Foliations, C*-algebras and index theory
Part I, II

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Contents

Chapter 1

Foliations

1.1 What is a foliation and why is it interesting ?

Question 1 *(H. Hopf).* Is there a completely integrable plane field on S^3 ? (Plane field - two dimensional subbundle $E \subset TS^3$).

Answer 1 *(G. Reeb).* Yes, it is a tangent bundle to a 2-dimensional Reeb's foliation of S^3, described in the example (1.2(6)).

Question 2 *(A. Haefliger).* Given a plane subbundle E of TM is it homotopic to an integrable one ?

Answer 2 *(R. Bott).* There exists at least one obstruction; not every subbundle has in its K-theory class an an integrable one.

Roughly speaking, a foliation is the decomposition of a manifold M^n into disjoint family of submanifolds (immersed injectively) of dimension $n-q$, which is locally trivial.

More precisely

Definition 1.1. *(1) A codimension q foliation of an manifold M^n is a family $\mathcal{F} = \{L_\alpha\}_{\alpha \in \mathcal{I}}$ of $n-q$-dimensional connected, injectively immersed submanifolds that satisfy*

1. $$L_\alpha \cap L_\beta \neq \emptyset \text{ iff. } \alpha = \beta \text{ and } \bigcup_{\alpha \in \mathcal{I}} L_\alpha = M.$$

2. *For all $p \in M$ there exist open $U \ni p$ and a diffeomorphism*

$$\varphi \colon U \to \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q,$$

such that for all $\alpha \in \mathcal{I}$

$$\varphi((U \cap L_\alpha) conn.\ comp.) = \{\underline{x}; x_{n-q+1} = c_{n-q+1}, \ldots, x_n = c_n\},$$

$$c_j = \text{ constant}, \quad j = n-q+1, \ldots, n.$$

Example 1.2. 1. Fibrations.

2. Surjective submersions.

3. The Kronecker foliation of $\mathbb{T} = S^1 \times S^1$, $S^1 = \mathbb{R}/\mathbb{Z}$.

   Solutions of differential equation $\mathrm{d}\,y = \lambda \,\mathrm{d}\,x$ with $\lambda = \tan(\theta)$ fixed. If a slope is rational then we get a closed curve - closed leaves of foliation. If $\lambda \notin \mathbb{Q}$ then leaves are dense - they are immersions of $\mathbb{R}$ which is not closed manifold.

   Rough quotient space $M/\mathcal{F}$. Two points are equivalent if and only if they belong to the same leaf. In the Kronecker foliation, when leaves are dense, we get a noncommutative torus.

4. The 1-dimensional Reeb foliation of $\mathbb{T}$.

   PICTURE

5. The 2-dimensional Reeb foliation of a solid torus $D^2 \times S^1$.

   In the universal cover $D^2 \times \mathbb{R} \to D^2 \times S^1$

   PICTURE

   We rotate these curves along vertical axis and define relation $(x, y, z) \sim (x, y, z+1)$. We have one closed leaf (boundary) and rest are open leaves (images of not closed manifolds).

6. The 2-dimensional Reeb foliation of S^3.

$$S^3 = D^2 \times S^1 \coprod S^1 \times D^2 / \sim$$

$$S^3 = \{\, (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

   The two tori in above decomposition are

$$\{\, x \in S^3 \mid x_1^2 + x_2^2 \leqslant \frac{1}{2}\}$$

$$\{\, x \in S^3 \mid x_1^2 + x_2^2 \geqslant \frac{1}{2}\}$$

   We put on each torus Reeb's foliation from preceeding example.

The notion of foliation is interesting for two reasons:

1. the definition is multifaceted

2. it gives rise to an interesting equivalence relation on M, which in turn gives rise to an interesting quotient "space" $M/\mathcal{F}$.

1.2 Equivalent definitions

Definition 1.3 (Manifold reformulation). *There exists covering of M by charts (U_i, φ_i) such that $\varphi(U_i) = V_i \times W_i$, where V_i and W_i are open subsets of $\mathbb{R}^{n-q}$ and $\mathbb{R}^q$, respectively, with the property that if $U_i \cap U_j \neq \emptyset$ then the diffeomorphism*

$$\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is of the form

$$(x, y) \mapsto (h_{ij}(x, y), g_{ij}(y)),\ g_{ij} \colon W_i^\circ \to W_j^\circ.$$

Definition 1.4 (1-cocycle reformulation). *There exists collection* (U_i, f_i, g_{ij})*, where* (U_i) *is a covering of* M*,* $f_i\colon U_i \to W_i$ *are surjective submersions onto open* q*-dimensional manifolds,* $g_{ij}\colon f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ *- diffeomorphisms satisfying*

$$f_i = g_{ij} \circ f_j \text{ on } U_i \cap U_j \text{ and } g_{ij} \circ g_{jk} = g_{ik} \text{ on } U_i \cap U_j \cap U_k.$$

Definition 1.5. *Let* $(M, \mathcal{F})$ *be manifold with foliation. The tangent bundle to* $\mathcal{F}$ *is*

$$\tau\mathcal{F} := \{\, X \in TM \bigm| X \text{ tangent to a leaf} \,\}.$$

Let $\mathcal{S}(\tau\mathcal{F})$ denote the space of smooth sections of this bundle. Clearly this is an involutive sub-bundle, i.e.

$$[\mathcal{S}(\tau\mathcal{F}), \mathcal{S}(\tau\mathcal{F})] \subset \mathcal{S}(\tau\mathcal{F}).$$

because this is local property, obvious on charts.

Conversely by Thm. of Frobenius we can take another

Definition 1.6. *Any involutive subbundle* $E \subset TM$ *is the tangent bubdle to a unique foliation.*

Equivalently we can say

Definition 1.7. *The ideal* $\mathcal{I}(E)$ *generated by the sections of*

$$\nu\mathcal{F} = \{\, \omega \in T^*M \bigm| \forall X \in \tau\mathcal{F}\ \omega(X) = 0\}$$

is closed under d*, i.e.* $\mathcal{I}(E)$ *is a differential ideal.*

1.3 Holonomy grupoid

Let $x, y \in L \subset M$ be points in a leaf of foliation, $\gamma\colon [0,1] \to M$ - path from x to y contained in L.

PICTURE

Let W -transversal through $\underline{x} = \varphi^{-1}(x_1 = c_1, \ldots, x_{n-q} = c_{n-q})$. If x' is close to x one can copy γ to γ', at least for a while. By the compactness of γ, there exists transversal $T_x \subset W$ such that we reach transversal T_y through y, starting from any $x' \in T_x$, and such that $x' \mapsto y' = \gamma'(1)$ is a diffeomorphism h_γ. We define holonomy of path γ as

$$\mathrm{Hol}(\gamma) := \text{ germ of } h_\gamma\colon \text{ germ of } T_x \to \text{ germ of } T_y$$

Obviously if $\gamma_1 \sim \gamma_2$ are homotopic, then $\mathrm{Hol}(\gamma_1) = \mathrm{Hol}(\gamma_2)$, i.e. holonomy factors through homotopy.

Recall that grupoid is a small category with all arrows invertible.

Definition 1.8. *Holonomy grupoid*

$$\mathcal{G}(\mathcal{F}) := \{\, (x, \mathrm{Hol}(\gamma), y) \bigm| \exists \text{ leaf } L \ni x, y, \text{ and path } \gamma\colon [0,1] \to L \text{ from } x \text{ to } y\}$$

with objects

$$\mathcal{G}^0 = M$$

and composition

$$(y, \mathrm{Hol}(\delta), z) \circ (x, \mathrm{Hol}(\gamma), y) = (z, \mathrm{Hol}(\delta \circ \gamma), z).$$

Interpretation:

- $(x, \mathrm{Hol}(\mathrm{const}), x)$ "reflexibility" = unit,
- $(x, \mathrm{Hol}(\gamma), y) = (y, \mathrm{Hol}(\gamma^{-1}), x)$ "symmetry"= inverse,
- $(y, \mathrm{Hol}(\delta), z) \circ (x, \mathrm{Hol}(\gamma), y) = (x, \mathrm{Hol}(\delta \circ \gamma), z)$ "transitivity"= composition.

Let T be a complete transversal to $\mathcal{F}$ i.e. T is an immersed submanifold, transverse to each leaf and intersecting each leaf at least once.

$$\mathcal{G}_T(\mathcal{F}) = \{ (x, \mathrm{Hol}(\gamma), y) \in \mathcal{G}(\mathcal{F}) \bigm| x, y \in T\}$$

$$C_c^\infty(\mathcal{G}_T(\mathcal{F})) \hookrightarrow C^*(\mathcal{G}_T(\mathcal{F}))$$

$$(f * g)(\mathrm{Hol}(\gamma)) = \sum_{\mathrm{Hol}(\gamma_1)\,\mathrm{Hol}(\gamma_2) = \mathrm{Hol}(\gamma)} f(\mathrm{Hol}(\gamma_1)) g(\mathrm{Hol}(\gamma_2))$$

1.4 How to handle "$M/\mathcal{F}$"

$$\text{“}M/\mathcal{F}'' = \text{ grupoid } \mathcal{G}(\mathcal{F})$$

(A) "Homotopy quotient" approach, or equivalently via classifying spaces. This is similar in spirit to

$$\text{“}M/\Gamma'' \leftrightarrow M \times_\Gamma \mathrm{E}\,\Gamma \to \mathrm{B}\,\Gamma,$$

where Γ is a group.

$$\text{“}M/\mathcal{F}'' \sim \mathrm{B}\,\mathcal{G}(\mathcal{F}) \to \mathrm{B}\,\Gamma_q$$

(B) "Topos" approach, by extending "duality"

Topological spaces $\leftrightarrow$ Sheaves of sets,

and associating a suitably defined topos to $\mathcal{G}(\mathcal{F})$.

(C) Connes noncommutative geometry approach, by extending the duality

Topological spaces $\leftrightarrow$ Commutative C*-algebras,

to include $C^*(\mathcal{G})$, for $\mathcal{G}$-grupoid.

1.5 Characteristic classes

All approaches produce cohomology groups for grupoids, equivalent for (A) & (B), and cyclic cohomology HC* for (C), as well as characteristic maps. They are all "huge" and not well understood. The ones which are best understood are the "geometric" characteristic classes.

1. Bott's construction a la Chern-Weil.
2. Gelfand-Fuks realization.
3. Hopf-cyclic cohomological construction.

Chapter 2

Characteristic classes

2.1 Preamble: Chern-Weil construction of Pontryagin ring

Let

$$E \to M$$

be a real vector bundle. A **connection** on E is a linear operator

$$\nabla \colon \mathcal{S}(E) \to \mathcal{S}(T^*M \otimes E) = \Omega^1(M) \otimes \mathcal{S}(E)$$

satisfying following rule

$$\nabla(fs) = df \otimes s + f\nabla(s).$$

Then ∇ extends to a graded $\Omega(M)$-module map

$$\nabla \colon \Omega^*(M) \otimes \mathcal{S}(E) \to \Omega^*(M) \otimes \mathcal{S}(E) = \Omega^*(M, E), \text{ by}$$

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \nabla(s).$$

The **Curvature** of ∇: we can view $\Omega^*(M, E)$ as a module over $\Omega^*(M)$ and then for any $\zeta \in \Omega^*(M, E)$ and any $\omega \in \Omega^*(M)$ we have

$$\nabla^2(\omega\zeta) = \nabla(d\omega\zeta + (-1)^{\partial\omega}\omega\nabla(\zeta)) =$$

$$= (-1)^{\partial\omega+1} d\omega\nabla(\zeta) + (-1)^{\partial\omega} d\omega\nabla(\zeta) + \omega\nabla^2(\zeta) = \omega\nabla^2(\zeta).$$

It means that ∇^2 is a local operator - multiplication by an element of the base ring. It follows that

$$\nabla^2(\zeta) = R \cdot \zeta, \quad R \in \Omega^2(M, \mathrm{End}(E)).$$

We call R a curvature form.

Explicit expression in terms of covariant derivative:

$$X - \text{ vector field }, \nabla_X(s) = \nabla s(X)$$

$$\nabla_X \colon \mathcal{S}(E) \to \mathcal{S}(E).$$

Let $\{X_i\}$ be basis of TM, i.e. linearly independent vector fields, $\{\omega^i\}$ - its dual basis of 1-forms. Then

$$\nabla(s) = \sum_i \omega^i \otimes \nabla_{X_i}(s), \text{ hence}$$

$$\nabla^2(s) = \sum_i d\omega^i \otimes \nabla_{X_i}(s) - \sum_i \omega^i \nabla(\nabla_{X_i}(s)) =$$

$$= \sum_i d\omega^i \otimes \nabla_{X_i}(s) - \sum_{i,j} \omega^i \wedge \omega^j \nabla_{X_j} \nabla_{X_i} s.$$

Where the second sum could be written as

$$\sum_{i,j} \omega^i \wedge \omega^j \nabla_{X_j} \nabla_{X_i} s = \sum_{i<j} \omega^i \wedge \omega^j [\nabla_{X_j}, \nabla_{X_i}] s.$$

Write

$$d\omega^i = \sum_{j<k} f^i_{jk} \omega^j \wedge \omega^k,$$

with $f^i_{jk} = d\omega^i(X_j, X_k) = -\omega^i([X_j, X_k])$. With that, we can rewrite first sum as

$$\sum_i d\omega^i \otimes \nabla_{X_i}(s) = -\sum_{j<k} \sum_i \omega^i([X_j, X_k]) \omega^j \wedge \omega^k \otimes \nabla_{X_i}(s) =$$

$$= -\sum_{j<k} \omega^j \wedge \omega^k \otimes \nabla_{\sum_i \omega^i([X_j, X_k]) X_i}(s) =$$

$$= -\sum_{j<k} \omega^j \wedge \omega^k \otimes \nabla_{[X_j, X_k]}(s).$$

We just proved

Lemma 2.1.

$$\nabla^2 s = \sum_{j<k} \omega^j \wedge \omega^k R_{X_j, X_k}(s) = R \cdot s, \textit{ where}$$

$$R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \mathrm{End}(E), \textit{ and}$$

$$R = \sum_{j<k} R_{X_j, X_k} \omega^i \wedge \omega^k.$$

For any Lie algebra $\mathfrak{g}$ of a Lie group G, we denote by $\mathcal{I}(\mathfrak{g})$ set of polynomials on $\mathfrak{g}$ which are invariant under adjoint action Ad_G. For

$$P \in \mathrm{Sym}(\mathfrak{g}^* \otimes \ldots \otimes \mathfrak{g}^*)$$

it means that

$$P(\mathrm{Ad}(g)x_1, \ldots, \mathrm{Ad}(g)x_r) = P(x_1, \ldots, x_r), \text{ where}$$

$$\mathrm{Ad}(g)(a) = gag^{-1}.$$

Let $\mathfrak{gl}_n(\mathbb{R})$ be the Lie algebra of $\mathrm{GL}_n(\mathbb{R})$. The set $\mathcal{I}(\mathfrak{gl}_n)$ is in fact ring, and is generated by elements

$$P_{2k}(A) = P_{2k}(A, \ldots, A) = \mathrm{tr}(A^k).$$

Theorem 2.2 (Chern-Weil). *Let $P \in \mathcal{I}(\mathfrak{gl}_n(\mathbb{R}))$ be an invariant polynomial of degree k, R - curvature of connection ∇ on real vector bundle $E \to M$.*

1. *Then $P(R) = P(R, \ldots, R) \in \Omega^{2k}(M)$ is closed and its de Rham cohomology class is independent of the connection.*

2. *More precisely, if* ∇_0*,* ∇_1 *are two connections, then*

$$P(R_1) - P(R_0) = k \cdot d \int_0^1 P(\alpha, R_t, \ldots, R_t)dt,$$

where $\alpha \in \Omega^1(M, \mathrm{End}(E))$ *is the difference* $\alpha = \nabla_1 - \nabla_0$*, and* R_t *is the curvature of a connection* $\nabla_t = (1-t)\nabla_0 + t\nabla_1$*.*

Proof. It is based on the two lemmas.

