcal{J}$-projective*
2. *for all $f \in \mathcal{J}(B,C)$ the map $\mathcal{T}(A,B) \xrightarrow{f_*} \mathcal{T}(A,C)$ vanishes*
3. *for all $C \in \mathcal{T}$ $\mathcal{J}(A,C) = 0$*

Definition 1.57. *A projective resolution of $A \in \mathcal{T}$ is a $\mathcal{J}$-exact chain complex*

$$\ldots \to P_2 \to P_1 \to P_0 \to A \to 0 \to \ldots$$

with P_i $\mathcal{J}$-projective.

Now we can ask the following questions:

- What are the projective objects in examples?
- Are there many of them? That is does every object have a $\mathcal{J}$-projective resolution?

We use (partially defined) left adjoints to decide this. Let $F\colon \mathcal{T} \to \mathcal{A}$ be stable homological with $\ker F = \mathcal{J}$. Its left adjoint $F^{\vdash}$ is defined on $B \in \mathcal{A}$ if there is $B' := F^{\vdash}(B)$ with $\mathcal{T}(B', D) \simeq \mathcal{A}(B, F(D))$ for all $D \in \mathcal{T}$, natural in D. This defines a functor on a subcategory of $\mathcal{A}$.

The functor $\mathcal{T}(F^{\vdash}(B), -)$ factors as follows

$$\mathcal{T} \xrightarrow{F} \mathcal{A} \xrightarrow{\mathcal{A}(B,-)} \mathbf{Ab}$$

$$D \mapsto F(D) \mapsto \mathcal{A}(B, F(D))$$

and therefore vanishes on $\mathcal{J} = \ker F$.

Examples 1.58. 1. Let $\mathcal{T} = \mathrm{Ho}(\mathcal{A})$, $F = \mathrm{H}_*\colon \mathrm{Ho}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$. Assume that $\mathcal{A}$ has enough projectives. Recall that if $P \in \mathcal{A}$ is projective, then $\mathcal{T}(P, C_\bullet) = \mathcal{A}(P, \mathrm{H}_*(C_\bullet))$. Thus $\mathrm{H}_*^{\vdash}$ is defined on projective objects of $\mathcal{A}$ or $\mathcal{A}^{\mathbb{Z}}$ and it produces a chain complex with vanishing boundary map.

2. Let $\mathcal{T} = \mathrm{KK}$, $F = \mathrm{K}_*\colon \mathrm{KK} \to \mathbf{Ab}^{\mathbb{Z}/2}$. Because

$$\mathrm{KK}(\mathbb{C}, A) = \mathrm{K}_*(A) = \mathrm{Hom}(\mathbb{Z}, \mathrm{K}_*(A))$$

we have

$$\mathrm{K}_*^{\vdash}(\underbrace{\mathbb{Z}[0]}_{\mathbb{Z}\text{ in degree }0}) = \mathbb{C}$$

$$\mathrm{K}_*^{\vdash}(\mathbb{Z}[1]) = \mathbb{C}[1] = C_0(\mathbb{R})$$

Left adjoints commute with direct sums, hence $\mathrm{K}_*^{\vdash}$ is defined on free $\mathbb{Z}/2$ graded abelian groups.

3. Let $\mathcal{T} = \mathrm{KK}^{\mathbb{Z}}$ be an equivariant KK-theory for integers, and $F\colon \mathrm{KK}^{\mathbb{Z}} \to \mathbf{Ab}^{\mathbb{Z}/2}$, $F(A, \alpha) = \mathrm{K}_*(A)$. If $A \in \mathrm{KK}$, $b \in \mathrm{KK}^{\mathbb{Z}}$ then

$$\mathrm{KK}^{\mathbb{Z}}(C_0(\mathbb{Z}) \otimes A, B) = \mathrm{KK}(A, B)$$

More generally, if $H \subseteq G$ is an open subgroup, then

$$\mathrm{KK}^G(\mathrm{Ind}_H^G A, B) \simeq \mathrm{KK}^H(A, \mathrm{Res}_G^H B)$$

Here we had $G = \mathbb{Z}$, $H = \{1\}$.

Since $(F \circ G)^{\vdash} = G^{\vdash} \circ F^{\vdash}$. $F^{\vdash}$ is defined on all free $\mathbb{Z}/2$-graded abelian groups, and given by

$$F^{\vdash}(\mathbb{Z}[0]) = C_0(G), \quad (G = \mathbb{Z})$$

Proposition 1.59. *Let $F\colon \mathcal{T} \to \mathcal{A}$ be a stable homological functor whose left adjoint is defined on all projective objects of an abelian category $\mathcal{A}$. If $\mathcal{A}$ has enough projectives, then there are enough $\ker F$-projective objects in $\mathcal{T}$, and any $\ker F$-projective object is a retract of $F^{\vdash}(B)$ for some projective object $B \in \mathcal{A}$.*

Proof. Let $D \in \mathcal{T}$, we need $B \in \mathcal{A}$ projective and a morphism $\pi \in \mathcal{T}(F^{\vdash}(B), D)$ such that $F(\pi)$ is an epimorphism. This is the beginning of a recursive construction of a projective resolution. We have

$$\mathcal{T}(F^{\vdash}(B), D) \simeq \mathcal{A}(B, F(D))$$
$$\rho* \leftarrow \rho$$

We claim that $F(\rho^*)$ is an epimorphism. There is a commutative diagram

$$\begin{array}{ccc} FF^{\vdash}(B) & \xrightarrow{F(\rho^*)} & F(D) \\ & {\scriptstyle \varepsilon_B} \nwarrow \quad \nearrow\!\!\!\nearrow {\scriptstyle \rho} & \\ & B & \end{array}$$

where $\varepsilon\colon \mathrm{Id} \to FF^{\vdash}$ is a unit of adjointness. □

Once we have $\mathcal{J}$-projective resolution, we get $\mathcal{J}$-derived functors. The question is how to compute them?

There are three conditions:

1. $F \circ F^{\vdash} = \mathrm{id}_{\mathrm{Proj}_A}$
2. $\mathrm{Proj}_J \xrightarrow{F} \mathrm{Proj}_A$
3. $$\left\{ \begin{array}{c} \mathcal{J} - \text{projective resolutions of } D \in \mathcal{T} \\ \text{up to isomorphism} \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{projective resolutions of } F(D) \\ \text{up to isomorphism} \end{array} \right\}$$

Example 1.60. Let $D \in \mathrm{KK}$, and there is a free resolution of its K-theory

$$\ldots \to 0 \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathrm{K}_*(D) \to 0$$

Then

$$\mathrm{KK}(\mathrm{K}_*^{\vdash}(P_1), \mathrm{K}_*^{\vdash}(P_0)) = \mathrm{Hom}_{\mathbf{Ab}^{\mathbb{Z}/2}}(P_1, P_0)$$

By (2) we can lift d_1 to $\widehat{d_1}\colon\ \mathrm{K}_*^{\vdash}(P_1) \to \mathrm{K}_*^{\vdash}(P_0)$

$$\mathrm{KK}(\mathrm{K}_*^{\vdash}(P_0), D) \simeq \mathrm{KK}(P_0, \mathrm{K}_*(D))$$
$$\widehat{d_0} \mapsto d_0$$

Then

$$0 \to \mathrm{K}_*^{\vdash}(P_1) \to \mathrm{K}_*^{\vdash}(P_0) \to 0 \to 0$$

is an $\mathcal{J}$-projective resolution, $\mathcal{J} = \ker(\mathrm{K}_*)$. Both $\mathrm{K}_*^{\vdash}(P_0)$ and $\mathrm{K}_*^{\vdash}(P_1)$ are direct sums of $\mathbb{C}$ and $C_0(\mathbb{R})$, and

$$\mathrm{K}_*(\mathrm{K}_*^{\vdash}(P_j)) = P_j$$

Hence we have lifted a projective resolution in $\mathbf{Ab}^{\mathbb{Z}/2}$ to one in KK.

In the nice case where (2) and hance (1) and (3) hold, the derived functors with respect to $\mathcal{J}$ are the same as derived functors in the abelian category $\mathcal{A}$ because resolutions are the same.

Proposition 1.61. *Assuming (1), any homological functor, $H\colon \mathcal{T} \to \mathcal{C}$ induces a right-exact functor $\overline{H}\colon \mathcal{A} \to \mathcal{C}$, and $\mathbb{L}^j_p H = \mathbb{L}^j_p \overline{H} \circ F$*

$$\operatorname{Ext}^n_{(\mathcal{T},\mathcal{J})}(D, E) \simeq \operatorname{Ext}^n_{\mathcal{A}}(F(D), F(E))$$

Example 1.62. Because

$$\operatorname{Ext}^n_{(\mathrm{KK},\ker(\mathrm{K}_*))}(D, E) = \operatorname{Ext}^n_{\mathbf{Ab}^{\mathbb{Z}/2}}(\mathrm{K}_*(D), \mathrm{K}_*(E))$$

for all $n \geq 1$, we have

$$\operatorname{Ext}^0_{(\mathrm{KK},\ker(\mathrm{K}_*))} = \mathrm{Hom}, \quad \operatorname{Ext}^n_{(\mathrm{KK},\ker(\mathrm{K}_*))} = 0$$

There is a canonical map

$$\mathcal{T}(D, E)/\mathcal{J}(D, E) \rightarrowtail \operatorname{Ext}^0_{(\mathcal{T},\mathcal{J})}(D, E)$$

The general feature is that $\mathcal{J}$ acts by 0 on all derived functors.

Definition 1.63. *Let $D \in \mathcal{T}$, (P_n, ∂_n) be an $\mathcal{J}$-projective resolution of D. Then $\operatorname{Ext}^n_{(\mathcal{T},\mathcal{J})}(D, E)$ is the n-th cohomology of*

$$\ldots \leftarrow sT(P_n, E) \leftarrow \mathcal{T}(P_{n-1}, E) \leftarrow \ldots \leftarrow \mathcal{T}(P_0, E) \leftarrow 0$$

For example

$$\operatorname{Ext}^0_{\mathcal{T},\mathcal{J}} = \ker(\mathcal{T}(P_0, E) \to \mathcal{T}(P_1, E))$$

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & D & \longrightarrow & 0 \\ & & & \searrow & \downarrow & & \\ & & & & E & & \end{array}$$

Assume we want to understand a triangulated category $\mathcal{T}$, that may have nothing to do with algebra, using the tools from homological algebra. We have been able to define projective resolutions and thus derived functors. How to achieve $F \circ F^{\vdash} = \mathrm{id}$? Is there abelian category that describes the derived functors?

Definition 1.64. *Let $\mathcal{J} \subseteq \mathcal{T}$ be a homological ideal. A stable homological functor $F\colon \mathcal{T} \to \mathcal{A}$ with $\ker F = \mathcal{J}$ is called universal (for $\mathcal{J}$) if any other stable homological functor $H\colon \mathcal{T} \to \mathcal{A}'$ with $\ker H \supseteq \mathcal{J}$ factors through F uniquely up to equivalence.*

Theorem 1.65. *If the left adjoint $F^{\vdash}$ is defined on all projective objects and $F \circ F^{\vdash} = \mathrm{id}_{\mathrm{Proj}_{\mathcal{A}}}$ then F is universal for $\ker F$.*

Conversely, if $\ker F$ has enough projectives, and F is universal, then $F^{\vdash}$ is defined on all projective objects and $F \circ F^{\vdash} = \mathrm{id}_{\mathrm{Proj}_{\mathcal{A}}}$.

Proof. Assume we have a functor $H\colon \mathcal{T} \to \mathcal{C}$

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{F} & \mathcal{A} \\ & \searrow^{H} & \downarrow{\scriptstyle \overline{H}} \\ & & \mathcal{C} \end{array}$$

We want to prove that there is a unique $\overline{H}\colon \mathcal{A} \to \mathcal{C}$. There is a following sequence of functors

$$\mathcal{A} \to \mathrm{Ho}(\mathrm{Proj}_A) \simeq \mathrm{Ho}(\mathrm{Proj}_{\mathcal{J}}) \subseteq \mathrm{Ho}(\mathcal{T}) \xrightarrow{H} \mathrm{Ho}(\mathcal{C}) \xrightarrow{\mathrm{H}_0} \mathcal{C}$$

First functor is taking the projective resolution, on objects $B \mapsto (P_n, \alpha_n)$. $\square$

Example 1.66. The functor

$$\begin{aligned} \mathrm{KK}^{\mathbb{Z}} &\to \mathbf{Ab}^{\mathbb{Z}/2} \\ (D,\alpha) &\mapsto \mathrm{K}_*(D) \end{aligned}$$

is not universal. The universal functor $\widetilde{F}$ here is defined on all projective objects and satisfies $\widetilde{F} \circ \tilde{F}^{\vdash} = \mathrm{id}_{\mathrm{Proj}_{\mathbf{Ab}^{\mathbb{Z}/2}}}$. Notice that the $\mathbb{Z}$-action on D induces an action on $\mathrm{K}_*(D)$. We enrich F to a functor

$$\begin{aligned} \widetilde{F}\colon \mathrm{KK}^{\mathbb{Z}} &\to \mathbf{Mod}(\mathbb{Z}[\mathbb{Z}])^{\mathbb{Z}/2} \\ \widetilde{F}(D) &:= \mathrm{KK}_*(\mathbb{C}, D) = \mathrm{KK}^{\mathbb{Z}}(C_0(\mathbb{Z}), (D,\alpha)) \end{aligned}$$

Then $\ker \widetilde{F}$ and $\widetilde{F}$ is universal. Furthermore

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}]}(\mathbb{Z}[\mathbb{Z}], \widetilde{F}(D)) = \tilde{F}(D)$$

Thus $\widetilde{F}(\mathbb{Z}[\mathbb{Z}]) = C_0(\mathbb{Z})$ and $\widetilde{F} \circ \widetilde{F}^{\vdash}(\mathbb{Z}[\mathbb{Z}]) = \mathbb{Z}[\mathbb{Z}]$.