Lemma 2.3. *If* $\deg(P)$ *is odd, then* $P(R) = 0$ *for any metric connection.*

Proof. By hypothesis we have using Euclidean structure $(E, \langle -, - \rangle)$

$$X\langle s, t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle.$$

This implies

$$XY\langle s, t\rangle = X(\langle \nabla_Y s, t\rangle + \langle s, \nabla_Y t\rangle) =$$
$$\langle \nabla_X \nabla_Y s, t\rangle + \langle \nabla_Y s, \nabla_X t\rangle + \langle \nabla_X s, \nabla_Y t\rangle + \langle s, \nabla_X \nabla_Y t\rangle,$$

and

$$[X, Y]\langle s, t\rangle = \langle [\nabla_X, \nabla_Y]s, t\rangle + \langle s, [\nabla_X, \nabla_Y]t\rangle =$$
$$= \langle \nabla_{[X,Y]}s, t\rangle + \langle s, \nabla_{[X,Y]}t\rangle.$$

We can write then

$$\langle R_{X,Y}s, t\rangle + \langle s, R_{X,Y}t\rangle = 0, \text{ i.e.}$$
$$R + R^t = 0, \text{ and } P(R) = P(R^t, \ldots, R^t) = (-1)^k P(R).$$

□

Lemma 2.4. *For* $\omega \in \mathcal{S}(M, \mathrm{End}(E))$ *one has*

$$d(\mathrm{tr}\, \omega) = \mathrm{tr}[\nabla, \omega].$$

Proof. Locally, on a chart U we have $\nabla = d + \alpha$, $\alpha \in \Omega^1(U, \mathrm{End}(E))$. Hence

$$[\nabla, \omega] = [d + \alpha, \omega] = d\omega + [\alpha, \omega], \text{ and}$$
$$\mathrm{tr}[\nabla, \omega] = \mathrm{tr}\, d\omega + \mathrm{tr}[\alpha, \omega] = d(\mathrm{tr}\, \omega).$$

□

In particular (Bianchi's identity)

$$d\, \mathrm{tr}(R^k) = \mathrm{tr}[\nabla, R^k] = \mathrm{tr}[\nabla, \nabla^{2k}] = 0.$$

This gives proof of the first part, because polynomials of the form $\mathrm{tr}(R^k)$ generate $\mathcal{I}(\mathfrak{gl}_n(\mathbb{R}))$.

For the second part, note that if $\nabla_t = (1-t)\nabla_0 + t\nabla_1$, we have

$$\frac{d}{dt}(R_t) = \frac{d}{dt}\left(\nabla_t^2\right) = \frac{d}{dt}(\nabla_t)\nabla_t + \nabla_t \frac{d}{dt}\nabla_t =$$
$$= \left[\frac{d}{dt}\nabla_t, \nabla_t\right] = [\alpha, \nabla_t] = [\nabla_t, \alpha],$$

where $\alpha = \nabla_1 - \nabla_0$. Now

$$\frac{d}{dt}\,\mathrm{tr}(R_t^k) = \mathrm{tr}\left(\frac{d}{dt}R_t^k\right) = k\,\mathrm{tr}\left(\frac{dR_t}{dt}R_t^{k-1}\right) =$$
$$= k\,\mathrm{tr}\left([\nabla_t, \alpha]\nabla_t^{2(k-1)}\right) = k\,\mathrm{tr}([\nabla_t, \alpha\nabla_t^{2(k-1)}]) = kd\,\mathrm{tr}(\alpha R_t^{k-1}).$$

□

2.2 Adapted connection and Bott's theorem

Let $E \subset TM$ be an involutive subbundle and let $Q = TM/E$ with $\pi\colon TM \to Q$ be the projection.

Definition 2.5. *An* **adapted** *(or E-***flat***)* **connection** *on Q is a connection ∇ such that*

$$\nabla_X \pi(Z) = \pi([X, Z]), \ \forall X \in \mathcal{S}(E).$$

This makes sense, since

$$\nabla_{fX}\pi(Z) = \pi([fX, Z]) = -\pi(Z(f)X) + f\pi([X, Z]) = f\nabla_X\pi(Z), \text{ and}$$

$$\nabla_X(f\pi(Z)) = \pi([X, fZ]) = \pi(X(f)Z) + f\pi([X, Z]) = X(f)\pi(Z) + f\nabla_X(\pi(Z)).$$

To construct such a connection, take a decomposition $TM = E \oplus Q$ and set

$$\nabla_X\pi(Z) = \nabla_{X_E}\pi(Z) + \nabla_{X_{E^\perp}}(Z) =$$

$$= \pi([X_E, Z]) + \nabla_{X_{E^\perp}}(Z)$$

where we take an arbitrary connection on $E^\perp$.

Lemma 2.6. *For any adapted connection*

$$R_{X,Y} = 0, \ \ \forall X, Y \in \mathcal{S}(E).$$

Proof.

$$R_{X,Y}\pi(Z) = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})(\pi(Z)) =$$

$$\pi([X, [Y, Z]] - [Y, [X, Z]] - [[X, Y], Z]) = 0.$$

□

Theorem 2.7 (Bott's vanishing theorem). *Given $E \subset TM$ which is involutive, we have for $Q = TM/E$,* $\dim Q = q$

$$\mathrm{Pont}^{>2q}(Q) = 0.$$

Proof. Let

$$P_{2k}(A) := \mathrm{tr}(A^k).$$

Then for

$$R = \sum_{i<j} R_{X_i,X_j}\omega^i \wedge \omega^j$$

we have

$$P_{2k}(R) = \mathrm{tr}(R^k) = \sum \mathrm{tr}(R_{X_{i_1},X_{j_1}}, \ldots, R_{X_{i_{2k}},X_{j_{2k}}})\omega^{i_1} \wedge \omega^{j_1} \wedge \ldots \wedge \omega^{i_{2k}} \wedge \omega^{j_{2k}}.$$

If $k > q$, at least one pair belongs to E, otherwise

$$\omega^{i_1} \wedge \ldots \wedge \omega^{i_{2k}} = 0.$$

□

Remark 2.8.

$$\mathrm{Pont}(Q) = \mathrm{Pont}(TM \ominus E),$$

hence the above is a restriction of $[E] \in \mathrm{K}^0(M)$.

2.3 The Godbillon-Vey class

Let $\mathcal{F}$ be a codimension q foliation of M^n, $E = \tau\mathcal{F}$, $Q = TM/E$. First, assume that $\mathcal{F}$ is **transversaly orientable** i.e. $\Lambda^q Q$ has nowhere zero section (giving trivialization $\Lambda^q Q \cong M \times \mathbb{R}$).

Lemma 2.9. *Let Ω be nonvanishing section of $\Lambda^q Q$. Then*

$$d\Omega = \alpha \wedge \Omega$$

for some $\alpha \in \Omega^1(M, \mathrm{End}(E))$.

Proof. It suffices to prove it locally, then patch by partition of unity.

On a chart U, choose a basis $\omega_1, \dots, \omega_q \in \mathcal{I}(E)$ such that

$$\Omega = \omega_1 \wedge \ldots \wedge \omega_q,$$

$$d\omega_i = \sum_{j=1}^{q} \alpha_{ij} \wedge \omega_j$$

Then

$$d\Omega = \sum_{i=1}^{q} (-1)^i \omega_1 \wedge \ldots \wedge d\omega_i \wedge \ldots \wedge \omega_q =$$

$$= \sum_{i=1}^{q} (-1)^i \omega_1 \wedge \ldots \wedge \left(\sum_{j=1}^{q} \alpha_{ij} \wedge \omega_j \right) \wedge \ldots \wedge \omega_q$$

Only $\alpha_{ii} \wedge \omega_i$ can contribute to the sum, so

$$d\Omega = \left(\sum_{i=1}^{q} \alpha_{ii} \right) \wedge \Omega.$$

□

Lemma 2.10. *For all α as above $(d\alpha)^{q+1} = 0$.*

Proof.

$$0 = d^2\Omega = d\alpha \wedge \omega - \alpha \wedge d\Omega = d\alpha \wedge \Omega + \alpha \wedge \alpha \wedge \Omega = d\alpha \wedge \Omega.$$

Write $d\alpha$ using basis of 2-forms extending $\{\omega_1, \dots, \omega_q\}$

$$d\alpha = \sum_{1 \leqslant i < j \leqslant n} f_{ij} \omega_i \wedge \omega_j.$$

Now take exterior product with $\Omega = \omega_1 \wedge \ldots \wedge \omega_q$

$$\sum_{1 \leqslant i < j \leqslant n} f_{ij} \omega_i \wedge \omega_j \wedge \omega_1 \wedge \ldots \wedge \omega_q = 0.$$

If at least one of $i, j \in \{1, \dots, q\}$ then corresponding summand is 0. Hence

$$\sum_{q+1 \leqslant i < j \leqslant n} f_{ij} \omega_i \wedge \omega_j \wedge \omega_1 \wedge \ldots \wedge \omega_q = 0,$$

so

$$f_{ij} = 0 \text{ for } q+1 \leqslant i < j \leqslant n.$$

Now we can write

$$d\alpha = \sum_{i<j;\text{ at least one } \leqslant q} f_{ij}\omega_i \wedge \omega_j =$$

$$\sum_{j=1}^{q} \alpha_j \wedge \omega_j \in \mathcal{S}(E),$$

and

$$(d\alpha)^{q+1} = \sum f_{i_1 j_1} \dots f_{i_{q+1} j_{q+1}} \omega_{i_1} \wedge \omega_{j_1} \wedge \dots \wedge \omega_{i_{q+1}} \wedge \omega_{j_{q+1}} = 0.$$

□

We just proved that form $\eta = \alpha \wedge (d\alpha)^q$ is closed.

Lemma 2.11. *The class*

$$[\eta] \in \mathrm{H}^{2q+1}(M, \mathbb{R})$$

is independent on all choices involved in definition.

Proof. First assume that $\Omega' = f\Omega$ for $f > 0$ everywhere. Then

$$d\Omega' = f d\Omega + df\Omega = f\alpha \wedge \Omega + df \wedge \Omega = \alpha \wedge \Omega' + \frac{df}{f} \wedge \Omega' =$$

$$= (\alpha + d(\log f)) \wedge \Omega' = \alpha' \wedge \Omega'.$$

Hence

$$\Omega' \wedge (d\Omega')^q = (\alpha + d(\log f)) \wedge (d\alpha)^q = \alpha \wedge (d\alpha)^{+} d(\log(f)(d\alpha)^q),$$

so η and $\eta' = \alpha' \wedge (d\alpha')$ differ by boundary.

Now assume that $d\Omega = \alpha' \wedge \Omega$, $\beta = \alpha - \alpha'$ sucht that $\beta \wedge \Omega = 0$. Hence $\beta \in \mathcal{S}(E)$, and recall that also $d\alpha, d\alpha' \in \mathcal{S}(E)$. Then we have

$$\eta' = \alpha' \wedge (d\alpha')^q = (\alpha + \beta) \wedge ((d\alpha)^q + d\beta \wedge \sigma)$$

with

$$\sigma = \sum_{i=0}^{q-1} c_i (d\alpha^i) \wedge (d\beta)^{q-i-1} \in \mathcal{S}(E)^{q-1}, \quad \text{and } d\sigma = 0.$$

Then

$$\alpha' \wedge (d\alpha')^q = \alpha \wedge (d\alpha)^q + \alpha \wedge d\beta \wedge \sigma + \beta \wedge (d\alpha)^q + \beta \wedge d\beta \wedge \sigma,$$

where the last two summands belong to $\mathcal{S}(E)^{q+1} = 0$, so in fact we have

$$\alpha' \wedge (d\alpha')^q = \alpha \wedge (d\alpha)^q + \alpha \wedge d\beta \wedge \sigma =$$

$$= \alpha \wedge (d\alpha)^q + \alpha \wedge d(\beta \wedge \sigma) = \alpha \wedge (d\alpha)^q - d(\alpha \wedge \beta \wedge \sigma) + d\alpha \wedge \beta\sigma,$$

where the last summand is from $\mathcal{S}(E)^{q+1} = 0$. Again we see, that $\eta' - \eta$ is a boundary. □

Definition 2.12. *The class* $\mathrm{gv}(\mathcal{F}) := [\eta] \in \mathrm{H}^{2q+1}(M; \mathbb{R})$ *is called* **Godbillon-Vey class** *of a manifold with foliation* $(M, \mathcal{F})$.

Remark 2.13. Nonorientable case. Lift $\mathcal{F}$ to $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ = orientable double covering with γ= the generator of $\mathbb{Z}/2$. Replacing $\widetilde{\Omega}$ by $\frac{1}{2}(\widetilde{\Omega} - \gamma^*\widetilde{\Omega}) \neq 0$ if needed, we can always assume

$$\gamma^*(\widetilde{\Omega}) = -\widetilde{\Omega}.$$

Then

$$d\tilde{\Omega} = \widetilde{\alpha} \wedge \tilde{\Omega}, \text{ and } d(\gamma^*\widetilde{\Omega}) = \gamma^*(\widetilde{\alpha}) \wedge \gamma^*(\widetilde{\Omega}).$$

Hence

$$d\tilde{\Omega} = \gamma^*(\widetilde{\alpha}) \wedge \widetilde{\Omega}, \text{ and}$$

$$\frac{1}{2}(\widetilde{\alpha} + \gamma^*(\widetilde{\alpha}))$$

drops down to M.

2.4 Nontriviality of Godbillon-Vey class

On $G = \mathrm{SL}(2, \mathbb{R})$, with $TG \simeq G \times \mathfrak{g}$, ($\mathfrak{g}$ - Lie slgebra of G = traceless matrices) take the foliation given by the subbundle E generated by the left invariant vector fields corresponding to

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with

$$[X, H] = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2X.$$

The third basis element is

$$Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with

$$[Y, H] = 2Y, \quad [X, Y] = H.$$

Take the dual basis $\{\zeta, \eta, \chi\}$ of $\mathfrak{g}^*$ and extend them as left-invariant 1-forms. Then η defines $\mathcal{F}$ (i.e. $E = \ker \eta$). One has

$$d\chi = a\chi \wedge \zeta + b\chi \wedge \eta + c\zeta \wedge \eta,$$

$$b = d\chi(H, Y) = -\chi([H, Y]) = 2\chi(Y) = 0$$

$$c = d\chi(X, Y) = -\chi([X, Y]) = -\chi(H) = -1$$

$$a = d\chi(H, X) = \chi([X, H]) = -2\chi(X) = 0,$$

hence

$$d\chi = -\zeta \wedge \eta.$$

Similarly

$$d\zeta = -2\chi \wedge \zeta,$$

$$d\eta = 2\chi \wedge \eta.$$

The last implies

$$\alpha = 4\chi \wedge d\chi = -4\chi \wedge \zeta \wedge \eta.$$

The form α drops down to $M = \Gamma \setminus G$ for any Γ cocompact giving a volume form, hence

$$[\alpha_\Gamma] = \text{ generator of } \mathrm{H}^3(M; \mathbb{R}).$$

More precisely, let Σ_g be the Riemann surface of genus $g \geqslant 2$. Then its universal cover is the upper half plane

$$\mathbb{H} = \mathrm{SL}(2, \mathbb{R}) / \mathrm{SO}(2),$$

on which $\Gamma = \pi_1(\Sigma_g)$ acts by Mobius transformation

$$\Gamma \subset \mathrm{PSL}(2, \mathbb{R}), \quad z \mapsto \frac{az+b}{cz+d}.$$

Let $\tilde{\Gamma}$ be the double cover of Γ. Then $\tilde{\Gamma}$ is cocompact. Morover $M \simeq S^1\Sigma_g$ (unit tangent bundle), hence

$$[\alpha_\Gamma]([M]) = 4 \int_{S^1\Sigma_g} \zeta \wedge \eta \wedge \chi = 4\pi \int_{\Sigma_g} \zeta \wedge \eta = 4\pi \operatorname{Area}(\Sigma_g) =$$

$$= -4\pi \int_{\Sigma_g} K d\sigma = -8\pi^2(2-2g).$$

2.5 Naturality under transversality

Let $\phi\colon N \to M$, $E \subset TM$ integrable subbundle, $\mathcal{F}$- codimension q foliation, $\tau\mathcal{F} = E$.

If $V \to M$ is a vector bundle, then for each invariant polynomial $P \in \mathcal{I}(\mathfrak{gl}_q(\mathbb{R}))$ of degree k, we have a class $P(V) \in \mathrm{H}^{2k}(M; \mathbb{R})$. It behaves naturally with respect to pullback

$$\begin{array}{ccc} \phi^*(V) & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \pi \\ N & \xrightarrow{\ \phi\ } & M \end{array}$$

$$P(\phi^*(V)) = \phi^*(P(V)).$$

By Bott's vanishing theorem (2.7), all classes for $Q = TM/E$ are 0 if $k > q$. The Godbillon-Vey class $\mathrm{gv}(M, \mathcal{F}) \in \mathrm{H}^{2q+1}(M; \mathbb{R})$ is a nontrivial invariant.