Example 1.67. Take the homology functor

$$F = \mathrm{H}_*\colon (R-\mathbf{Mod}) \to \mathbf{Ab}^{\mathbb{Z}}$$

Passing from F to the universal functor for $\ker F$ reconstructs $\mathrm{H}_*\colon \mathrm{Ho}(R-\mathbf{Mod}) \to (R-\mathbf{Mod})^{\mathbb{Z}}$. The left adjoint $\mathrm{H}_*^{\vdash}$ is defined on projective modules, and $\mathrm{H}_* \circ \mathrm{H}_*^{\vdash} = \mathrm{id}$.

Example 1.68. Let (C,Δ) be a discrete quantum group, $\mathcal{T} = \mathrm{KK}^{(C,\Delta)}$, $F(A,\Delta_A) = \mathrm{K}_*(A)$ for a separable C*-algebra with coaction $\Delta_A\colon A \to \mathcal{M}(A \otimes C)$.

F is a poor invariant - it forgets too much. Say $C = C^*(G)$ for finite G. Then

$$\mathrm{KK}^{(C,\Delta)}(C \otimes A, B) \simeq \mathrm{KK}(A,B)$$

The left adjoint $F^{\vdash}$ is defined on free abelian groups. From Baaj-Skandalis duality

$$\mathrm{KK}^{(C,\Delta)}(A,B) = \mathrm{KK}^{(\widehat{C},\widehat{\Delta})}(A \rtimes \widehat{C}, B \rtimes \widehat{C})$$

$$A \rtimes \widehat{C} \rtimes C \simeq A \otimes \mathcal{K}(\mathcal{H}_C) \sim A$$

There turns out to be a canonical $\mathrm{Rep}(\widehat{C})$-module structure on $\mathrm{K}_*(A \rtimes C) =: \mathrm{K}_*^{\widehat{C}}(A)$.

In Baaj-Skandalis duality example

$$\mathrm{KK}_*^{\mathbb{Z}}(A,B) \simeq \mathrm{KK}^{\mathrm{U}(1)}(A \rtimes \mathbb{Z}, B \rtimes \mathbb{Z})$$

Let $\mathcal{T}$ be a triangulated category (with direct sums), and $F\colon \mathcal{T} \to \mathcal{A}$ be a stable homological functor into some abelian category (commuting with direct sums). The left adjoint of F is defined on all projecive objects in $\mathcal{A}$.

Examples 1.69.

- $\mathcal{T} = \mathrm{Ho}(\widetilde{\mathcal{A}})$, $F\colon \mathcal{T} \to \widetilde{\mathcal{A}}^{\mathbb{Z}}$, $F(C_\bullet) = \mathrm{H}_*(C_\bullet)$
- $\mathcal{T} = \mathrm{KK}$, $F\colon \mathrm{KK} \to \mathbf{Ab}^{\mathbb{Z}/2}$, $F(B) = \mathrm{K}_*(B)$
- $\mathcal{T} = \mathrm{KK}^{\mathbb{Z}}$, $F\colon \mathrm{KK} \to \mathbf{Ab}^{\mathbb{Z}/2}$, $F(B,\beta) = \mathrm{K}_*(B)$, $F^{\vdash}(\mathbb{Z}) = C_0(\mathbb{Z})$ with free action of $\mathbb{Z}$

Let $\mathcal{L}$ be the smallest subcategory of $\mathcal{T}$ that is thick, contains all $\ker F$-projective objects, and is closed under direct sums. Let $\mathcal{N} = \{A \in \mathcal{T} \mid F(A) = 0\}$. Then if $L \in \mathcal{L}$, $N \in \mathcal{N}$ we have $\mathcal{T}(L, N) = 0$ because it holds if L is $\ker F$-projective, and $\{A \mid \mathcal{T}(A, N) = 0\}$ is localising. For $\mathcal{L}$, $\mathcal{N}$ to be complementary, we need that any $B \in \mathcal{T}$ can be embedded in an exact traingle

$$L \to B \to N \to L[1], \quad L \in \mathcal{L},\ N \in \mathcal{N}$$

Theorem 1.70. *If $F : \mathcal{T} \to \mathcal{A}$ commutes with direct sums and $\mathcal{T}$ has enough* $\ker F$*-projectives, then $(\mathcal{L}, \mathcal{N})$ are complementary.*

Example 1.71. For K_* on KK

$$\mathcal{L} = \langle \mathbb{C} \rangle$$

$$\mathcal{N} = \{B \in \mathrm{KK} \mid \mathrm{K}_*(B) = 0\}$$

Example 1.72. For K_* on $\mathrm{KK}^{\mathbb{Z}}$

$$\mathcal{L} = \langle C_0(\mathbb{Z}) \rangle = \{(B, \beta) \in \mathrm{KK}^{\mathbb{Z}} \mid B \text{ is the bootstrap class}\}$$

The inclusion $\subset$ is obvious, and $\supset$ is closely related to the Pimsner-Voiculescu sequence and the Baum-Connes conjecture for $\mathbb{Z}$. We will give a sketch of the proof.

Take $(B, \beta) \in \mathrm{KK}^{\mathbb{Z}}$. Look at the extension

$$C_0(\mathbb{R}, B) \rightarrowtail C_0(\mathbb{R} \cup \{+\infty\}, B) \twoheadrightarrow B$$

Here we have an action of $\mathbb{Z}$ on $\mathbb{R}$ by translation. This extension does not have an equivariant completely positive section. But an argument by Baaj-Skandalis shows that it yields an extension triangle nevertheless.

$$C_0(\mathbb{Z} \times (0,1)) \rightarrowtail C_0(\mathbb{R}, B) \twoheadrightarrow C_0(\mathbb{Z}, B)$$

If $B \in \langle \mathbb{C} \rangle$, then $C_0(\mathbb{Z}, B)$ and $C_0(\mathbb{Z} \times (0,1))$ belong to $\langle C_0(\mathbb{Z}) \rangle$, hence so does $C_0(\mathbb{R}, B)$.