Definition 2.14. *We say that ϕ is transversal to E (or to $\mathcal{F}$), $\phi \pitchfork E$, if for each $x \in N$*

$$T_{\phi(x)}M = \phi_*(T_xN) \oplus E_{\phi(x)}.$$

Equivalently

$$\pi \circ \phi_{*x}\colon T_xN \to T_{\phi(x)}M/E$$

is surjective.

Lemma 2.15. *$\widetilde{E} := \phi_*^{-1}(E)$ is involutive, hence defining a foliation $\widetilde{\mathcal{F}} = \phi^{-1}(\mathcal{F})$, whose leaves are the connected components of $\phi^{-1}(L)$, $L \subset \mathcal{F}$.*

Proof. (Short) Let $E = \tau\mathcal{F}$ be given by a cocycle $\{(U_i, f_i, g_{ij}) \mid i,j \in I\}$, $f_i\colon U_i \to \mathbb{R}^q$ submersions, $g_{ij}\colon f_j(U_i \cap U_j) \xrightarrow{\simeq} f_i(U_i \cap U_j)$. Then $\{(\phi^{-1}(U_i), f_i \circ \phi, g_{ij}) \mid i,j \in I\}$ define $\widetilde{\mathcal{F}}$. ☐

Proof. (More useful) Any map ϕ can be decomposed as a composition

$$N \xrightarrow{\mathrm{id}\times\phi} N \times M \xrightarrow{\mathrm{pr}_M} M,$$

$$x \mapsto (x, \phi(x)); \ \ (x, y) \mapsto y.$$

It is sufficient to prove the lemma for

(a) $\mathrm{id}\times\phi$ - injective immersion,

(b) pr_M - projection.

For each map in this composition the statement is obvious.

(a) $\widetilde{E} = E \cap TN$,

(b) $\widetilde{E} = TN \oplus E$.

□

Definition 2.16. *A* **characteristic class for foliation** $\mathcal{F}$ *is an assignment*

$$(M, \mathcal{F}) \mapsto \gamma(M, \mathcal{F}) \in \mathrm{H}^*(M; \mathbb{R})$$

such that if $\phi \colon N \to M$ *is transversal to* $\mathcal{F}$*, then*

$$\gamma(N, \phi^*(\mathcal{F})) = \phi^*(\gamma(M, \mathcal{F})).$$

Example 2.17. If $(M, \mathcal{F})$ is transversally oriented, i.e. there exists nowhere zero section Ω of $\Lambda^q Q$, then we have Godbillon-Vey class. On local chart U

$$\Omega = \omega_1 \wedge \ldots \wedge \omega_q, \ \ \{\omega_1, \ldots, \omega_q\} - \text{ generators of } \mathcal{S}(E\big|_U),$$

$$d\Omega = \alpha \wedge \Omega, \ \ \mathrm{gv}(M, \mathcal{F}) = [\alpha \wedge (d\alpha)^q] \in \mathrm{H}^{2q+1}(M; \mathbb{R}).$$

For $\phi \colon N \to M$

$$\{\phi^*(\omega_1), \ldots, \phi^*(\omega_q)\} - \text{ generators of } \mathcal{S}(\phi^*(E)\big|_{\phi^{-1}(U)})$$

and therefore

$$d\phi^*(\Omega) = \phi^*(d\Omega) = \phi^*(\alpha) \wedge \phi^*(\Omega),$$

and thus

$$\mathrm{gv}(N, \phi^*(\mathcal{F})) = \phi^*(\alpha) \wedge (d\phi^*(\alpha))^q = \phi^*(\alpha \wedge (d\alpha)^q) = \phi^*(\mathrm{gv}(M, \mathcal{F})).$$

Example 2.18. Pontryagin classes are characteristic classes of for foliation, since for $P \in \mathcal{I}^k(\mathfrak{gl}_q(\mathbb{R}))$ we have

$$P(\phi^*(\mathcal{F})) = \phi^*(P(\mathcal{F})),$$

where $P(\mathcal{F}) = P(Q)$ for $Q = TM/\tau\mathcal{F}$.

2.6 Transgressed classes

Let $(M, \mathcal{F})$ be a manifold with foliation, ∇_0, ∇_1 two connections on $Q = TM/E$, $E = \tau\mathcal{F}$. Then

$$\nabla_1 - \nabla_0 = \alpha \in \Omega^1(M, \operatorname{End}(E)).$$

Let $\nabla_t := t\nabla_1 + (1-t)\nabla_0$ be linear homotopy between connections, and R_0, R_1, R_t corresponding curvatures. Then by the theorem of Chern-Weil (2.2) for $P \in \mathcal{I}^k(\mathfrak{gl}_q(\mathbb{R}))$

$$P(R_1) - P(R_0) = dTP(\nabla_1, \nabla_0), \text{ where}$$

$$TP(\nabla_1, \nabla_0) := k\int_0^1 P(\alpha, R_t, \ldots, R_t)dt.$$

Let $\nabla_1 = \nabla^\flat$ be the E-flat connection (or Bott connection) (def. (2.5)), i.e.

$$\nabla^\flat_X(\pi(Y)) = \pi([X, Y]), \ \ \forall X \in \mathcal{S}(E), \pi\colon TM \to TM/E = Q.$$

The corresponding curvature satisfies (lemma (2.6))

$$R^\flat(X_1, X_2) = 0, \ \ \forall X_1, X_2 \in \mathcal{S}(E).$$

As a second connection ∇_0 we take metric (or Riemannian) connection $\nabla^\sharp$, i.e.

$$X\langle s_1, s_2\rangle = \langle \nabla^\sharp_X s_1, s_2\rangle + \langle s_1, \nabla^\sharp_X s_2\rangle,$$

for $s_1, s_2 \in \mathcal{S}(Q)$. Then

- $P(R^\flat) = 0$ if $k > q$, by Bott's theorem (2.7),
- $P(R^\sharp) = 0$ if k is odd, by lemma (2.3).

In particular for $k > q$ odd form $TP(\nabla^\flat, \nabla^\sharp)$ is closed, $dTP(\nabla^\flat, \nabla^\sharp) = 0$, so

$$TP(M, \mathcal{F}) := [TP(\nabla^\flat, \nabla^\sharp)] \in \mathrm{H}^{2k-1}(M, \mathbb{R}).$$

Definition 2.19. *We call $TP(M, \mathcal{F})$ a* **transgressed class**.

Proposition 2.20. *For foliation $\mathcal{F}$ on a manifold M and $P \in \mathcal{I}^k(\mathfrak{gl}_q(\mathbb{R}))$, $k > q = \dim TM/\tau\mathcal{F}$, class $[TP(M, \mathcal{F})] \in \mathrm{H}^{2k-1}(M; \mathbb{R})$ is independent of choices $\nabla^\flat$ and $\nabla^\sharp$, and therefore is an invariant of foliation.*

Proof. Let ${}^i\nabla^\flat$, ${}^i\nabla^\sharp$, $i = 0, 1$ be two different choices of connections, and let

$${}^t\nabla^\flat := \psi(t)^1\nabla^\flat + (1 - \psi(t))^0\nabla^\flat,$$

$${}^t\nabla^\sharp := \psi(t)^1\nabla^\sharp + (1 - \psi(t))^0\nabla^\sharp,$$

where in both cases $\psi\colon [0, 1] \to [0, 1]$ is a smooth function such that $\psi \equiv 0$ near 0 and $\psi \equiv 1$ near 1.

Now take the bundle $\widetilde{E} = E \oplus \mathbb{R}$ on $M \times \mathbb{R}$ (as a integrable bundle of foliation on $M \times \mathbb{R}$). On the quotient $\mathrm{pr}_M^*(Q)$ we define the connections $\widetilde{\nabla^\flat}$ and $\widetilde{\nabla^\sharp}$.

$$\begin{array}{ccc} \mathrm{pr}_M^*(Q) = T(M \oplus \mathbb{R})/\widetilde{E} & & Q = TM/\tau\mathcal{F} \\ \downarrow & & \downarrow \\ M \times \mathbb{R} & \xrightarrow{\ \mathrm{pr}_M\ } & M \end{array}$$

Sections of bundles over $M \times \mathbb{R}$ can be represented as follows

$$\mathcal{S}(T(M\times\mathbb{R})) = \{f(x,s)Y + g(x,s)\frac{\partial}{\partial s} \mid Y \in \mathcal{S}(TM), f,g \in C^\infty(M\times\mathbb{R})\}.$$

$$\mathcal{S}(\mathrm{pr}_M^*(Q)) = \{f(x,s)\pi(Y) \mid Y \in \mathcal{S}(TM), \pi\colon TM \to Q, f \in C^\infty(M\times\mathbb{R})\}$$

It suffices to define

$$\widetilde{\nabla}_{(X,\frac{\partial}{\partial t})}(\pi(Y)) :=^s \nabla_X(\pi(Y)).$$

for $\widetilde{\nabla} = \widetilde{\nabla^\flat}$ or $\widetilde{\nabla^\sharp}$.

We have

$$\widetilde{\nabla}_X(f(x,s)\pi(Y)) = X(f)\pi(Y) + f^s\nabla_X(\pi(Y)),$$

$$\widetilde{\nabla}_{\frac{\partial}{\partial s}}(f(x,s)\pi(Y)) = \frac{\partial f}{\partial s}\pi(Y),$$

where ${}^s\nabla^\flat = s^0\nabla^\flat + (1-s)^0\nabla^\flat$, ${}^s\nabla^\sharp = s^0\nabla^\sharp + (1-s)^0\nabla^\sharp$. Using inclusions $i_s\colon M \to M\times\mathbb{R}$, $i_s(x) = (x,s)$, we can write

$$i_0^*(\widetilde{R^\flat}) =^0 R^\flat, \quad i_1^*(\widetilde{R^\flat}) =^1 R^\flat$$

and analogously for $\nabla^\sharp, R^\sharp$. Similarly

$$i_0^*(\widetilde{\alpha}) =^0 \alpha, \quad i_1^*(\widetilde{\alpha}) =^1 \alpha$$

for corresponding differences ${}^0\alpha =^0 \nabla^\flat -^0 \nabla^\sharp$ and ${}^1\alpha =^1 \nabla^\flat -^1 \nabla^\sharp$. Hence

$$i_0^*(TP(\widetilde{\nabla}^\flat, \widetilde{\nabla}^\sharp)) = TP({}^0\nabla^\flat, {}^0\nabla^\sharp), \text{ and}$$

$$i_1^*(TP(\widetilde{\nabla}^\flat, \widetilde{\nabla}^\sharp)) = TP({}^1\nabla^\flat, {}^1\nabla^\sharp).$$

Note that $\widetilde{\nabla}^\flat$ is $\widetilde{E}$-flat, and $\widetilde{\nabla}^\sharp$ is Riemannian for $\mathrm{pr}_M^*(Q)$.

The proof is completed by the elementary lemma (homotopy invariance of de Rham cohomology)

Lemma 2.21. *Let $\omega \in \Omega^k(M\times\mathbb{R})$, $d\omega = 0$. Then $i_1^*(\omega) - i_0^*(\omega)$ is exact.*

Proof. We can write

$$\omega = \pi^*(\alpha) \wedge f(x,t)dt + g(x,t)\pi^*(\beta),$$

with $\alpha \in \Omega^{k-1}(M)$, $\beta \in \Omega^k(M)$.

One has

$$\mathcal{L}_{\partial_t}(\omega) = d\iota_{\partial t} + \iota_{\partial t}d\omega = \mathcal{L}_{\partial_t}(\omega) = d((-1)^{k-1}f(x,t)\,\mathrm{pr}_M^*(\alpha)) =$$

$$= (-1)^{k-1}f(x,t)d\,\mathrm{pr}_M^*(\alpha) + \mathrm{pr}_M^*(\alpha)\wedge d_x f + \mathrm{pr}_M^*(\alpha)\wedge \partial_t f dt,$$

where $\partial_t := \frac{\partial}{\partial t}$. On the other hand

$$\mathcal{L}_{\partial_t}\big|_{s=t_0}(\omega) = \frac{\partial}{\partial s}\big|_{s=t_0}(i_s(\mathrm{pr}_M^*(\alpha)\wedge f(x,t)dt + g(x,t)\,\mathrm{pr}_M^*(\beta))) =$$

$$= \partial_t f(x,t)\big|_{t_0}\mathrm{pr}_M^*(\alpha)\wedge dt + \partial_t g(x,t)\big|_{t_0}\mathrm{pr}_M^*(\beta).$$

Comparing both sides one gets

$$\partial_t g(x,t)\wedge \mathrm{pr}_M^*(\beta) = (-1)^{k-1}(f(x,t)d\,\mathrm{pr}_M^*(\alpha) + d_x f(x,t)\wedge \mathrm{pr}_M^*(\alpha)) =$$

$$= (-1)^{k-1} d_x(f(x,t)\,\mathrm{pr}_M^*(\alpha)).$$

Hence

$$g(x,1)\,\mathrm{pr}_M^*(\beta) - g(x,0)\,\mathrm{pr}_M^*(\beta) = (-1)^{k-1} d_x\left(\int_0^1 f(x,t)dt \cdot \mathrm{pr}_M^*(\alpha)\right),$$

so

$$i_1^*(\omega) - i_0^*(\omega) = d\left((-1)^{k-1}\int_0^1 f(x,t)dt \cdot \alpha\right).$$

□

□

Proposition 2.22. *For any $P \in \mathcal{I}^k(\mathfrak{gl}_n(\mathbb{R}))$ with $k > q$ odd, $TP(M\mathcal{F})$ is a characteristic class.*

Proof. It is sufficient to prove the naturality in two special cases

1. $i\colon N \to M$ is injective immersion,
2. $p\colon N \times M \to M$ a projection.

Case. 1 We have $i^*(E) = E \cap TN$, $i^*(Q) = Q\big|_N$, hence $\nabla^\flat$, $\nabla^\sharp$ restrict to the same kind of connections. Thus one has

$$TP(N, i^*(\mathcal{F})) = i^*(TP(M,\mathcal{F})).$$

Case. 2 We lift $\nabla^\flat$, $\nabla^\sharp$ to the same kind of connections on $N \times M$. $\widetilde{R}_t = p^*(R_t)$, $\widetilde{\alpha} = p^*(\alpha)$.