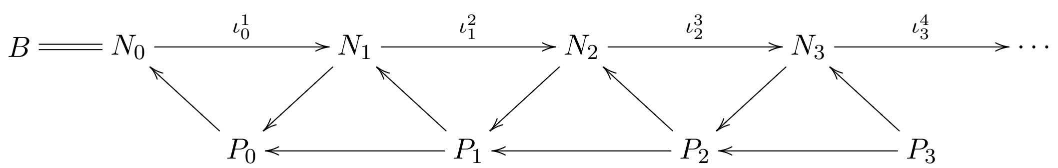
Theorem 1.73. *$C_0((-\infty, \infty], B) \simeq 0$ in $\mathrm{KK}^{\mathbb{Z}}$ with diagonal action.*

This is where the work has to be done. More generally, if $(B, \beta) \in \mathrm{KK}^{\mathbb{Z}}$ satisfies $B \simeq 0$ in KK, then $(B, \beta) \simeq 0$ in $\mathrm{KK}^{\mathbb{Z}}$. Equivalently if $f \in \mathrm{KK}^{\mathbb{Z}}(B_1, B_2)$ is invertible in $\mathrm{KK}(B_1, B_2)$, then f is invertible in $\mathrm{KK}^{\mathbb{Z}}$.

More generally we can replace $\mathbb{Z}$ by any torsion-free (that is without compact subgroups) a-T-menable locally compact group. It is implied by the proof of the Baum-Connes conjecture by Higson and Kasparov.

The full proof of the fact that $(\mathcal{L}, \mathcal{N})$ are complementary is in Ralf Meyer, "Homological algebra in triangulated category", part II. We will prove a weaker fact, that is $(\mathcal{N}^{\vdash}, \mathcal{N})$ are complementary. The proof uses phantom tower (maps in $\ker F$ are called phantom maps).

Definition 1.74. *Let $B \in \mathcal{T}$. Phantom tower is a diagram of the form*



(1.6)

where all P_n are $\ker F$-projective, $\iota_n^{n+1} \in \ker F$, and all triangles

$$N_n \xrightarrow{\iota_n^{n+1}} N_{n+1} \to P_n \to N_n$$

are exact. This means that the maps $N_{n+1} \to P_{n-1}$ are of degree 1, that is actually $N_{n+1} \to P_{n-1}[1]$. The bottom row

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \dots$$

is a chain complex with differential of degree 1.

Proposition 1.75. *Given a phantom tower (1.6), the complex*

$$B \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \dots$$

is a projective resolution. Conversely, any projective resolution embeds uniquely in a phantom tower.

Proof. The sequence

$$B \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \dots$$

is $\ker F$ -exact. We know that

$$F_{*+1}(N_{j+1}) \rightarrowtail F_*(P_j) \twoheadrightarrow F_*(N_j)$$

is a short exact sequence because $F(\iota_j^{j+1}) = 0$. The Yoneda product of these extensions is the chain complex

$$F(B) \leftarrow F(P_0) \leftarrow F(P_1) \leftarrow \dots$$

This is exact as a Yoneda product of extensions. Now take a projective resolution

$$B \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \dots$$

Recursively construct N_j starting with $N_0 = B$. Now embed $N_j \leftarrow P_j$ in an exact triangle $P_j \to N_j \xrightarrow{\iota_j^{j+1}} N_{j+1} \to P_j[1]$.

Induction assumption tells that $N_j \leftarrow P_j$ is $\ker F$-epimorphism, that is $F(P_j) \to F(N_j)$ is an epimorphism. Then $F(\iota_j^{j+1}) = 0$ because F is homological. Now we must lift the boundary map $P_{j+1} \to P_j[1]$ to a map $P_{j+1} \to N_{j+1}$. and check that then it is $\ker F$-epimorphism.

In the sequence

$$\mathcal{T}(P_{j+1}, N_j) \to \mathcal{T}(P_{j+1}, N_{j+1}) \to \mathcal{T}(P_{j+1}, P_{j+1}[1]) \to \mathcal{T}(P_{j+1}, N_j[1])$$

the first map is zero, because P_{j+1} is projective and i_j^{i+1} is phantom.

Because the composition

$$P_{j+1} \to P_j[1] \to P_{j-1}[2]$$

vanishes, the boundary map goes to 0 in $\mathcal{T}(P_{j+1}, N_j[1])$, hence comes from $\mathcal{T}(P_{j+1}, N_{j+1})$.

Now routine check that it is an epimorphism. □

Now we will prove that for any $B \in \mathcal{T}$ there is $N \in \mathcal{N}$ and a map $f\colon B \to N$ such that

$$\mathcal{T}_*(N, M) \to \mathcal{T}_*(B, M)$$

is invertible for all $M \in \mathcal{N}$. Then $B \mapsto N$ is a functor $\mathcal{T} \to \mathcal{N}$ that is left adjoint to the embedding functor $\mathcal{N} \to \mathcal{T}$. We let N to be the homotopy direct limit of the phantom tower.

$$\bigoplus_j N_j \xrightarrow{\mathrm{id}-S} \bigoplus_j N_j \to \underrightarrow{\mathrm{holim}}\, N_j \to \bigoplus_j N_j[1], \quad S = \bigoplus_j i_j^{j+1}$$

Since F commutes with direct sums, and $i_j^{j+1} \in \ker F$, $F(S) = 0$. Therefore $F(\mathrm{id}-S) = F(\mathrm{id})$ is invertible so that $F(\underrightarrow{\mathrm{holim}}\, N_j) = 0$.

Let $M \in \mathcal{N}$. Then $\mathcal{T}_*(P_j, M) = 0$ because P_j is $\ker F$-projective. Therefore i_j^{j+1} induces an isomorphism

$$\mathcal{T}_*(N_{j+1}, M) \xrightarrow{\simeq} \mathcal{T}_*(N_j, M)$$

There is an extension

$$\begin{array}{ccccc} \underleftarrow{\lim}^1 \mathcal{T}_{*-1} & \rightarrowtail & \mathcal{T}_*(\underrightarrow{\mathrm{holim}}\, N_j, M) & \twoheadrightarrow & \underleftarrow{\lim}\, \mathcal{T}_*(N_j, M) & & \\ \| & & & & \| & & \\ 0 & & & & \mathcal{T}_*(N_0, M) & = & \mathcal{T}_*(B, M) \end{array}$$

1.6 Index maps in K-theory and K-homology

Consider the following extension of C*-algebras

$$I \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$$

There are long exact sequences in K-theory and in K-homology:

$$\begin{array}{ccccc} \mathrm{K}_0(I) & \longrightarrow & \mathrm{K}_0(E) & \longrightarrow & \mathrm{K}_0(Q) \\ \uparrow \partial & & & & \downarrow \partial \\ \mathrm{K}_1(Q) & \longleftarrow & \mathrm{K}_1(E) & \longleftarrow & \mathrm{K}_1(I) \end{array} \tag{1.7}$$

$$\begin{array}{ccccc} \mathrm{K}^0(Q) & \longrightarrow & \mathrm{K}^0(E) & \longrightarrow & \mathrm{K}^0(I) \\ \uparrow \delta & & & & \downarrow \delta \\ \mathrm{K}^1(I) & \longleftarrow & \mathrm{K}^1(E) & \longleftarrow & \mathrm{K}^1(Q) \end{array} \tag{1.8}$$

and we have pairings between K-theory and K-homology. We will prove that

$$-\langle \partial(x),\, y\rangle = \langle x,\, \delta(y)\rangle, \quad x \in \mathrm{K}_1(Q),\ y \in \mathrm{K}^0(I) \tag{1.9}$$

We will use only formal properties of the boundary maps.

Theorem 1.76. *Let*

$$\partial\colon\ \mathrm{K}_1(Q) \to \mathrm{K}_0(I)$$
$$\delta\colon\ \mathrm{K}^0(I) \to \mathrm{K}^1(Q)$$

be natural for morphisms of extensions. Then there is $\varepsilon \in \{\pm 1\}$ *such that*

$$\langle \partial(x),\, y\rangle = \varepsilon\langle x,\, \delta(y)\rangle$$

for all extensions and all $x \in \mathrm{K}_1(Q)$, $y \in \mathrm{K}^0(I)$.

Remark 1.77. The sign ε is fixed by looking at the extension

$$\mathcal{K} \rightarrowtail \mathcal{T} \twoheadrightarrow C(S^1)$$

and the generators of $\mathrm{K}_1(C(S^1)) = \mathbb{Z}$, $\mathrm{K}^0(\mathcal{K}) = \mathbb{Z}$.

$$[\mathcal{K} \rightarrowtail \mathcal{T} \rightarrowtail C(S^1)] \in \mathrm{K}^1(C(S^1)) \simeq \mathrm{Hom}(\mathrm{K}_1(C(S^1))) \simeq \mathbb{Z}$$

$$[\mathcal{K} \rightarrowtail \mathcal{T} \rightarrowtail C(S^1)] \mapsto -1 \in \mathbb{Z}$$

Even more, up to sign there is only one natural boundary map.

Theorem 1.78. *Let* $\partial\colon \mathrm{K}_{*+1}(Q) \to \mathrm{K}_*(I)$ *be a natural boundary map. Then there is* $\varepsilon \in \{\pm 1\}$ *such that for all extensions* $\varepsilon \cdot \partial$ *is the composition*

$$\mathrm{K}_{*+1}(Q) \simeq \mathrm{KK}_{*+1}(\mathbb{C}, Q) \to \mathrm{KK}_*(\mathbb{C}, I) \simeq \mathrm{K}_*(I)$$

where the middle map is the Kasparov product with the class of the extension in $\mathrm{KK}_1(Q, I)$. *The same holds in K-homology.*

1.7 Mayer-Vietoris sequences

Consider the category of pullback diagrams

$$\begin{array}{ccccc} I & \rightarrowtail & A & \twoheadrightarrow & B \\ \| & & \downarrow & & \downarrow \\ I & \rightarrowtail & A' & \twoheadrightarrow & B \end{array}$$

A natural Mayer-Vietoris sequence is a functor from this category to the category of exact chain complexes, whose entries are $\mathrm{K}_*(A)$, $\mathrm{K}_*(A') \oplus \mathrm{K}_*(B)$, $\mathrm{K}_*(B')$.

Theorem 1.79. *Let* $d\colon \mathrm{K}_*(B') \to \mathrm{K}_{*+1}(A)$ *be a boundary map in a natural Mayer-Vietoris sequence. Then there is a sign* $\varepsilon_* \in \{\pm\}$ *such that for any pullback diagram* $\varepsilon \cdot d$ *is the composition*

$$\begin{array}{ccc} \mathrm{K}_*(B') & \xrightarrow{\delta} & \mathrm{K}_*(\ker(A' \to B')) \\ & & \| \\ \mathrm{K}_*(A) & \longleftarrow & \mathrm{K}_*(\ker(A \to B)) \end{array}$$

Remark 1.80. To fix sign, one can look at pullback

$$\begin{array}{ccc} C_0((0,1)) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ C_0((0,1]) & \longrightarrow & \mathbb{C} \end{array}$$

or its suspension.

Let F be a homological functor on separable C*-algebras, and let $d\colon F_1(B') \to F_0(A)$ be a natural transformation on pullback diagrams

$$\begin{array}{ccc} A & \twoheadrightarrow & B \\ \downarrow & & \downarrow \\ A' & \twoheadrightarrow & B' \end{array}$$

We compare a given structure to simpler one

$$\begin{array}{ccc} \ker p' & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow \\ A' & \overset{p'}{\twoheadrightarrow} & B' \end{array} \quad \to \quad \begin{array}{ccc} A & \twoheadrightarrow & B \\ \downarrow & & \downarrow \\ A' & \twoheadrightarrow & B' \end{array}$$

$$\begin{array}{ccc} F(B') & \xrightarrow{d_2} & F(\ker p') \\ \| & & \downarrow {\scriptstyle F(can)} \\ F(B') & \xrightarrow{d_1} & F(?) \end{array}$$

As a consequence, a natural transformation for pullback diagrams reduces to a natural transformation $E_1(Q) \to F_0(I)$ for extensions

$$I \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$$

Next we compare this extension with mapping cylinder extension

$$\begin{array}{ccccc} I & \overset{i}{\rightarrowtail} & E & \overset{p}{\twoheadrightarrow} & Q \\ \downarrow & & \downarrow & & \| \\ C_p & \rightarrowtail & Z_p & \twoheadrightarrow & Q \end{array}$$

where

$$Z_p := \{(e, q) \in E \oplus C([0,1], Q) \mid p(e) = q(1)\}$$

Now there are

$$\begin{array}{ccc} F_1(Q) & \xrightarrow{d_3} & F_0(I) \\ \| & & \downarrow {\scriptstyle F(can)} \\ F_1(Q) & \xrightarrow{d_4} & F_0(C_p) \end{array} \quad , \quad d_3 = F_0(can)^{-1} \circ d_4$$

If p has a completely positive contractive section, then $F_0(I) \xrightarrow{\simeq} F_0(C_p)$. Actually if F is exact, this is true without completely positive contractive section. Then the class of the extension in $\mathrm{KK}_1(Q, I)$ is the product of

$$C_0((0,1)) \otimes Q \hookrightarrow C_p \xleftarrow{\simeq} I$$

The map $I \hookrightarrow C_p$ has to be an E-equivalence because it is part of an extension

$$I \rightarrowtail C_p \twoheadrightarrow C_0((0,1], Q)$$

and $C_0((0,1], Q)$ is contractible.