□

Definition 2.23. *Two vector bundles $E_0, E_1 \subset TM$ of* codim $= q$ *are transversaly homotopic if there exists $\widetilde{E} \subset T(M \times \mathbb{R})$ of* codim $= q$*, such that*

1. *$\widetilde{E}$ is involutive,*
2. *$\widetilde{E}$ is transversal to $M \times \{0\}$ and $M \times \{1\}$,*
3. *$i_0^*(\widetilde{E}) = E_0$ and $i_1^*(\widetilde{E}) = E_1$.*

Proposition 2.24. *The class $TP(M,\mathcal{F})$ depends only on transverse homotopy class of foliation $\mathcal{F}$.*

Chapter 3

Weil algebras

3.1 The truncated Weil algebras and characteristic homomorphism

The set of invariant polynomials $\mathcal{I}(\mathfrak{gl}_q(\mathbb{R}))$ is generated by $P_{2k}(A) := \mathrm{tr}(A^k)$, $A \in \mathfrak{gl}_q(\mathbb{R})$. Alternatively we have

$$\det(I + tA) = \sum_{i=0}^{q} c_i(A)t^i.$$

Coefficients $c_i(A)$ are symmetric functions of eigenvalues. If

$$A \sim \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_q \end{pmatrix}$$

then

$$\det(I + tA) = (1 + t\lambda_1)(1 + t\lambda_2)\dots(1 + t\lambda_q) =$$

$$= 1 + t(\lambda_1 + \lambda_2 + \dots + \lambda_q) + t^2(\sum \lambda_i\lambda_j) + \dots + t^q\lambda_1\lambda_2\dots\lambda_q.$$

$$c(A) := \det(I + A) = 1 + c_1(A) + \dots + c_q(A),$$

$$c(A \oplus B) = c(A)c(B).$$

The set $\mathcal{I}(\mathfrak{gl}_q(\mathbb{R}))$ can be presented as polynomial ring

$$\mathcal{I}(\mathfrak{gl}_q(\mathbb{R})) = \mathbb{R}[c_1, \dots, c_q].$$

For manifold with foliation $(M, \mathcal{F})$, $Q = TM/E$, $E = \tau\mathcal{F}$, we have

$$c_k(R^\flat) = 0, \ \ \forall k > q.$$

Moreover for each $P \in \mathbb{R}^k[c_1, \dots, c_q]$, $k > q$

$$P(R^\flat) = 0 \in \Omega^{2k}(M).$$

Define

$$\mathbb{R}[c_1, \dots, c_q]_q := \mathbb{R}[c_1, \dots, c_q]/(\text{weight } > 2q), \ \ \deg(c_i) = 2i.$$

For any connection ∇ on E we have a map

$$\lambda_E(\nabla)\colon \mathbb{R}[c_1,\dots,c_q] \to \Omega^\bullet(M),$$

$$\lambda_E(\nabla)(P) := P(\nabla^2).$$

Proposition 3.1. *1. $\lambda_E(\nabla^\flat)$ annihilates all polynomials of degree $> q$, so it induces a map*

$$\lambda_E(\nabla^\flat)\colon \mathbb{R}[c_1,\dots,c_q]_q \to \Omega^\bullet(M).$$

2. $\lambda_E(\nabla^\sharp)$ annihilates all polynomials of odd degree, in particular

$$\lambda_E(\nabla^\sharp)(c_{2i-1}) = 0.$$

3. There is a third map

$$T\lambda_E(\nabla^\flat,\nabla^\sharp)\colon \mathbb{R}[c_1,\dots,c_q] \to \Omega^*(M)$$

satisfying

$$dT\lambda_E(\nabla^\flat,\nabla^\sharp)(P) = \lambda_E(\nabla^\flat)(P) - \lambda_E(\nabla^\sharp)(P).$$

In particular

$$dT\lambda_E(\nabla^\flat,\nabla^\sharp)(c_{2i-1}) = \lambda(\nabla^\flat)(c_{2i-1}).$$

This can be summarized in the following cochain complex. First form a differential graded algebra (DGA)

$$WO_q := \Lambda\langle u_1, u_3, \dots, u_{2l-1}\rangle \otimes \mathbb{R}[c_1,\dots,c_q]_q,$$

where the first algebra in the tensor product is an exterior algebra generated by elements u_{2i-1} of degree $4i-3$, and l is maximal integer such that $2l-1 \leqslant q$. Generators of second algebra c_j have degree $2j$, and this is a quotient of polynomial algebra by the ideal of polynomials of degree $> q$ (weight $> 2q$). Now define $d\colon WO_q \to WO_q$ as the diferenital of degree 1 given on generators by the formula

$$du_{2i-1} = c_{2i-1},\ \ 1 \leqslant i \leqslant l,$$

$$dc_j = 0,\ \ 1 \leqslant i \leqslant q.$$

Definition 3.2. *Define a map $\lambda_E\colon WO_q \to \Omega^\bullet(M)$ by*

$$\lambda_E(u_{2i-1}) := T\lambda_E(\nabla^\flat,\nabla^\sharp)(c_{2i-1}),$$

$$\lambda_E(c_j) := \lambda_E(\nabla^\flat)(c_j),\ \ 1 \leqslant j \leqslant q.$$

Then $\lambda_E\colon WO_q \to \Omega^\bullet(M)$ is a map of DGA's, hence it induces a map

$$\lambda_E^*\colon\ \mathrm{H}^*(WO_q) \to \mathrm{H}^*(M;\mathbb{R})$$

of cohomology algebras.

We call λ_E^* a characteristic map in analogy to

$$\chi_E\colon\ \mathrm{H}^*(\mathrm{B}\,\mathrm{GL}_n(\mathbb{R})) = \mathcal{I}(\mathfrak{gl}_n(\mathbb{R})) \to \mathrm{H}^*(M;\mathbb{R})$$

for a n-dimesional vector bundle $E \to M$.

Theorem 3.3 (Bott). *1. λ_E^* depends only on E, and not on the choice of connections.*

2. λ_E^* is natural, i.e. for $\phi\colon N \to M$, $\phi \pitchfork \mathcal{F}$, one has

$$\lambda^*_{\phi^*(E)} = \phi^* \circ \lambda^*_E.$$

3. λ_E^* depends only on the transverse homotopy class of E (def. (2.23)).

Proof. Theorem has essentially been proved.

1. This has been proved in proposition (2.20).
2. This has been proved in proposition (2.22).
3. The same proof as in proposition (2.20) and lemma (2.21) with $\widetilde{\nabla}_t$ on $M \times I$ inducing ∇_t^0 on E_0 and ∇_t^1 on E_1.

□

Example 3.4 *(WO_1 and Godbillon-Vey class).* For $q = 1$ we have

$$WO_1 = \Lambda\langle u_1\rangle \otimes \mathbb{R}[c_1]_1,$$

hence $\{1, u_1, c_1, u_1c_1\}$ form a $\mathbb{R}$-basis and $du_1 = c_1$, $dc_1 = 0$. Clearly

$$\mathrm{H}^0(WO_1) = \mathbb{R} \cdot 1,$$
$$\mathrm{H}^1(WO_1) = 0,$$
$$\mathrm{H}^2(WO_1) = 0,$$
$$\mathrm{H}^3(WO_1) = \mathbb{R} \cdot u_1c_1.$$

Let (M, E) be a manifold with codim $= 1$ foliation $\mathcal{F}$, $\tau\mathcal{F} = E$, and assume that $Q = TM/E$ is trivializable (i.e. E transversaly oriented).

$$\lambda_E(c_1) = \lambda_E(\nabla^\flat)(c_1),$$
$$\lambda_E(u_1) = T\lambda_E(\nabla^\flat, \nabla^\sharp)(c_1).$$

Let $\Omega \in \Omega^1(M)$ be the orientation form of Q^*, so $E = \ker \Omega$. Let Z be a vector field with $\Omega(Z) = 1$, which gives trivialization of Q. Then

$$TM = E \oplus \mathbb{R}Z.$$

Let Ω be defined by

$$\Omega(X) = 0, \text{ for } X \in E,$$
$$\Omega(Z) = 1.$$

Then

$$d\Omega = \alpha \wedge \Omega, \ \ \alpha \in \Omega^1(M).$$

Form α defines a Bott connection by

$$\nabla^\flat(\pi(Z)) = -\alpha \otimes \pi(Z),$$

$$\nabla^\flat_X(\pi(Z)) = -\alpha(X)(\pi(Z)) = \pi([X, Z]).$$

Indeed, one has for all $X \in E$

$$d\Omega(X,Z) = -\Omega([X,Z]) = -\Omega(\pi([X,Z])), \text{ and}$$

$$\alpha \wedge \Omega(X,Z) = \alpha(X)\Omega(Z) - \alpha(Z)\Omega(X) = \alpha(X).$$

Thus

$$\alpha(X) = -\Omega(\pi([X,Z])).$$

Godbillon-Vey class is a class of $\alpha \wedge d\alpha$ in $\mathrm{H}^3(M;\mathbb{R})$. One the other hand one has

$$(\nabla^\flat)^2(\pi(Z)) = \nabla^\flat(-\alpha \otimes \pi(Z)) = -d\alpha \otimes \pi(Z) + \alpha \wedge \alpha \otimes \pi(Z) =$$

$$= d\alpha \otimes \pi(Z),$$

hence

$$R^\flat = d\alpha, \text{ so}$$

$$\lambda_E(c_1) = d\alpha.$$

Define a Riemannian connection on Q by

$$\nabla^\sharp_X(\pi(Z)) = 0, \ \ \forall X \in E,$$

$$\nabla^\sharp_Z(\pi(Z)) = 0, \text{ where } ||Z|| = 1.$$

Then $\nabla^\flat - \nabla^\sharp = -\alpha \in \Omega^1(M, \mathrm{End}(Q)) = \Omega^1(M)$, hence

$$\lambda_E(u_1) = T\lambda_E(\nabla^\flat, \nabla^\sharp)(c_1) = -\alpha.$$

This implies

$$\lambda_E(u_1 c_1) = \alpha \wedge d\alpha = \mathrm{gv}(M, \mathcal{F}).$$

Proposition 3.5. *If $E = \tau\mathcal{F}$ is of* codim $= q$, *transversally oriented, then*

$$\lambda_E(u_1 c_1^q) = \mathrm{gv}(E).$$

Proof. We have nonvanishing form $\Omega \in \mathcal{S}((Q^*)^q)$. Locally it can be written as

$$\Omega = \omega_1 \wedge \ldots \wedge \omega_q,$$

with $\{\omega_1, \ldots, \omega_q\}$- generators of $\mathcal{S}(E)$. Write

$$d\omega_i = \sum_j \alpha_{ij} \wedge \omega_j,$$

and define $\nabla^\flat \colon \mathcal{S}(Q) \to \mathcal{S}(T^*M \otimes Q)$ by

$$\nabla^\flat(\pi(Z_i)) = -\sum_j \alpha_{ji} \otimes \pi(Z_j),$$

where $\{Z_1, \ldots, Z_q\}$ is a dual basis to $\{\omega_1, \ldots, \omega_q\}$ on a complement of E. One has for all $X \in E$

$$d\omega_i(X, Z_k) = \sum_j (\alpha_{ij}(X)\omega_j(Z_k) - \alpha_{ij}(Z_k)\omega_j(X)).$$

But

$$d\omega_i(X, Z_k) = -\omega_i([X, Z_k]) = \pi([X, Z_k]),$$

and on the right hand side we have only $\alpha_{ik}(X)$, so

$$\pi([X, Z_k]) = \sum_i \alpha_{ik}(X)\pi(Z_i),$$

while

$$\nabla^\flat_X(\pi(Z_k)) = -\sum_j \alpha_{jk}(X)\pi(Z_j) = \pi([X, Z_k]),$$

hence it is a Bott connection. Its curvature is

$$(\nabla^\flat)^2(\pi(Z_i)) = -\sum_j \nabla^\flat(\alpha_{ij} \otimes \pi(Z_j)) =$$

$$= -\sum_j d\alpha_{ji} \otimes \pi(Z_j) + \sum_j \alpha_{ji}(-\sum_k \alpha_{kj} \otimes \pi(Z_k)) =$$

$$= -\sum_k (d\alpha_{ki} - \sum_j \alpha_{kj} \wedge \alpha_{ji})\pi(Z_k),$$

i.e.

$$R = d\alpha - \alpha \wedge \alpha.$$

This implies

$$c_1(R) = \mathrm{tr}(d\alpha) - \mathrm{tr}(\alpha \wedge \alpha) = \mathrm{tr}(d\alpha) = d(\mathrm{tr}\,\alpha),$$

hence

$$c_1(R)^q = d(\mathrm{tr}\,\alpha)^q.$$

Take Riemannian connection given by an orthogonal matrix form

$$\nabla^\sharp(\pi(Z_i)) = \sum_j \beta_{ij} \otimes \pi(Z_j).$$

Now

$$(\nabla^\flat - \nabla^\sharp)(\pi(Z_i)) = \sum_j (\alpha_{ij} + \beta_{ij}) \otimes \pi(Z_j),$$

hence

$$\nabla^\flat - \nabla^\sharp = -\alpha - \beta, \quad \mathrm{tr}\,\beta = 0$$

so the transgressed form is

$$Tc_1(\alpha + \beta) = \mathrm{tr}\,\alpha.$$

Now

$$\mathrm{gv}(E) = [\mathrm{tr}\,\alpha \wedge (\mathrm{tr}(d\alpha))^q] = [u_1 c_1(R)^q].$$

□

3.2 W_q and framed foliations

Definition 3.6. *Differential graded algebra* W_q

$$W_q := \Lambda\langle u_1, \dots, u_q\rangle \otimes \mathbb{R}[c_1, \dots, c_q]_q$$
$$du_i = c_i, \quad dc_i = 0, \ \forall i = 1, \dots, q.$$

These algebras are useful for foliation $(M, \mathcal{F})$ with Q trivializable, when one can transgress to a flat Riemannian connection and get

$$\mu_E \colon W_q \to \Omega^\bullet(M),$$
$$\mu_E(u_i) := T\lambda_E(\nabla^\flat, \nabla^{\sharp,0})(c_i),$$
$$\mu_E(c_i) := \lambda_E(\nabla^\flat)(c_i).$$

Notation: for $\underbrace{i_1 < \dots < i_r}_{I}$, $\underbrace{j_1 \leqslant \dots \leqslant j_s}_{J}$ we denote

$$u_I c_J = u_{i_1} \dots u_{i_r} c_{j_1} \dots c_{j_r}.$$

Proposition 3.7. *The elements*

(a)

$$1 \cup \{\, u_I c_J \mid |J| \leqslant q, i_1 + |J| > q, i_1 \leqslant j_1\}$$

form a basis of $\mathrm{H}^*(W_q)$.

(b)

$$1 \cup \{\, u_I c_J \mid i_k \ \textit{odd}\ , |J| < q, i_1 + |J| > q, \ \textit{and} \begin{cases} \textit{if } r = 0 \textit{ then all } j_k \textit{ even}, \\ \textit{if } r \neq 0 \textit{ then } i_1 \leqslant \min_{odd}\{j_k\} \end{cases} \}$$

form a basis of $\mathrm{H}^*(WO_q)$.

Proof. (sketch)

Ad.(a)

$$d(u_I c_J) = \sum_{k=1}^{r} (-1)^{k-1} u_{i_1} \dots du_{i_k} \dots u_{i_r} c_J =$$
$$= \sum_{k=1}^{r} (-1)^{k-1} u_{i_1} \dots \widehat{u_{i_k}} \dots u_{i_r} c_{i_k} c_J = 0,$$

because $\deg c_{i_k} c_J \geqslant 2(|J| + i_1) > 2q$.

Ad.(b) If $r = 0$ then $d(c_J) = 0$. The case $r \neq 0$ is treated as above.

□

Consequences of (a) for $\mathrm{H}^*(W_q)$.

1.

$$\deg(u_I c_J) = (2i_1 - 1) + \dots + (2i_r - 1) + (2j_1 + \dots + 2j_s) \leqslant$$
$$\leqslant 2(1 + \dots + q) - q + 2|J| \leqslant q(q+1) - q + 2q = q^2 + 2q.$$

Hence

$$\mathrm{H}^m(W_q) = 0, \text{ for } m > q^2 + 2q.$$

2. On the other hand
$$\deg(u_I c_J) \geqslant 2|J| > 2q,$$
hence
$$\mathrm{H}^m(W_q) = 0, \text{ for } 1 \leqslant m < 2q.$$
With a little more work we can elliminate $m = 2q$ which can occur only if $|I|$ even.
3. The product structure is trivial.
4. In $\mathrm{H}^{2q+1}(W_q)$ the classes $u_1 c_1^{\alpha_1} \dots c_k^{\alpha_k}$ with $\sum_{i=1}^k \alpha_i = q$ are linearly independent

Similar conclusions hold for $\mathrm{H}^*(WO_q)$:

1. $$\mathrm{H}^m(WO_q) = 0, \text{ for } m > q^2 + 2q.$$
2. For $m \leqslant 2q$ one gets the Pontryagin classes
$$\{1, p_1, \dots, p_{[\frac{q}{2}]}\}.$$
3. The product structure is trivial in 'high degree'.
4. In $\mathrm{H}^{2q+1}(WO_q)$ the classes $u_1 c_1^{\alpha_1} \dots c_k^{\alpha_k}$ with $\sum_{i=1}^k \alpha_i = q$ are linearly independent.

Chapter 4

Gelfand-Fuks cohomology

4.1 Cohomology of Lie algebras

Recall the formula for the exterior derivation

$$d\colon \Omega^p(M) \to \Omega^{p+1}(M)$$

$$d\omega(X_0, \dots, X_p) = \sum_{i=0}^{p} (-1)^i X_i \omega(X_0, \dots, \widehat{X_i}, \dots, X_p) +$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p).$$

$$\mathrm{H}^*(\Omega^\bullet(M), d) = \mathrm{H}^*_{dR}(M; \mathbb{R}).$$

We can view $\Omega^\bullet(M)$ as a $C^\infty(M)$ linear homomorphisms

$$\Omega^\bullet(M) \simeq \mathrm{Hom}_{C^\infty(M)}(\Lambda^\bullet V_M, C^\infty(M)),$$

where V_M is a Lie algebra of vector fields on M with

$$[X, Y] = XY - YX.$$

More general context consists of

- $\mathfrak{g}$ - a Lie algebra of finite dimension over a field k,
- A - $\mathfrak{g}$-module
- Cochains $C^\bullet(\mathfrak{g}; A) := \mathrm{Hom}_k(\Lambda^\bullet \mathfrak{g}, A)$ with differential

  $$d\colon C^p(\mathfrak{g}; A) \to C^{p+1}(\mathfrak{g}; A),$$

  given by the same formula as above.
- Cohomology

  $$\mathrm{H}^*(\mathfrak{g}; A) := \mathrm{H}^*(C^\bullet(\mathfrak{g}; A), d).$$

Relative Lie algebra cohomology is defined as follows. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie subalgebra. Define relative cochains as

$$C^\bullet(\mathfrak{g}, \mathfrak{h}; A) := \{c \in C^\bullet(\mathfrak{g}; A) \mid \iota_X c = 0 \text{ and } \iota_X dc = 0 \; \forall X \in \mathfrak{h}\}.$$

By definition it is a subcomplex and its cohomology is

$$\mathrm{H}^*(\mathfrak{g}, \mathfrak{h}; A) := \mathrm{H}^*(C^\bullet(\mathfrak{g}, \mathfrak{h}; A), d).$$

Since

$$\mathcal{L}_X = d\iota_X + \iota_X d, \; \mathcal{L}_X \omega = d\iota_X \omega + \iota_X d\omega = 0,$$

alternatively we can put

$$C^\bullet(\mathfrak{g}, \mathfrak{h}; A) := \{c \in C^\bullet(\mathfrak{g}; A) \mid c \textbf{ basic} \text{ i.e. } \iota_X c = 0 \text{ and } \mathcal{L}_X c = 0 \; \forall X \in \mathfrak{h}\}.$$

One has

$$C^\bullet(\mathfrak{g}, \mathfrak{h}; A) = \mathrm{Hom}_k(\Lambda^\bullet(\mathfrak{g}/\mathfrak{h}), A)^{\mathfrak{h}}.$$

Slightly more generally, if H is a Lie group with $\mathfrak{h} = \mathrm{Lie}(H)$, acting on $\mathfrak{g}$ and A such that, the differential of the action on $\mathfrak{g}$ is $\mathrm{ad}_{\mathfrak{g}} \mathfrak{h}$, then

$$C^\bullet(\mathfrak{g}, H; A) := \{c \in \mathrm{Hom}_H(\Lambda^\bullet \mathfrak{g}, A) \mid \iota_X c = 0 \; \forall X \in \mathfrak{h}\},$$

and its cohomology is

$$\mathrm{H}^*(\mathfrak{g}, H; A).$$

Example 4.1. Let $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{R})$. Its complexification is $\mathfrak{g}_{\mathbb{C}} := \mathfrak{gl}_n(\mathbb{C})$. We have

$$\mathrm{H}^*(\mathfrak{g}_{\mathbb{C}}) = \mathrm{H}^*(\mathfrak{g}) \otimes \mathbb{C}.$$

Also one has for $\mathfrak{u}_n := \mathrm{Lie}(U(n))$

$$\mathrm{H}^*(\mathfrak{gl}_n(\mathbb{R})) = \mathrm{H}^*(\mathfrak{u}_n) = \Lambda\langle u_1, u_3, \ldots, u_{2l+1}\rangle, l = \left[\frac{n}{2}\right].$$

Furthermore for $g \in U(n)$ and k odd

$$d\,\mathrm{tr}((g^{-1}dg)^k) = -\,\mathrm{tr}((g^{-1}dg)^{k+1}) = 0.$$

The class $u_k := [\mathrm{tr}((g^{-1}dg)^k)]$ is called a Chern-Simons class.

4.2 Gelfand-Fuks cohomology

Let V_M be the algebra of vector fields on a manifold M, that is $\mathcal{S}(TM)$. C^∞ topology on V_M is given by C^∞ convergence on compacta of the local components (which are functions), and their derivatives.

$$X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}, \; f^i \in C^\infty(M).$$

Definition 4.2. *Define the Gelfand-Fuks cohomology as the cohomology of the algebra V_M continuous with respect to the C^∞ topology on V_M*

$$\mathrm{H}^*_{GF}(V_M) := \mathrm{H}^*_{cont}(V_M; \mathbb{R}).$$

Here $C^{\bullet}_{cont}(V_M;\mathbb{R})$ are continuous functionals on V_M with respect to C^∞ topology.

The remarkable fact [Gelfand-Fuks] is that H^*_{GF} is finite dimensional. An important step in the proof of this is played by an algebra of formal vector fields on M

$$\mathfrak{A}_n := \{X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \mid f^i \in \mathbb{R}[[x^1, \dots, x^n]]\}.$$

The dual algebra of vector fields

$$V_M^* := \mathrm{Hom}_{cont}(V_M, \mathbb{R})$$

consists of distributions with compact support. The notion of support makes sense for the cochains

$$C^{\bullet}_{cont}(V_M, \mathbb{R}) := \Lambda^{\bullet} V_M^*$$

and is preserved by

$$d\colon \Lambda^{\bullet} V_M^* \to \Lambda^{\bullet+1} V_M^*.$$

In particular one can take for $p_0 \in M$ the subcomplex

$$\Lambda^{\bullet} V^*_{M,p_0} := \text{ distributions supported at } p_0.$$

Then V^*_{M,p_0} is a real vector space spanned by ∇_{p_0} and its partial derivatives

$$X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i}$$

$$X \mapsto (-1)^{|\alpha|} \frac{\partial^{|\alpha|} f^i}{\partial x^\alpha}.$$

They only depend on the jet of X at p_0. Thus we are dealing with the continuous Lie algebra complex of

$$\mathfrak{A}_n := \{X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \mid f^i \in \mathbb{R}[[x^1, \dots, x^n]]\}.$$

with the $\mathcal{I}$-adic topology (since the elements of the dual depend on finite set).

In $\mathfrak{A}_n^*$ we have following forms

$$\theta^i(X) := f^i(0),\ 1 \leqslant i \leqslant n,$$

$$\theta^i_j(X) := -\frac{\partial f^i}{\partial x^j}\Big|_{x=0},\ 1 \leqslant i, j \leqslant n,$$

$$\theta^i_{jk}(X) := \frac{\partial^2 f^i}{\partial x^j \partial x^k}\Big|_{x=0},\ 1 \leqslant i, j, k \leqslant n,$$

and generally for multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$

$$\theta^i_\alpha := (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha}\Big|_{x=0}.$$

We make $\Lambda^{\bullet}\mathfrak{A}_n^*$ into a complex by defining the differential

$$d\omega(X_0, \dots, X_n) := \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_n).$$

1. The elements
$$\{\theta^i_\alpha \mid 1 \leqslant i \leqslant n, \alpha \in (\mathbb{Z}_+)^n\}$$
span $C^1(\mathfrak{A}_n) = \mathfrak{A}_n^*$, hence generate all of
$$C^\bullet(\mathfrak{A}_n) = \bigoplus_{k=0}^{\infty} \Lambda^k \mathfrak{A}_n^*.$$
Note that $\theta^i_\alpha = \theta^i_\beta$ if $\alpha = \beta$ as an unordered sets.

2. The Lie derivative
$$\mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\theta^i = \theta^i_j, \text{ and}$$
$$\mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\mathcal{L}\left(\frac{\partial}{\partial x^k}\right)\theta^i = \theta^i_{jk}, \text{ etc.}$$
Indeed
$$\mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\theta^i(X) = \left(\frac{d}{dt}\big|_{t=0}\tau^j_t\theta^i\right)(X) = \theta^i\left(\frac{d}{dt}\big|_{t=0}\tau^j_{-t}(X)\right) =$$
$$= \frac{d}{dt}\big|_{t=0} f^i(x^1, \dots, x^j - t, \dots, x^n) = -\frac{\partial f^i}{\partial x^i}\big|_{x=0} = \theta^i_j(X).$$
In general
$$\mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\theta^i_\alpha = \theta^i_{\alpha\cup j}$$
Since
$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0,$$
we have
$$\left[\mathcal{L}\left(\frac{\partial}{\partial x^i}\right), \mathcal{L}\left(\frac{\partial}{\partial x^j}\right)\right] = 0,$$
whence

3. $$C^1(\mathfrak{A}_n) \simeq \mathbb{R}\left[\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right]\{\theta^1, \dots, \theta^n\}$$
i.e. is a free module with n generators over the polynomial ring in n generators.

Proposition 4.3. *We have following identities in* $C^\bullet(\mathfrak{A}_n)$

1. $$d\theta^i + \sum_j \theta^i_j \wedge \theta^j = 0,$$

2. $$d\theta^i_k + \sum_j \left(\theta^i_{jk} \wedge \theta^j + \theta^i_j \wedge \theta^j_k\right) = 0,$$

3. $$d\theta^i_{kl} + \sum_j \left(\theta^i_{jkl} \wedge \theta^j + \theta^i_{jk} \wedge \theta^j_l + \theta^i_{jl} \wedge \theta^j_k + \theta^i_j \wedge \theta^j_{kl}\right) = 0.$$

Proof.

$$d\theta^i(X,Y) = \underbrace{X\theta^i(Y) - Y\theta^i(X)}_{=0} - \theta^i([X,Y]) = -\theta^i([X,Y]),$$

where $X = \sum_j f^j \frac{\partial}{\partial x^j}$, $X = \sum_j g^k \frac{\partial}{\partial x^k}$.

$$[X,Y] = \sum_{j,k} \left(f^j \frac{\partial g^k}{\partial x^j} \frac{\partial}{\partial x^k} - g^k \frac{\partial f^j}{\partial x^k} \frac{\partial}{\partial x^j} \right) =$$

$$= \sum_k \left(\sum_j \left(f^j \frac{\partial g^k}{\partial x^j} - g^j \frac{\partial f^k}{\partial x^j} \right) \right) \frac{\partial}{\partial x^k}.$$

Hence

$$d\theta^i(X,Y) = \sum_j \left(\underbrace{f^j \frac{\partial g^i}{\partial x^j} - g^j \frac{\partial f^i}{\partial x^j}}_{=0} - f^j \frac{\partial g^i}{\partial x^j} + g^j \frac{\partial f^i}{\partial x^j} . \right)$$

On the other hand

$$\theta^i_j \wedge \theta^j(X,Y) = \theta^i_j(X)\theta^j(Y) - \theta^i_j(Y)\theta^j(X) =$$

$$= \sum_j \left(-\frac{\partial f^i}{\partial x^j} g^j + \frac{\partial g^i}{\partial x^j} f^j \right).$$

This proves (1). To obtain (2) we apply $\mathcal{L}\left(\frac{\partial}{\partial x_k}\right)$, and applying $\mathcal{L}\left(\frac{\partial}{\partial x_l}\right)$ to (2) we obtain (3) etc. These equations completely determine differential d. □

Denote

$$R^i_j := d\theta^i_j + \sum_k \theta^i_k \wedge \theta^k_j \in C^2(\mathfrak{A}_n) = \Lambda^2 \mathfrak{A}_n^*.$$

Then equation (2) becomes

2'

$$R^i_j = -\sum_k \theta^i_{jk} \wedge \theta^k.$$

Proposition 4.4. *1.*

$$R^i_j \wedge \theta^j = 0,$$

2.

$$dR^i_j = \sum_k \left(R^i_k \wedge \theta^k_j - \theta^i_k \wedge R^k_j \right).$$

Proof. From (2')

$$R^i_j \wedge \theta^j = -\sum_k \theta^i_{jk} \wedge \theta^k \wedge \theta^j = 0$$

since $\theta^i_{jk} = \theta^i_{kj}$.

From (2)

$$dR^i_j = \sum_k \left(d\theta^i_k \wedge \theta^k_j - \theta^i_k \wedge d\theta^k_j \right) =$$

$$= \sum_k \left(-\sum_l (\theta^i_{lk} \wedge \theta^l + \theta^i_l \wedge \theta^l_k) \wedge \theta^k_j + \sum_l \theta^i_k \wedge (\theta^k_{lj} \wedge \theta^l + \theta^k_l \wedge \theta^l_j) = \right)$$

$$= \sum_{k,l} \left(R^i_k \wedge \theta^k_j - \theta^i_l \wedge \theta^l_k \wedge \theta^k_j + \theta^i_k \wedge R^k_j + \theta^i_k \wedge \theta^k_l \wedge \theta^l_j \right) =$$

$$= \sum_k \left(R^i_k \wedge \theta^k_j - \theta^i_k \wedge R^k_j \right).$$

□

Corollary 4.5. *The subalgebra* $\widetilde{W_n} := \mathbb{R}\{\theta^i_j, R^i_j\}$ *is closed under* d *and finite dimensional.*

Proof. Finite dimension follows from (2’). □

4.3 Some "soft" results

We describe the grading on an algebra $\mathfrak{A}_n$.

$$\mathfrak{A}_n = \{X = \sum_{i=1}^n f^i \frac{\partial}{\partial x^i} \mid f^i(x) = \sum_\alpha c^i_\alpha x^\alpha \in \mathbb{R}[[x_1, \dots, x_n]], \alpha = (\alpha_1, \dots, \alpha_n)\}.$$

$$\mathfrak{A}_n = \mathbb{R}^n \oplus \mathfrak{gl}_n(\mathbb{R}) \oplus \dots$$

One has

$$[x^i \frac{\partial}{\partial x^j}, x^k \frac{\partial}{\partial x^l}] = \delta^k_j x^i \frac{\partial}{\partial x^l} - \delta^i_l x^k \frac{\partial}{\partial x^j},$$

To see grading we take $E = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i} \in \mathfrak{A}_n$. Then

$$[E, X] = \sum_j \sum_i \left(x^i \frac{\partial f^j}{\partial x^i} - f^j \right) \frac{\partial}{\partial x^j}$$

and if $f^j = c^j_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $|\alpha| = r$, then

$$[E, c^j_\alpha x^\alpha \frac{\partial}{\partial x^j}] = [\sum_i x^i \frac{\partial}{\partial x^i}, c^j_\alpha x^\alpha \frac{\partial}{\partial x^j}] =$$

$$= \sum_i \alpha_i x^\alpha \frac{\partial}{\partial x^j} - \sum_i x^\alpha \delta^i_j \frac{\partial}{\partial x^i} = (|\alpha| - 1) x^\alpha \frac{\partial}{\partial x^j}.$$

Thus each monomial is an eigenvector for E, and we can write $\mathfrak{A}_n$ as a sum of eigenspaces

$$\mathcal{L}_E(x^\alpha \frac{\partial}{\partial x^j}) = (|\alpha| - 1) x^\alpha \frac{\partial}{\partial x^j},$$

$$\mathfrak{A}_n^{(p)} := \{X \in \mathfrak{A}_n \mid \mathcal{L}_E(X) = pX\},$$

$$\mathfrak{A}_n = \bigoplus_{p=-1}^{\infty} \mathfrak{A}_n^{(p)}, \quad E|_{\mathfrak{A}_n^{(p)}} = p \cdot \mathrm{Id}\,.$$

It is a grading, i. e.

$$[\mathfrak{A}_n^{(p)}, \mathfrak{A}_n^{(q)}] \subset \mathfrak{A}_n^{(p+q)}.$$

We have a dual grading on the Gelfand-Fuks complex $C^\bullet(\mathfrak{A}_n) = \Lambda^\bullet\mathfrak{A}_n^*$. One has the Lie derivative

$$\mathcal{L}_E\colon \mathfrak{A}_n^* \to \mathfrak{A}_n^*.$$

$$\mathcal{L}_E = d\iota_E + \iota_E d,$$

The dual grading on $\mathfrak{A}_n^*$ can be described as

$$(\mathfrak{A}_n^*)^{(p)} := \{\omega \in \mathfrak{A}_n^* \mid \mathcal{L}_E(\omega) = -p\omega\}.$$