Next we consider

$$\begin{array}{ccccc} C_p & \rightarrowtail & Z_p & \twoheadrightarrow & Q \\ \uparrow & & \uparrow & & \| \\ SQ & \rightarrowtail & C_0([0,1],Q) & \twoheadrightarrow & Q \end{array}$$

and

$$\begin{array}{ccc} F_1(Q) & \xrightarrow{d_4} & F_0(C_p) \\ \| & & \uparrow{\scriptstyle F_0(can)} \\ F_1(Q) & \xrightarrow{d_5} & F_0(SQ) \end{array} \quad , \quad d_5 = F_0(can)^{-1} \circ d_4$$

This is d_5 composed with the class of the extension $I \rightarrowtail E \twoheadrightarrow Q$ in $\mathrm{KK}_0(SQ, I)$ or rather $E_0(SQ, I)$ if there is no completely positive contractive section.

Now assume $F_* = \mathrm{K}_*$. We want to get rid of Q. Now the boundary map for the cone extension of Q is a natural transformation $\mathrm{K}_1(Q) \to \mathrm{K}_0(SQ)$. We have naturality of *-homomorphisms to begin with, but this implies naturality of KK_0-morphisms. Any $x \in \mathrm{K}_1(Q)$ is of the form $\tilde{x}_*(g)$, where $g \in \mathrm{K}_1(C_0(\mathbb{R}))$ is the canonical generator, and $\tilde{x} \in \mathrm{KK}_0(C_0(\mathbb{R}), Q)$.

$$\begin{gathered} \mathrm{K}_1(Q) \simeq \mathrm{KK}_0(C_0(\mathbb{R}), Q) \\ x \mapsto \tilde{x} \end{gathered}$$

$$\begin{array}{c} x \\ \uparrow \\ g \end{array} \qquad \begin{array}{ccc} \mathrm{K}_1(Q) & \xrightarrow{d} & \mathrm{K}_0(SQ) \\ \uparrow{\scriptstyle \tilde{x}} & & \uparrow{\scriptstyle \tilde{x}} \\ \mathrm{K}_1(C_0(\mathbb{R})) & \xrightarrow{d} & \mathrm{K}_0(SC_0(\mathbb{R})) \end{array}$$

We conclude that $d(x) = (\tilde{x})_*(d(g))$, so d is fixed completely once we know $d(g) \in \mathrm{K}_0(C_0(\mathbb{R}^2)) = \mathbb{Z}$. If we use an exact sequence

$$\underbrace{\mathrm{K}_1(C_0([0,1)), C_0(\mathbb{R}))}_{=0} \to \underbrace{\mathrm{K}_1(C_0(\mathbb{R}))}_{\simeq \mathbb{Z}} \xrightarrow{\simeq} \mathrm{K}_0(C_0(\mathbb{R}^2)) \to 0$$

we conclude that $d(g)$ has to be a generator of $\mathrm{K}_0(C_0(\mathbb{R}^2)) \simeq \mathbb{Z}$, and

$$\mathrm{K}_1(Q) \simeq \mathrm{KK}_0(C_0(\mathbb{R}), Q) \simeq \mathrm{KK}_0(\mathbb{C}, C_0(\mathbb{R}, Q)) \simeq \mathrm{K}_0(C_0(\mathbb{R}, Q))$$

$$x \mapsto \tilde{x}$$

We fix natural isomorphisms

$$\mathrm{K}_1(Q) \simeq \mathrm{KK}_0(C_0(\mathbb{R}), Q) \simeq \mathrm{KK}_0(\mathbb{C}, C_0(\mathbb{R}) \otimes Q) \simeq \mathrm{K}_1(SQ)$$

which are unique up to sign. Then d is this isomorphism up to sign.

For the boundary map $\mathrm{K}_0(Q) \to \mathrm{K}_1(SQ)$ the same thing happens, but replacing g by the generator of $\mathrm{K}_0(\mathbb{C})$.

Let $x \in \mathrm{K}_1(Q)$, $y \in \mathrm{K}^0(I)$.

$$\mathbb{C} \to Q \xrightarrow{[E]} I$$

Using Kasparov product $\circ$ we write

$$\begin{aligned}
\partial(x) &= \varepsilon_\partial [E] \circ x \\
\delta(y) &= \varepsilon_\delta y \circ [E] \\
\langle x,\, \delta y\rangle &= \delta(y) \circ x = \varepsilon_\delta(y) \circ [E] \circ x \in \mathrm{KK}_0(\mathbb{C},\mathbb{C}) \simeq \mathbb{Z} \\
\langle \partial(x),\, y\rangle &= y \circ \partial(x) = \varepsilon_\partial(y) \circ [E] \circ x
\end{aligned}$$

1.8 Localisation of functors

Assume we have a triangualted category $\mathcal{T}$ with $\oplus$, a localising subcategory $\mathcal{N}$ and a class of objects $\mathcal{P}$ such that $(\langle\mathcal{P}\rangle, \mathcal{N})$ is complementary. For example we can take $\mathcal{T} = \mathrm{KK}$, $\mathcal{N} = \{B \in \mathrm{KK} \mid \mathrm{K}_*(B) = 0\}$, $\mathcal{P} = \{\mathbb{C}\}$. Furthermore, let $F\colon \mathcal{T} \to \mathcal{A}$ be a homological functor commuting with $\oplus$. Recall that there are functors

$$P\colon \mathcal{T} \to \langle\mathcal{P}\rangle, \quad N\colon \mathcal{T} \to \mathcal{N}$$

and natural exact triangles

$$P(B) \to B \to N(B) \to P(B)[1]$$

Definition 1.81. *The* ***localisation of functor*** *F at $\mathcal{N}$, denoted $\mathbb{L}F$, is a functor*

$$F \circ P\colon \mathcal{T} \to \mathcal{A}$$

We may also view this as a functor on $\mathcal{T}/\mathcal{N}$. There is a natural transformation $\mathbb{L}F \to F$.

Proposition 1.82. *$\mathbb{L}F \to F$ is universal among natural transformations $G \to F$ with G homological and $G/\mathcal{N} = 0$*

$$\begin{array}{ccc}
G & \longrightarrow & F \\
 & \searrow & \uparrow \\
 & & \mathbb{L}F
\end{array}$$

Proof. There is an isomorphism

$$G(P(B)) \xrightarrow{\simeq} G(B)$$

and a map

$$G(P(B)) \to F(P(B)) = \mathbb{L}F(B)$$

□

Roughly speaking, $\mathbb{L}F$ is the best approximation to F that vanishes on $\mathcal{N}$.