This induces a grading on G-F complex

$$C^m(\mathfrak{A}_n)^{(p)} = (\Lambda^m\mathfrak{A}_n^*)^{(p)} = \bigoplus \Lambda^{k_{-1}}(\mathfrak{A}_n^*)^{(-1)} \otimes \Lambda^{k_0}(\mathfrak{A}_n^*)^{(0)} \otimes \ldots \otimes \Lambda^{k_r}(\mathfrak{A}_n^*)^{(r)},$$

where

$$k_{-1} + k_0 + \ldots = m, \quad -k_{-1} + k_1 + 2k_2 + \ldots + rk_r = p.$$

We have $\mathcal{L}_E d = d\mathcal{L}_E$ (so $\mathcal{L}_E$ is a map of complexes). We can restrict to degree p

$$\mathcal{L}_E\big|_{C^\bullet(\mathfrak{A}_n)^{(p)}} = -p \cdot \mathrm{Id}$$

Proposition 4.6.

$$\dim \mathrm{H}^*_{GF}(\mathfrak{A}_n) < \infty, \ \forall n \geqslant 0,$$

$$\mathrm{H}^m_{GF}(\mathfrak{A}_n) = 0, \ \forall m > n^2 + 2n.$$

Proof. One has

$$\mathcal{L}_E(\omega) = d\iota_E(\omega) + \iota_E d\omega$$

so any $\omega \in C^m(\mathfrak{A}_n)^{(p)}$ with $p \neq 0$ such that $d\omega = 0$ is exact, since then

$$d\iota_E(\omega) = \mathcal{L}_E(\omega) = -p\omega.$$

This gives on cohomology

$$\mathrm{H}^m_{GF}(\mathfrak{A}_n) = \mathrm{H}^m_{GF}(\mathfrak{A}_n)^{(0)} := \mathrm{H}^m(C^\bullet(\mathfrak{A}_n)^{(0)}),$$

where

$$C^m(\mathfrak{A}_n)^{(0)} = (\Lambda^m\mathfrak{A}_n^*)^{(0)} = \bigoplus \Lambda^{k_{-1}}(\mathfrak{A}_n^*)^{(-1)} \otimes \Lambda^{k_0}(\mathfrak{A}_n^*)^{(0)} \otimes \ldots \otimes \Lambda^{k_r}(\mathfrak{A}_n^*)^{(r)},$$

$$-k_{-1} + k_1 + 2k_2 + \ldots + rk_r = 0,$$

$$k_{-1} + k_0 + k_1 + \ldots + k_r = m.$$

Since

$$\dim \mathfrak{A}_n^{(-1)} = \dim \mathbb{R}^n = n \implies k_{-1} \leqslant n,$$

$$\dim \mathfrak{A}_n^{(0)} = n^2 \implies k_0 \leqslant n^2.$$

Furthermore

$$k_1 \leqslant n, k_2 \leqslant \frac{n}{2}, \ldots, k_n \leqslant 1.$$

Hence

$$\dim C^m(\mathfrak{A}_n)^{(0)} < \infty \text{ for } m \geqslant 0,$$

$$C^m(\mathfrak{A}_n)^{(0)} = 0 \text{ for } m > n^2 + 2n.$$

□

Example 4.7. For $n = 1$ we have following

$$k_1 + 2k_2 + \ldots k_r = k_{-1},$$

$$k_{-1} + k_0 + k_1 + \ldots + k_r \leqslant 3.$$

This gives

$$k_1 \leqslant 1, k_2 \leqslant \frac{1}{2} \text{ etc.} \quad \Longrightarrow \; k_2 = \ldots = k_r = 0.$$

The dual algebra

$$\mathfrak{A}_n^* \simeq \underbrace{\mathbb{R}\theta^1}_{\text{deg}=-1} \oplus \underbrace{\mathbb{R}\theta_1^1}_{deg=0} \oplus \underbrace{\mathbb{R}\theta_{11}^1}_{deg=1} \oplus \ldots$$

If $k_{-1} = 0$ then $k_1 = k_2 = \ldots = 0$ hence the only one allowed is

$$\Lambda^\bullet(\mathfrak{A}_1^*)^{(0)} = \mathbb{R} \oplus \mathbb{R}\theta_1^1.$$

For $k_{-1} = 1$ we have $k_1 = 1$ and

$$\underbrace{\Lambda^1(\mathfrak{A}_1^*)^{(-1)}}_{=\mathbb{R}\theta^1} \otimes \underbrace{\Lambda^\bullet(\mathfrak{A}_1^*)^{(0)}}_{=\mathbb{R}\oplus\mathbb{R}\theta_1^1} \otimes \underbrace{\Lambda^1(\mathfrak{A}_1^*)^{(1)}}_{=\mathbb{R}\theta_{11}^1}$$

Thus we need only to look at the subcomplex

$$\mathbb{R}\{1, \theta_1^1, \theta^1 \wedge \theta_{11}^1, \underbrace{\theta^1 \wedge \theta_1^1 \wedge \theta_{11}^1}_{=\theta_1^1 \wedge R_1^1}\}$$

because $R_1^1 = d\theta_1^1 = -\theta_{11}^1 \wedge \theta^1 \neq 0$, so the cohomology is

$$\mathrm{H}_{GF}^* = \underbrace{\mathbb{R}}_{\text{dim}=0} \oplus \underbrace{\mathbb{R}(\theta_1^1 \wedge R_1^1)}_{\text{dim}=3}.$$

4.4 Spectral sequences

The algebra generated by $\{\theta_j^i, R_j^i\}$ is closed under the differential d, so we have a subcomplex

$$(\mathbb{R}\{\theta_j^i, R_j^i\}, d) =: (\widetilde{W_n}, d) \subset (C^\bullet(\mathfrak{A}_n), d).$$

where

$$\mathbb{R}\{\theta_j^i, R_j^i\} \simeq \Lambda^\bullet \mathfrak{gl}_n(\mathbb{R})^* \otimes S_n(\mathfrak{gl}_n(\mathbb{R})^*)$$

Theorem 4.8. *The inclusion*

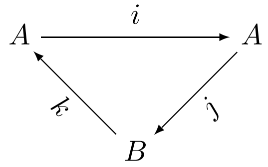
$$(\widetilde{W_n}, d) \hookrightarrow (C^\bullet(\mathfrak{A}_n), d)$$

is a quasi-isomorphism (induces isomorphism on cohomology).

The proof uses Hochschild-Serre spectral sequence, which we describe next.

4.4.1 Exact couples

Assume we have an exact sequence of the form

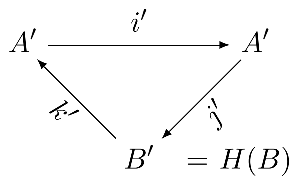


It is called an **exact couple**. Define

$$d\colon B \to B, \ \ d := jk, \ \ d^2 = jkjk = 0, \ \text{and}$$

$$\mathrm{H}(B) := \ker d / \operatorname{im} d.$$

Now we can form derived couple taking



where

- $A' := i(A)$,
- $B' := \mathrm{H}(B)$,
- $i'(a') = i(a') = i(i(a))$,
- $j'(a') = [j(a)]$ for $a' = i(a)$,
- $k'([b]) = k(b)$.

Check this definitions for independence of representatives. The derived couple is again exact couple.

4.4.2 Filtered complexes

Let $(C^\bullet, d)$ be a filtered complex i.e. there is a sequence of subcomplexes

$$C^\bullet = C_0^\bullet \supset C_1^\bullet \supset C_2^\bullet \supset \dots$$

Let

$$A := \bigoplus_{p\in\mathbb{Z}} C_p, \quad B := \bigoplus_{p\in\mathbb{Z}} C_p/C_{p+1}$$

Inclusions $C_{p+1} \hookrightarrow C_p$ induce exact sequence

$$0 \to A \xrightarrow{i} A \xrightarrow{B}\to 0,$$

a long exact sequence of homology

$$\dots \mathrm{H}(A) \xrightarrow{i_*} \mathrm{H}(A) \xrightarrow{j_*} \mathrm{H}(B) \xrightarrow{k_*} A \to \dots,$$

and an exact couple

$$
\begin{array}{ccc}
A_1 := \mathrm{H}(A) & \xrightarrow{i_*} & \mathrm{H}(A) \\
{\scriptstyle k_*}\nwarrow & & \swarrow{\scriptstyle j_*} \\
& \mathrm{H}(B) =: \ B_1 &
\end{array}
$$

4.4.3 Illustration of convergence

Consider simple case, filtration of a complex $\mathrm{H}(C^\bullet)$

$$\ldots = C_{-2} = C_{-1} = C_0 \supset C_1 \supset C_2 \supset 0 = \ldots$$

$$
\begin{array}{ccccccccccccccccccc}
\ldots & = & C_{-2} & = & C_{-1} & = & C_0 & \supset & C_1 & \supset & C_2 & = & 0 & = & \ldots \\
& \swarrow & & \swarrow & & \swarrow & & \swarrow & & \swarrow & & & & & \\
\ldots & = & C_{-2} & = & C_{-1} & = & C_0 & \supset & C_1 & \supset & C_2 & = & 0 & = & \ldots
\end{array}
$$

Here

$$B = \ldots \oplus 0 \oplus 0 \oplus C_0/C_1 \oplus C_1/C_2 \oplus C_2 \oplus 0 \oplus \ldots$$

Taking homology we get sequences

$$\mathrm{H}(C^\bullet) = \mathrm{H}(C_0) \leftarrow \mathrm{H}(C_1) \leftarrow \mathrm{H}(C_2) \leftarrow 0 \leftarrow \ldots$$

$$A_1 := \bigoplus_{p\in\mathbb{Z}} \mathrm{H}(C_p)$$

$$\mathrm{H}(C^\bullet) = \mathrm{H}(C_0) \supset i_* \,\mathrm{H}(C_1) \leftarrow i_* \,\mathrm{H}(C_2) \leftarrow 0 \leftarrow \ldots$$

$$A_2 := \bigoplus_{p\in\mathbb{Z}} i_* \,\mathrm{H}(C_p)$$

$$\mathrm{H}(C^\bullet) = \mathrm{H}(C_0) \supset i_* \,\mathrm{H}(C_1) \supset i_* i_* \,\mathrm{H}(C_2) \leftarrow 0 \leftarrow \ldots$$

$$A_3 := \bigoplus_{p\in\mathbb{Z}} i_* i_* \,\mathrm{H}(C_p).$$

When we reach the stage in wich all maps become inclusions, process is stationary i.e.

$$A_3 = A_4 = \ldots$$

$$
\begin{array}{ccc}
A_3 & \xrightarrow{i} & A_3 \\
{\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\
& B_3 \quad = \mathrm{H}(A_3) &
\end{array}
$$

where i is inclusion, $\operatorname{im} k = \ker i = 0$ so $k = 0$. This means that also

$$B_3 = B_4 = \ldots$$

since $d = kj = 0$

4.4.4 Hochschild-Serre spectral sequence

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra of a Lie algebra $\mathfrak{g}$.

$$C^\bullet(\mathfrak{g}; M) = \mathrm{Hom}(\Lambda^\bullet \mathfrak{g}, M), \;\; d\colon C^\bullet(\mathfrak{g}; M) \to C^{\bullet+1}(\mathfrak{g}; M)$$

$$d\omega(X_0, X_1, \ldots, X_r) = \sum_i (-1)^i X_i \omega(X_0, \ldots, \widehat{X_i}, \ldots, X_r) +$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_r).$$

Define the filtration on the above complex by

$$F^p C^{p+q}(\mathfrak{g}; M) := \{\omega \in C^{p+q} \mid \iota_{X_1} \ldots \iota_{X_q} \omega = 0 \;\forall X_1, \ldots, X_q \in \mathfrak{h}\}.$$

This means that we can associate with $\omega \in F^p C^{p+q}$ an element

$$\phi(\omega) \in \mathrm{Hom}(\Lambda^q \mathfrak{h}, \mathrm{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{h}), M))$$

given by the formula

$$\phi(\omega)(X_1, \ldots, X_q)(\underbrace{\widehat{Y_1}, \ldots, \widehat{Y_p}}_{\text{classes}}) = \omega(X_1, \ldots, X_q, Y_1, \ldots, Y_p).$$

Then

$$\ker \phi = F^{p+1} C^{p+q},$$

Hence there is a spectral sequence with

$$E_0^{p,q} \simeq C^q(\mathfrak{h}; \mathrm{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{h}), M)), \;\; d_0 = d,$$

$$E_1^{p,q} \simeq \mathrm{H}^q(\mathfrak{h}; \mathrm{Hom}(\Lambda^p(\mathfrak{g}/\mathfrak{h}), M)),$$

$$E_2^{p,0} \simeq \mathrm{H}^p(\mathfrak{g}, \mathfrak{h}; M),$$

$$E_\infty^* \implies \mathrm{H}^*(\mathfrak{g}; M)$$

Now we are ready to prove that the inclusion

$$i\colon \widetilde{W_n} \hookrightarrow C^\bullet(\mathfrak{A}_n)$$

induces an isomorphism

$$\mathrm{H}^*(\widetilde{W_n}, d) \simeq \mathrm{H}^*_{GF}(\mathfrak{A}_n)$$

that is theorem (4.8).

Proof. Both $\widetilde{W_n}$ and $C^\bullet(\mathfrak{A}_n)$ are filtered differential graded algebras, and their associated spectral sequences converge to $\mathrm{H}^*(\widetilde{W_n})$ and respectively to $\mathrm{H}^*_{GF}(\mathfrak{A}_n)$. On the other hand i induces isomorphism on the level of E_1.

First $\widetilde{W_n}$ is graded by

$$\widetilde{W_n}^p = \bigoplus_{r+2s=p} \Lambda^r \langle \theta^i_j \rangle \otimes S^s_n[R^i_j]$$

and then

$$F^p \widetilde{W_n}^{p+q} := \{\omega \in \widetilde{W_n}^{p+q} \mid \iota_{X_0} \ldots \iota_{X_q} \omega = 0 \;\forall X_0, \ldots, X_q \in \mathfrak{A}_n^{(0)}\}$$

Fact 4.9.

$$E_0^{p,q} \simeq \begin{cases} 0, & p \text{ odd or } p > 2n, \\ C^q(\mathfrak{A}_n^{(0)}; S_n^{\frac{p}{2}}[R^i_j]), & p \text{ even and } p \leqslant 2n. \end{cases}$$

$$E_1^{p,q} \simeq \begin{cases} 0, & p \text{ odd or } p > 2n, \\ \mathrm{H}^q_{GF}(\mathfrak{A}_n^{(0)}; S_n^{\frac{p}{2}}[R^i_j]), & p \text{ even and } p \leqslant 2n. \end{cases}$$

The filtration on $C^\bullet(\mathfrak{A}_n) = \bigoplus_p C^p(\mathfrak{A}_n)$ is the Hochschild-Serre filtration relative to $\mathfrak{A}_n^{(0)}$.

$$F^pC^{p+q}(\mathfrak{A}_n) = \begin{cases} C^{p+q}(\mathfrak{A}_n), & p \leqslant 0 \\ \{\omega \in C^{p+q}(\mathfrak{A}_n) \mid \iota_{X_0} \dots \iota_{X_q}\omega = 0 \ \forall X_0, \dots, X_q \in \mathfrak{A}_n^{(0)}\}, & p > 0, q \geqslant 0. \end{cases}$$

Fact 4.10.

$$E_1^{p,q} \simeq \mathrm{H}^q_{GF}(\mathfrak{A}_n^{(0)}; F^pC^p(\mathfrak{A}_n)).$$

It is a filtration, so

$$[\mathfrak{A}_n^{(0)}, \mathfrak{A}_n^{(p)}] \subset \mathfrak{A}_n^{(p)}$$

and we have an action of $\mathfrak{gl}_n(\mathbb{R}) = \mathfrak{A}_n^{(0)}$ on $\mathfrak{A}_n^{(p)}$ for each p. Since $\mathfrak{A}_n^{(0)}$ acts semisimply on the coefficients one gets further

$$E_1^{p,q} \simeq \mathrm{H}^q_{GF}\left(\mathfrak{A}_n^{(0)}, \left(\Lambda^p(\mathfrak{A}_n^{(0)})\right)^*\right) \simeq \mathrm{H}^q_{GF}(\mathfrak{A}_n^{(0)}; B^p),$$

where

$$B^p := \{\omega \in C^p(\mathfrak{A}_n) \mid \iota_X\omega = 0 = \mathcal{L}_X\omega \ \forall X \in \mathfrak{A}_n^{(0)}\}$$

are the **basic** elements with respect to $\mathfrak{A}_n^{(0)}$. Note that if $Y = Y^r_s = X^r\frac{\partial}{\partial x^s}$

$$\iota_Y R^i_j = -\iota_Y(\theta^i_{jk} \wedge \theta^k) = 0,$$

whence the map

$$E_1^{p,q}(\widetilde{W_n}) \to E_1^{p,q}(C^\bullet(\mathfrak{A}_n)).$$

Lemma 4.11. *The inclusion* $i\colon \widetilde{W_n} \hookrightarrow C^\bullet(\mathfrak{A}_n)$ *induces an isomorphism between the* $\mathfrak{A}_n^{(0)}$*-basic elements of* $\widetilde{W_n}$ *and* $C^\bullet(\mathfrak{A}_n)$*.*

Proof. Elementary invariance theory to eliminate the form θ^i_α with $|\alpha| > 2$. □

□

Again let

$$W_n = \Lambda\langle u_1, \dots, u_n\rangle \otimes S_n[c_1, \dots, c_n]$$

$$\deg(u_i) = 2i-1, \ \deg(c_i) = 2i, \ du_i = c_i, \ dc_i = 0.$$

$$\widetilde{W_n} = \Lambda\langle\theta^i_j\rangle \otimes S_n[R^i_j]$$

Proposition 4.12. *The map*

$$c_i \mapsto c_i(R), \ R = (R^i_j)$$

has an extension to a map of complexes $W_n \to \widetilde{W_n}$*. Any such extension induces isomorphism in cohomology*

$$\mathrm{H}^*(W_n) \xrightarrow{\simeq} \mathrm{H}^*(\widetilde{W_n}).$$

For example if $n = 1$ we have

$$c_1 \mapsto c_1(R) = R_1^1,$$

$$u_1 \mapsto \theta_1^1.$$

Proof.

$$E_1^{0,2q-1}(\widetilde{W_n}) = \mathrm{H}^{2q-1}(\mathfrak{gl}_n(\mathbb{R}); \mathbb{R}) \ni u_j,$$

where u_j is a generator for $j = 1, \ldots, n$. Now each u_j has a representative $[w_j]$ such that

$$w_j \in F^0\widetilde{W_n}^{2q-1},\ dw_j = c_j \in F^{2q}\widetilde{W_n}^{2q}$$

thus giving a basic element of $\widetilde{W_n}$ in

$$E_1^{2q,0} \simeq S^q(R_j^i)_{inv}.$$

The basic elements of $\widehat{W_n}$ form an algebra isomorphic to $\mathbb{R}[c_1, \ldots, c_n]$.

The extesnsion is given by

$$u_j \mapsto w_j,$$

$$c_j \mapsto d\omega_j.$$

Filtering W_n by the ideals F^pW_n generated by polynomials of degree at least p in the c_i's one obtains a morphism of complexes compatible with filtrations, which induces isomorphism on the level of E_1. □

In the relative case $\mathfrak{o}_n \subset \mathfrak{gl}_n(\mathbb{R}) = \mathfrak{A}_n^{(0)}$ gives actions of $\mathfrak{o}_n$ on $\widetilde{W_n}$ and $C^\bullet(\mathfrak{A}_n)$. Passing to the subalgebras of $\mathfrak{o}_n$-basic elements, then restricting the filtrations one obtains isomorphisms

$$\mathrm{H}^*(WO_n) \simeq \mathrm{H}^*(\widetilde{W_n}, \mathfrak{o}_n) \simeq \mathrm{H}^*_{GF}(\mathfrak{A}_n, \mathfrak{o}_n),$$

where

$$WO_n = \Lambda\langle u_1, u_3, \ldots u_k\rangle \otimes S_n[c_1, \ldots, c_n],$$

$$du_{2j-1} = c_{2j},\ dc_j = 0.$$

Corollary 4.13. *Any class in* $\mathrm{H}^*(\mathfrak{A}_n)$ *(respectively* $\mathrm{H}^*(\mathfrak{A}_n, \mathfrak{o}_n)$*) has a representative which depends only on the second jet.*

Chapter 5

Characteristic maps and Gelfand-Fuks cohomology

5.1 Jet groups

Definition 5.1. *Let $x \in \mathbb{R}^n$ and let $f\colon U \to \mathbb{R}^n$ be a C^∞-function. Then $j^k_x(f)$ is an equivalence class with respect to*

$$f \sim_k g \quad \textit{iff} \quad \frac{\partial^{|\alpha|} f}{\partial x^\alpha}\Big|_x = \frac{\partial^{|\alpha|} g}{\partial x^\alpha}\Big|_x,\ \forall |\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant k.$$

Then

$$G_k(n) := \{ j^k_0(f) \bigm| f \textit{ local diffeomorphism of } \mathbb{R}^n,\ f(0)=0\}$$

is a Lie group under composition

$$j^k_0(f) \circ j^k_0(g) := j^k_0(f \circ g).$$

Identifying with polynomial representatives

$$j^k_0(f) \simeq \{ \sum_{1 \leqslant |\alpha| \leqslant k} a^j_\alpha x^\alpha \in \mathcal{P}^k_0[x_1, \ldots, x_n] \bigm| 1 \leqslant j \leqslant n\}$$

Then $j^k_0(f) \in G_k(n)$ means $a^j_\alpha \in \mathrm{GL}_n(\mathbb{R})$.

One has a sequence of projections

$$G_\infty(n) := \ldots \to G_{k+1}(n) \to G_k(n) \to \ldots \to G_1(n).$$

If $h = f \circ g$

$$h^i(x^1, \ldots, x^n) = f^i(g^1(x^1, \ldots, x^n), \ldots, g^n(x^1, \ldots, x^n))$$

$$c^i_k := \frac{\partial h^i}{\partial x^k}\Big|_0 = \sum_l \frac{\partial f^i}{\partial x^l}\Big|_0 \frac{\partial g^l}{\partial x^k}\Big|_0 = \sum_l a^i_l b^l_k.$$

$$c^i_{jk} := \frac{\partial^2 h^i}{\partial x^j \partial x^k}\Big|_0 = \sum_{l,s} \frac{\partial^2 f^i}{\partial x^s \partial x^l}\Big|_0 \frac{\partial g^s}{\partial x^j}\Big|_0 \frac{\partial g^l}{\partial x^k}\Big|_0 + \sum_l \frac{\partial f^i}{\partial x^l}\Big|_0 \frac{\partial^2 g^l}{\partial x^j \partial x^k}\Big|_0$$

so

$$c^i_{jk} = \sum_{l,s} a^i_{sl} b^s_j b^l_k + \sum_l a^i_l b^l_{jk}$$

etc. In particular $\ker(G_2(n) \to G_1(n))$ has multipllication

$$c^i_{jk} = a^i_{jk} + b^i_{jk}.$$

In general

$$N_k(n) := \ker(G_k(n) \to G_1(n))$$

is a vector space equipped with a polynomial multiplication which implies that $N_k(n)$ is a nilpotent Lie subgroup, and

$$G_k(n) = G_1(n) \ltimes N_k(n).$$

$$\mathfrak{g}_k(n) := \mathrm{Lie}(G_k(n)) \simeq \{j^k_0 X \mid X = \sum_i \frac{\partial}{\partial x^i},\ X(0) = 0\}$$

with the bracket

$$[j^k_0(X), j^k_0(Y)] = -j^k_0([X, Y]).$$

5.2 Jet bundles

Definition 5.2. *Let M^n be a C^∞-manifold. The* **jet bundle** *on M*

$$J^k(M) := \{j^k_0(f) \mid f\colon U \subset \mathbb{R}^n \to M \text{ local diffeomorphism at } 0 \in U\}.$$

It has a tautological C^∞-structure modelled on

$$J^k(\mathbb{R}^n) = \mathcal{P}_k(n) \simeq \text{ polynomial jets}$$

Again one has a sequence of natural projections

$$J^\infty(M) := \ldots \to J^{k+1}(M) \to J^k(M) \to \ldots \to J^1(M) \to M,$$

which are principal bundles with structure groups

$$G_\infty(n) := \ldots \to G_{k+1}(n) \to G_k(n) \to \ldots \to G_1(n).$$

$J^1(M) = F(M) \to M$ is a frame bundle with the structure group $\mathrm{GL}_n(\mathbb{R}) = G_1(n)$.
There is a natural (commuting with Diff_M) map

$$\mathfrak{A}_n \xrightarrow{\simeq} T_{j^\infty_0(\phi)} J^\infty(M)$$

For

$$X \in \mathfrak{A}_n,\ X = \sum_i f^i \frac{\partial}{\partial x^i}$$

and a 1-parameter family ψ_t of local diffeomorphism of $\mathbb{R}^n$ such that

$$\psi_t(0) = 0,\ \ \psi_0 = \mathrm{Id},\ \ X = j^\infty_0 \left(\frac{d\psi_t}{dt}\Big|_{t=0} \right),$$

we have a curve in a manifold of jets $j^\infty_0(\psi_t)$. For a local diffeomorphism $\phi\colon \mathbb{R}^q \to M^n$ we have a curve passing through ϕ

$$j^\infty_0 \left(\frac{d}{dt}(\phi \circ \psi_t)\Big|_{t=0} \right)$$

and

$$X = \frac{d}{dt} j_0^\infty(\psi_t)\big|_{t=0} = j_0^\infty\left(\frac{d\psi_t}{dt}\big|_{t=0}\right).$$

Let $u = j_0^\infty(\phi) \in J^\infty(M)$, and define

$$\widetilde{X_u} := j_0^\infty\left(\frac{d}{dt}\phi\circ\psi_t\big|_{t=0}\right) = \frac{d}{dt}(\phi\circ\psi_t)\big|_{t=0} \in T_u J^\infty(M), \;\; \phi\circ\psi_t\big|_{t=0} = \phi.$$

The map

$$\mathfrak{A}_n \to T_u J^\infty(M), \;\; X \mapsto \widetilde{X_u}$$

is natural i.e. it commutes with the action of the diffeomorphisms

$$\begin{array}{ccc} & & T_{j_0^\infty(\rho\circ\phi)} J^\infty(M) \\ & \overset{\simeq}{\nearrow} & \uparrow \rho^* \\ \mathfrak{A}_n & \xrightarrow{\simeq} & T_{j_0^\infty(\phi)} J^\infty(M) \end{array}$$

Proposition 5.3. *We have a natural isomorphism of differential graded algebras*

$$(C^\bullet(\mathfrak{A}_n), d) \xrightarrow{\simeq} (\Omega^\bullet(J^\infty(M))^{\mathrm{Diff}_M}, -d).$$

Proof. We take for $u = j_0^\infty(\phi)$

$$\widetilde{\omega_u}(\widetilde{X_u}^1, \ldots, \widetilde{X_u}^p) := \omega(X_1, \ldots, X^p).$$

$$[\widetilde{X}, \widetilde{Y}] := -\widetilde{[X, Y]}.$$

In particular if we set for a basis $\{\theta^i_\alpha\}$ of $\mathfrak{A}_n^*$

$$\widetilde{\theta}^i_\alpha(\widetilde{X_u}) = \frac{\partial^{|\alpha|} f^i}{\partial x^\alpha}\big|_{x=0} = (-1)^{|\alpha|}\theta^i_\alpha(X)$$

then they satisfy the same differential equations as θ^i_α. ∎

Example 5.4. In local coordinates $(v_1, \ldots, v_n)$ around $u = j_0^\infty(\phi)$

$$\left\{ v_i\big|_u, v^i_j := \frac{\partial(v^i\circ\phi)}{\partial x^j}\big|_u, v^i_{jk} := \frac{\partial^2(v^i\circ\phi)}{\partial x^j \partial x^k}\big|_u, \ldots, v^i_\alpha = \frac{\partial^{|\alpha|}(v^i\circ\phi)}{\partial x^\alpha}\big|_u \right\}$$

one has

$$dv^i_\alpha = \sum_{\beta+\gamma=\alpha} v^i_{\beta[k]}\widetilde{\theta}^k_\gamma, \;\; \beta[k] := (\beta_1, \ldots, \beta_k + 1, \ldots, \beta_n).$$

5.3 Characteristic map for foliation

Let $(M, \mathcal{F})$ be a manifold with foliation, which we can describe by a 1-cycle with values in Γ_q given by the following data

1. an open cover $M = \bigcup_\alpha U_\alpha$,
2. $\forall\alpha$ there is a submersion $f_\alpha \colon U_\alpha \to V_\alpha \in \mathbb{R}^q$,

3. $\forall x \in U_\alpha \cap U_\beta$ there is a local diffeomorphism $g_{\alpha\beta}\colon V_\alpha \to V_\beta$ (neighbourhoods of $f_\alpha(x)$ and $f_\alpha(x)$ rspectively) such that $f_\beta = g_{\beta\alpha} \circ f_\alpha$ near x.

Then

$$f_\alpha^*(J^\infty(V_\alpha)) \to U_\alpha, \text{ and } f_\beta^*(J^\infty(V_\beta)) \to U_\beta$$

can be identified over $U_\alpha \cap U_\beta$ via $j_0^\infty(g_{\beta\alpha})$, giving the principal $G^k(q)$-bundles over M:

$$J^\infty(\mathcal{F}) := \ldots \to J^{k+1}(\mathcal{F}) \to J^k(\mathcal{F}) \to \ldots \to J^2(\mathcal{F}) \to J^1(\mathcal{F}) \to M.$$

This are jet bundles of "transverse local diffeomorphisms". In particular $J^1(\mathcal{F})$ is a principal $\mathrm{GL}_q(\mathbb{R})$-bundle associated to the transverse bundle $Q(\mathcal{F}) = TM/\mathcal{F}$ - bundle of transverse frames.

The forms θ^i_κ on $J^\infty(V_\alpha)$ are invariant under Diff hence they also define forms on $J^\infty(\mathcal{F})$. They are the "canonical forms" on $J^\infty(\mathcal{F})$.

The characteristic homomorphisms

$$\chi_{GF}\colon C^\bullet(\mathfrak{A}_q) \to \Omega^\bullet(J^\infty(\mathcal{F}))$$

is defined by sending ω to the lift to M of the Diff-invariant forms $\widetilde{\omega}_\alpha$ on V_α. It is a homomorphism of DGA's inducing

$$\chi^*_{GF}\colon\ \mathrm{H}^*_{GF}(\mathfrak{A}_q) \to \mathrm{H}^*(J^\infty(\mathcal{F})) \simeq \mathrm{H}^*(J^1(\mathcal{F})).$$

Remark 5.5 *(Bott's vanishing theorem revisited).* Any E-flat (Bott) connection (def. (2.5)) $\nabla^\flat$ on Q is given by a $\mathfrak{gl}_n(\mathbb{R})$-valued form on $J^1(\mathcal{F})$ which is of the form $\omega^i_j = s^*(\widetilde{\theta}^i_j)$ for some $\mathrm{GL}_n(\mathbb{R})$ -equivariant section $s\colon J^1(\mathcal{F}) \to J^2(\mathcal{F})$. Then its curvature form

$$\Omega^i_j = s^*(R^i_j) \implies \Omega^i_j \wedge \omega^j = s^*(R^i_j \wedge \theta^j) = 0$$

hence

$$\Omega^{i_1}_{j_1} \wedge \ldots \wedge \Omega^{i_p}_{j_p} = 0,\ \forall p > q.$$

Assume the normal bundle $Q = Q(\mathcal{F})$ is trivializable and choose a global section $s\colon M \to \mathcal{F}$. Then the diagram

$$\begin{array}{ccccc} s^*\circ\chi^*_{GF}\colon \mathrm{H}^*_{GF}(\mathfrak{A}_q) & \longrightarrow & \mathrm{H}^*(M) & \xrightarrow[\simeq]{\mathrm{pr}^*} & \mathrm{H}^*(J^1(\mathcal{F})) \\ \uparrow{\scriptstyle\simeq} & \nearrow{\scriptstyle\mu_E^*} & & & \\ \mathrm{H}^*(W_q) & & & & \end{array}$$

is commutative.

Passing to the relative subcomplex one gets

$$\chi^{rel}_{GF}\colon C^\bullet(\mathfrak{A}_n, O(n)) \to \Omega^\bullet(J^\infty/O(n))$$

which induces

$$\chi^{rel}_{GF}\colon\ \mathrm{H}^*(\mathfrak{A}_n, O(n)) \to \mathrm{H}^*(J^1(\mathcal{F})/O(n)) \xrightarrow{\simeq} \mathrm{H}^*(M).$$

The isomorphism

$$\sigma^*\colon\ \mathrm{H}^*(J^1(\mathcal{F})/O(n)) \to \mathrm{H}^*(M)$$

is implemented by a metric on Q (i.e. a section $\sigma\colon M \to J^1(\mathcal{F})/O(n)$). Then the diagram

$$\begin{array}{ccc} \mathrm{H}^*(\mathfrak{A}_n, O(n)) & \xrightarrow{\chi_{GF}^{rel*}} & \mathrm{H}^*(M) \\ & \nwarrow \qquad \nearrow {\scriptstyle \chi_E^*} & \\ & \mathrm{H}^*(WO_n) & \end{array}$$

is again commutative.