Corollary 1.83. *If $\mathbb{L}F \to F$ is invertible, then $F|_\mathcal{N} = 0$.*

Proposition 1.84. *Let G, F be homological, commuting with $\oplus$, $G/\mathcal{N} = 0$, and let $\Phi\colon G \to F$ be a natural transformation. Then if Φ_B is invertible for all $B \in \mathcal{P}$, then Φ induces a natural isomorphism $G \simeq \mathbb{L}F$.*

Proof. We get a transformation $\Psi\colon G \to \mathbb{L}F$ by the previous proposition. Ψ is invertible on $\mathcal{P}$ because $\mathbb{L}F(B) \simeq F(B)$ for $B \in \mathcal{P}$. Since G and $\mathbb{L}F$ are homological and commuting with $\oplus$, the class of objects where Ψ is invertible is localising. Hence contains $\mathcal{P}$. It also contains $\mathcal{N}$ because G and $\mathbb{L}F$ vanish on $\mathcal{N}$. Thus it contains $\mathcal{T}$. □

Usually we do not expect the map $\mathbb{L}F \to F$ to be an isomorphism. But sometimes in noncommutative topology this happens for rather deep reason. For example the Baum-Connes assembly map is of this form for suitable choice of $\mathcal{N}$ and $F(B) = \mathrm{K}_*(G \ltimes_r B)$.

Let $\mathcal{T} = \mathrm{KK}^G$, G locally compact group. How to chose $\mathcal{N}$? In the group case the following choice is most useful

$$B \in \mathcal{N} \text{ if and only if } \mathrm{Res}_G^H(B) \simeq 0 \text{ in } \mathrm{KK}^H, \text{ for all compact subgroups } H \leq G$$

This definition contains the insight that the K-theory for crossed products by compact groups has to be computes by hand, whereas those for non-compact groups often reduce to compact groups.

Theorem 1.85. *Let $\mathcal{T} = \mathrm{KK}^G$ for a Lie group G, and $F(B) = \mathrm{K}_*(G \ltimes_r B)$, $\mathcal{N}$ as above. Then the natural transformation $\mathbb{L}F \to F$ is naturally isomorphic to the Baum-Connes assembly map with coeffictients.*

Proof. The domain of the Baum-Connes map

$$\mathrm{K}_*^{top}(G, B) = \varinjlim_{X \subset \underline{E}G,\ X\ G-\text{compact}} \mathrm{KK}^G(C_0(X), B)$$

has two properties

- it vanishes for $B \in \mathcal{N}$

$$\mathrm{KK}^G(C_0(X), B) \to \mathrm{KK}(G \ltimes_r C_0(X), G \ltimes_r B) \to \mathrm{K}_*(G \ltimes_r B)$$

- the Baum-Connes assembly map is invertible

□

Definition 1.86. *A G-algebra is called proper Husdorff if there is a proper G-space X and a continuous G-map $\mathrm{Prim}(A) \to X$ (equivalently $C_0(X) \to A$ is central).*

1.9 Towards an analogue of the Baum-connes conjecture for quantum groups

The main question is: what are good choices for $\mathcal{P}$, $\mathcal{N}$? We must choose $\mathcal{N}$, $\mathcal{P}$ so that the resulting assembly map is invertible for "nice" quantum groups. first approach is to use restriction functors to all compact quantum subgroups.

Definition 1.87. *A **quantum group** is a C^*-algebra A with a comultiplication $\Delta\colon A \to A \otimes A$ satisfying certain properties.*

*A quantum group is **compact** if A is unital.*

Example 1.88. Right now, we had only two examples: groups and their duals

1. $A = C_0(G)$
$$\Delta\colon C_0(G) \to C_0(G \times G), \quad (\Delta f)(x, y) = f(xy).$$

2. $A = C_r^*(G)$,
$$\Delta\colon C_r^*(G) \to M(C_r^*(G) \otimes C_r^*(G)), \quad \Delta\left(\int_G f(t)\lambda_t dt\right) = \int_G f(t)\lambda_t \otimes \lambda_t dt.$$

Group actions on C*-algebras become coactions of (A, Δ)

$$\delta_B \colon B \to M(B \otimes A)$$

coassociative plus technical conditions.

Example 1.89.

1. Group actions as usual.
2. Grading by G.

Definition 1.90. *A closed quantum subgroup of (A, Δ) is a quotient A/I to which Δ descends.*

Example 1.91.

1. Closed quantum subgroups of $C_0(G)$ are $C_0(H)$ for $H \leq G$ closed subgroup.
2. Closed quantum subgroups of $C_r^*(G)$ are too few. The candidates are $C_r^*(G/N)$, where $N \leq G$ is a closed normal subgroup.

Many locally compact groups such as $\mathrm{GL}_2(\mathbb{Q}_p)$ have many open subgroups but no open normal subgroup.

Definition 1.92. *A* ***quantum homogeneous space*** *for (A, Δ) is a C^*-subalgebra B of $M(A)$ that is a left Δ-coideal ($\Delta(B) \subseteq M(B \otimes A)$). It is proper if $B \subsetneq A$.*

Example 1.93.

1. $B = C_0(G/H)$, $H \subseteq G$ closed subgroup.
2. $C_r^*(H)$, for any closed subgroup of $H \subseteq G$ is even a two-sided Cl-coideal. Proper homogeneous spaces are open subgroups here.

Let us look at $C_r^*(G)$ when G is a compact Lie group. Then the following conditions are equivalent

1. G is connected.
2. G has no open subgroups.
3. $C_r^*(G)$ has no non-trivial proper homogeneous spaces.

But $G = \mathrm{SO}(3)$ creates a problem because it has projective representations. G acts on $M_2(\mathbb{C})$ because of the representation of $\widetilde{\mathrm{SO}(3)}$ on $\mathbb{C}^2$. G coacts on $G \rtimes_r M_2(\mathbb{C})$.

What are particularly simple actions of a quantum group?

$$C_0(G/H) \ltimes_r G \sim_{M.E.} C^*(H) \simeq \bigoplus_{\pi \in \widehat{G}} M_{d_\pi}(\mathbb{C})$$

A necessary condition for a torsion coefficient algebra is that the crossed product $B \ltimes_r A$ be a sum of matrix algebras (compact operators).

Theorem 1.94. *Let G be a locally compact group.*

$$\mathcal{P} := \{C_0(G/H) \mid H \leq G, \quad \textit{compact}\}$$

$$\widetilde{\mathcal{N}} = P^{\dashv} := \{B \mid \mathrm{KK}^G(P, B) = 0 \textit{ for all } P \in \mathcal{P}\}$$

The localisation of $\mathrm{K}_*(G \rtimes B)$ at $\widetilde{N}$ and $\mathcal{N}$ agree with the domain of the Baum-Connes assembly map

$$\mathcal{N} = \{B \mid \mathrm{Res}_G^H B \simeq 0 \textit{ for all compact } H \leq G\}.$$

1.10 Quantum groups

Definition 1.95. *A* ***quantum group*** *is a C*-algebra A with a comultiplication $\Delta \in \operatorname{Mor}(A, A \otimes A)$ such that*

$$\overline{\Delta(A)(A \otimes A)} = A \otimes A$$

$$\Delta\colon \quad \begin{array}{ccc} A & \longrightarrow & M(A \otimes A) \\ \downarrow & \nearrow & \\ A & & \end{array}$$

$$\otimes = \otimes_{min}, \quad 1_A \in M(A)$$

and for all $a, b \in A$

$$\Delta(a)(1 \otimes b) \in A \otimes A$$
$$(a \otimes 1)\Delta(b) \in A \otimes A$$

$$\operatorname{span}\{\Delta(a)(1 \otimes b) \mid a, b \in A\} \text{ is dense in} A \otimes A$$
$$\operatorname{span}\{(a \otimes 1)\Delta(b) \mid a, b \in A\} \text{ is dense in} A \otimes A$$

in the compact case, that is when $1_A \in A$ we have

$$(\Delta \circ \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$$

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & M(A \otimes A) \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ M(A \otimes A) & \xrightarrow[\Delta\otimes\mathrm{id}]{} & M(A \otimes A \otimes A) \end{array}$$

Theorem 1.96. *There is a unique state h on A such that*

$$(\mathrm{id} \otimes h)\Delta(A) = h(a)1_A = (h \otimes \mathrm{id})\Delta(a)$$

Let G be a locally compact quantum group, $A = C_0(G)$, $(\Delta f)(x, y) = f(xy)$. Here $\Delta \in \operatorname{Mor}(A, A \otimes A)$ is induced by the group multiplication $\mu\colon G \times G \to G$. Multiplication μ is associative if Δ is coassociative. The conditions

$$\operatorname{span}\{\Delta(a)(1 \otimes b) \mid a, b \in A\} \text{ is dense in} A \otimes A$$
$$\operatorname{span}\{(a \otimes 1)\Delta(b) \mid a, b \in A\} \text{ is dense in} A \otimes A$$

can be written as

$$\exists\, x \quad \mu(xy) = \mu(xz) \implies y = z$$
$$\exists\, x \quad \mu(yx) = \mu(zx) \implies y = z$$

On a group Haar measure h satisfies

$$\int_G f(st)dh(s) = \int_G f(s)dh(s)$$

Definition 1.97. *A function $h\colon A_+ \to [0, \infty]$ such that $h(a+b) = h(a)+h(b)$, $h(\lambda a) = \lambda h(a)$ for $\lambda \geq 0$ is called a* ***weight****.*

We define

$$\begin{aligned}\mathcal{N}_h &:= \{a \mid h(a^*a) < \infty\} \quad (L^2)\\ \mathcal{M}_h &:= \operatorname{span}\{a \geq 0 \mid h(a) < \infty\}\\ &= \operatorname{span}\{a^*b \mid a, b \in \mathcal{N}_h\}\end{aligned}$$

Then $\overline{\mathcal{M}_h} = A$ (h locally finite), and $(\mathrm{id} \otimes h)\Delta(a) = h(a)1_A$ (h lower semicontinuous).

Let $\varphi \in A^*$, $a \in A$. Then

$$\varphi * a := (\mathrm{id} \otimes \varphi)\Delta(a).$$

In particular, for $\varphi = \delta_t$

$$(\varphi * a)(s) = a(st).$$

Right invariance of h means that

$$h(\varphi * a) = h(a)\varphi(1_A)$$

for all $\varphi \in A^*_+$ and all $a \geq 0$.

We say that h is strictly faithful if

$$h(a^*a) = 0 \implies a = 0.$$

There exists κ - closed densely defined map $A \to A$, such that

$$\kappa = R \circ \tau_{i/2},$$

where R is an antiautomorphism, and $\tau_{i/2}$ is an analitic extension of a 1-parameter group $(\tau_t)_{t\in\mathbb{R}}$ of automorphisms of A. There exists $\lambda > 0$ such that $h \circ \tau_t = \lambda^t h$.

For all $\varphi \in A^*$, $\varphi \circ \kappa \in A$ and all $a, b \in \mathcal{N}_h$

$$h((\varphi * a^*)b) = h(a^*((\varphi \circ \kappa) * b))$$

Strong right invariance means that

$$\mu(\kappa \otimes \mathrm{id})\Delta(a) = \varepsilon(1)1_A = \mu(\mathrm{id} \otimes \kappa)\Delta(a)$$

The maps

$$\begin{aligned}\Phi &\colon a \otimes b \mapsto \Delta(a)(1_A \otimes b)\\ \Psi &\colon r \otimes s \mapsto (\mathrm{id} \otimes \kappa)(\Delta(r))(1 \otimes s)\end{aligned}$$

are inverse to each other.

We can embed A in a Hilbert space $\mathcal{H}$ and extend Φ, Ψ to

$$\begin{aligned}W &\colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}\\ V &\colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}\end{aligned}$$

Strong right invariance means that $W^* = V$

$$\langle W(a \otimes b),\, c \otimes d\rangle = \langle a \otimes b,\, V(c \otimes d)\rangle.$$

1.11 The Baum-Connes conjecture

Let G be a torsion-free group, that is without compact subgroups. The Baum-Connes conjecture with coefficients for G means that $\mathrm{K}_*(G \rtimes_r A) = 0$ whenever $\mathrm{K}_*(A) = 0$. If G has torsion, then the statement is: if $\mathrm{K}_*(A \rtimes_r H) = 0$ for all $H \leq G$ compact, then $\mathrm{K}_*(A \rtimes_r G) = 0$.

Theorem 1.98 (Higson-Kasparov)**.** *The Baum-Connes conjecture with coefficients holds for all amenable groups.*

In particular it holds if $G = \mathbb{Z}^n$ for some $n \in \mathbb{N}$.
Let

$$\mathcal{N} := \{A \in \mathrm{KK}^G \mid \mathrm{K}_*(A \rtimes H) = 0 \text{ for all compact } H \leq G\}$$
$$\mathcal{N}^{\vdash} := \{A \in \mathrm{KK}^G \mid \mathrm{KK}^G(A, B) = 0 \text{ for all } B \in \mathcal{N}\}$$

for a discrete G. Then $(\mathcal{N}^{\vdash}, \mathcal{N})$ are complementary.