Chapter 6

Index theory and noncommutative geometry

6.1 Classical index theorems

Let (M, g) be a Riemannian manifold, g-metric. Index theorems describe properties of geometric elliptic operators in terms of topological characteristic classes.

For a selfadjoint elliptic operator $D = D^*$

$$\mathrm{Index}(D) := \dim\ker D - \dim\operatorname{coker} D \in \mathbb{Z}$$

We give a few examples of index theorems.

Example 6.1. Take a de Rham complex $\Omega^\bullet(M)$ with

$$d\colon \Omega^i(M) \to \Omega^{i+1}(M)$$

and its adjoint

$$d^*\colon \Omega^i(M) \to \Omega^{i-1}(M).$$

One has even/odd grading on forms ($\gamma = (-1)^{\deg}$), and the operator

$$d + d^*\colon \Omega^{ev} \to \Omega^{odd}$$

is selfadjoint elliptic operator. Furthermore

$$\mathrm{Index}(d + d^*)^{ev} = \dim\ker(d + d^*)^{ev} - \dim\operatorname{coker}(d + d^*)^{ev}$$

and

$$\ker(d + d^*) = \mathrm{H}^*_{dR}(M; \mathbb{R}),$$

$$\ker(d + d^*)^{ev} = \mathrm{H}^{ev}_{dR}(M; \mathbb{R}), \ \ \operatorname{coker}(d + d^*)^{odd} = \mathrm{H}^{odd}_{dR}(M; \mathbb{R}).$$

This means

$$\mathrm{Index}(d + d^*) = \dim \mathrm{H}^{ev}(M; \mathbb{R}) - \dim \mathrm{H}^{odd}(M; \mathbb{R}) = \chi(M)$$

- the Euler characteristic of a manifold M.

Theorem 6.2 *(Gauss-Bonnet).*

$$\chi(M) = \mathrm{Index}(d + d^*)^{ev} = \int_M \mathrm{Pf}(R),$$

where $Pf(M)$ is a Pffafian i.e. the square root of the determinant, and R - a curvature.

This theorem gives topological constraints on Gaussian curvature, for if $n = 2$ one has $\mathrm{Pf}(R) = K$. The right hand side depends on the metric, while on the left we have topological invariant.

Example 6.3. In the example above lets take different grading. Assume that $\dim M = 4n$. Take a Hodge star operator

$$*\colon \Omega^k(M) \to \Omega^{4n-k}.$$

One has $*^2 = (-1)^{k(4n-k)}$ so it gives rise to another grading γ on $\Omega^\bullet(M)$. It splits the complex into $\Omega^-(M)$ and $\Omega^+(M)$ (negative and positive eigenspaces). Furthermore

$$\mathrm{Index}(d + d^*)^+ = \dim \mathrm{H}^{2n}(M)^+ - \dim \mathrm{H}^{2n}(M) = \sigma(M)$$

- the signature of M i.e. a signature of bilinear form

$$\mathrm{H}^{2n}(M) \times \mathrm{H}^{2n}(M) \to \mathbb{R}, \;\; (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta.$$

On the other side

Theorem 6.4 *(Hirzebruch signature thm.).*

$$\sigma(M) = \mathrm{Index}(d + d^*) = \int_M L(R), \;\; L(R) := (\det)^{\frac{1}{2}} \left(\frac{\frac{R}{2}}{\tanh \frac{R}{2}} \right)$$

as a formal series. $L(R)$ is a L-genus of a manifold.

$L(R)$ is a combination of Pontryagin classes which depends on a metric structure of a manifold.

Example 6.5. Let E be a holomorphic Hermitian bundle on a manifold M. One has an operator $\bar\partial_E \oplus \bar\partial_E^*$ on $\Omega^{0,\bullet} \otimes \mathcal{S}(E)$. Its index

$$\mathrm{Index}(\bar\partial_E \oplus \bar\partial_E^*) = \chi(E)$$

- the Euler characteristic of a bundle E. On the other hand

Theorem 6.6 *(Riemann-Roch-Hirzebruch).*

$$\chi(E) = \mathrm{Index}(\bar\partial_E \oplus \bar\partial_E^*) = \int_M \mathrm{Td}(M)\,\mathrm{ch}(E),$$

where the Todd class of M and Chern character of E are given by

$$\mathrm{Td}(M) = \det \frac{R^{hol}}{e^{R^{hol}} - 1}, \;\; \mathrm{ch}(E) = \mathrm{Tr}(e^{F_E}).$$

Example 6.7. The most general example one has for Dirac operator $D\!\!\!/$. One has a grading $D\!\!\!/^+$, $D\!\!\!/^-$ from Spin-bundle.

$$\mathrm{Index}\, D\!\!\!/ = \dim\ker D\!\!\!/ - \dim\mathrm{coker}\, D\!\!\!/ = S(M)$$

- the spinor number of a manifold M. On the other side

Theorem 6.8 *(Atiyah-Singer).*

$$S(M) = \mathrm{Index}\, D\!\!\!/ = \int_M \widehat{A}(R), \;\; \widehat{A}(R) := (\det)^{\frac{1}{2}} \left(\frac{\frac{R}{2}}{\sinh \frac{R}{2}} \right)$$

$\widehat{A}(R)$ is another combination of Pontryagin classes. Together with Lichnerowicz theorem it gives constraints on scalar curvature.

Summarizing

Elliptic operator and grading	Analitic index	Index formula (characteristic classes)	Corollaries

6.2 General formulation and proto-index formula

Let A be a $C*$-algebra and $\mathfrak{A}$ its dense subalgebra such that if $a \in \mathfrak{A}$ has an inverse $a^{-1} \in A$, then $a^{-1} \in \mathfrak{A}$

Example 6.9. M - closed manifold, $A = C(M)$, $\mathfrak{A} = C^\infty(M)$. Then

$$\mathrm{K}^*(M) = \mathrm{K}_*(C(M)) = \mathrm{K}_*(C^\infty(M)),$$

(via Serre-Swan theorem) where the right hand side has algebraic definition (purely for $* =$ *even* and almost for $* =$ *odd*).

In general

$$\mathrm{K}_0(\mathfrak{A}) := \mathrm{Idemp}(M_\infty(\mathfrak{A}))/\sim \simeq \pi_1(\mathrm{GL}_\infty(\mathfrak{A})),$$

where $\sim$ is some equivalence relation,

$$\mathrm{K}_1(\mathfrak{A}) := \mathrm{GL}_\infty(\mathfrak{A})/\,\mathrm{GL}_\infty(\mathfrak{A})^0 \simeq \pi_0(\mathrm{GL}_\infty(\mathfrak{A})),$$

where $\mathrm{GL}_\infty(\mathfrak{A})^0$ is a group of connected components. For the definition of $\mathrm{K}_1(\mathfrak{A})$ we need a topology on $\mathfrak{A}$. We can replace $\mathrm{GL}_\infty(\mathfrak{A})$ by $U_\infty(\mathfrak{A})$ (unitary matrices). From Bott periodicity $\mathrm{K}_2(\mathfrak{A}) = \mathrm{K}_0(A)$ and so on.

What is the dual (homology) theory ? K-homology.

Assume $A \subset B(\mathcal{H})$ (bounded operators on Hilbert space $\mathcal{H}$). Let $F = F^* \in A$, Fredholm operator, such that

$$[F, A] \subset \mathcal{K}(\mathcal{H}), \text{ (compact operators)},$$

and moreover

$$[F, \mathfrak{A}] \subset \mathcal{L}^p(\mathcal{H}), \text{ (Schatten class)}$$

for some $p \geqslant 1$. The triple $(\mathfrak{A}, \mathcal{H}, F)$ is a **p-summable Fredholm module**. Together with grading γ such that

$$\gamma^2 = \mathrm{Id},\ \gamma = \gamma^*,\ \gamma a = a\gamma\ \forall\ a \in \mathfrak{A},$$

$$\gamma F + F\gamma = 0,$$

the quadruple $(\mathfrak{A}, \mathcal{H}, \gamma, F)$ is a K-cycle. The Hilbert space $\mathcal{H}$ decomposes into positive and negative eigenspaces of γ

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$$

and there is a decomposition of F

$$F = \begin{pmatrix} 0 & F^+ \\ F^- & 0 \end{pmatrix}.$$

Lemma 6.10. *Let F be bounded selfadjoint involution on $\mathcal{H}$ (i.e. $F^2 = \mathrm{Id}$). Then*

1. *If $e^2 = e \in \mathfrak{A}$ then*

$$F_e := eFe$$

is Fredholm operator.

2. *If* $g \in \mathrm{GL}_1(\mathfrak{A})$ *and* $P = \frac{1+F}{2}$ *then*

$$F_g := PgP$$

is Fredholm operator.

Proof. Ad. 1
$$F_e^2 = eFeFe = e([F,e] + eF)Fe$$
which is a sum of e and compact operator on $e\mathcal{H}e$.

Ad. 2
$$F_g F_{g^{-1}} = PgPg^{-1}P = Pg([P,g^{-1}] + g^{-1}P)P$$
which is a sum of P and compact operator on $P\mathcal{H}P$.

□

If $e^2 = e \in M_N(\mathfrak{A}) = \mathfrak{A} \otimes M_N(\mathbb{C})$ then we can form

$$\mathcal{H}_N := \mathcal{H} \otimes \mathbb{C}^N, \quad F_N := F \otimes \mathrm{Id}\,.$$

For an idempotent e, assignment

$$(F,e) \mapsto \mathrm{Index}(F_e^+) \in \mathbb{Z}$$

extends to a pairing
$$\mathrm{K}^0(\mathfrak{A}) \times \mathrm{K}_0(\mathfrak{A}) \to \mathbb{Z}.$$
Similarly for $g \in \mathrm{GL}_1(\mathfrak{A})$, assignment

$$(P,g) = \left(\frac{1+F}{2}, g\right) \mapsto \mathrm{Index}(F_g) \in \mathbb{Z}$$

extends to a pairing
$$\mathrm{K}^1(\mathfrak{A}) \times \mathrm{K}_1(\mathfrak{A}) \to \mathbb{Z}.$$

Lemma 6.11 (Well known). *Let* P, Q *be bounded operators on a Hilbert space* $\mathcal{H}$*, such that*

$$\mathrm{Id} - QP, \mathrm{Id} - PQ \in \mathcal{L}^p.$$

Then P, Q *are Fredholm operatos and*

$$\mathrm{Index}(P) = \mathrm{Tr}((\mathrm{Id} - QP)^n) - \mathrm{Tr}((\mathrm{Id} - PQ)^n), \ \forall\, n \geqslant p.$$

Proposition 6.12. *Assume* $[F, \mathfrak{A}] \in \mathcal{L}^p$ *(that is* $(\mathfrak{A}, \mathcal{H}, F)$ *is* p*-summable Fredholm module). Then*

1. *In the graded case, that is given* $\gamma\colon \mathcal{H} \to \mathcal{H}$*, one has for all projections* e

$$\mathrm{Index}(F_e^+) = (-1)^m \,\mathrm{Tr}(\gamma e [F,e]^{2m}), \ \forall\, 2m \geqslant p.$$

2. *In the ungraded case one has for all* $g \in \mathrm{GL}_1(\mathfrak{A})$

$$\mathrm{Index}(F_g) = \frac{1}{2^{2m+1}} \,\mathrm{Tr}(g[F, g^{-1}])^{2m+1}, \ \forall\, 2m \geqslant p.$$

Proof. In the graded case

$$\mathrm{Index}(F_e^+) = \mathrm{Tr}(\gamma P_{\ker F_e}) = \mathrm{Tr}(\gamma(e - F_e^2)^m) = \mathrm{Tr}(\gamma(e - eFeFe)^m)$$

for $2m = n \geqslant p$. Now as above

$$e - eFeFe = -e[F,e]Fe = -e[F,e]([F,e] + eF) = -e[F,e][F,e] - \underbrace{e[F,e]e}_{=0} F =$$

$$= -e[F,e]^2 = [F,e]^2 e$$

since

$$[F,e] = [F,e^2] = [F,e]e + e[F,e].$$

Thus

$$\mathrm{Tr}(\gamma(e - eFeFe)^m) = (-1)^m \,\mathrm{Tr}(\gamma(e[F,e]^2)^m) = (-1)^m \,\mathrm{Tr}(\gamma e([F,e])^{2m}).$$

In the ungraded case one has

$$\mathrm{Index}(F_g) = \mathrm{Tr}((P - Pg^{-1}PgP)^m) - \mathrm{Tr}((P - PgPg^{-1}P)^m)$$

for m sufficiently large. Furthermore

$$P - Pg^{-1}PgP = P + P([P,g^{-1}] - Pg^{-1})gP =$$

$$= P[P,g^{-1}]gP = -P[P,g^{-1}]([P,g] - Pg) =$$

$$= -P[P,g^{-1}][P,g] + \underbrace{P[P,g^{-1}]P}_{=0} g$$

because

$$P^2 = P \implies [g^{-1},P]P + P[g^{-1},P] = [g^{-1},P] \implies P[P,g^{-1}]P = 0.$$

Hence

$$\mathrm{Tr}((P - Pg^{-1}PgP)^m) = (-1)^m \,\mathrm{Tr}(P([P,g^{-1}][P,g])^m).$$

Writig again

$$[P,g^{-1}] = P[P,g^{-1}] + [P,g^{-1}]P,$$

$$[P,g] = P[P,g] + [P,g]P$$

one has

$$P[P,g^{-1}][P,g] = P[P,g^{-1}][P,g]P = [P,g^{-1}][P,g]P.$$

Therefore

$$\mathrm{Tr}((P - Pg^{-1}PgP)^m) = (-1)^m \,\mathrm{Tr}(P([P,g^{-1}][P,g])^m) =$$

$$= (-1)^m \,\mathrm{Tr}\left(\frac{1+F}{2}\left(\frac{1}{2}[F,g^{-1}]\frac{1}{2}[F,g]\right)^m\right) =$$

$$= \frac{(-1)^m}{2^{2m+1}}\left(\mathrm{Tr}(([F,g^{-1}][F,g])^m) + \mathrm{Tr}(F([F,g^{-1}][F,g])^m)\right).$$

Changing g to g^{-1} one gets

$$\mathrm{Tr}((P - PgPg^{-1}P)^m) = \frac{(-1)^m}{2^{2m+1}}\left(\mathrm{Tr}(([F,g][F,g^{-1}])^m) + \mathrm{Tr}(F([F,g][F,g^{-1}])^m)\right).$$

Noting that

$$[F,g^{-1}][F,g] = (-g^{-1}[F,g]g^{-1})(-g[F,g^{-1}]g) = g[F,g][F,g^{-1}]g$$

one has

$$\mathrm{Tr}(([F,g^{-1}][F,g])^m) = \mathrm{Tr}(([F,g][F,g^{-1}])^m).$$

Now

$$([F,g^{-1}][F,g])^m = (-g^{-1}[F,g^{-1}]g^{-1}[F,g])^m = (-1)^m(g^{-1}[F,g])^{2m},$$

whence

$$\mathrm{Index}(F_g) = \frac{1}{2^{2m+1}}(\mathrm{Tr}(F(g^{-1}[F,g])^{2m}) - \mathrm{Tr}(F(g[F,g^{-1}])^{2m})).$$

The second term can be written as

$$\mathrm{Tr}(F(g[F,g^{-1}])^{2m}) = \mathrm{Tr}(F([F,g]g^{-1})^{2m}) =$$

$$\mathrm{Tr}(Fg(g^{-1}[F,g]g^{-1}g)^{2m}g^{-1}) = \mathrm{Tr}(g^{-1}Fg(g^{-1}[F,g])^{2m}).$$

So the difference gives

$$\mathrm{Index}(F_g) = \frac{1}{2^{2m+1}}\,\mathrm{Tr}((F - g^{-1}Fg)(g^{-1}[F,g])^{2m}) =$$

$$= \frac{1}{2^{2m+1}}\,\mathrm{Tr}(g^{-1}[g,F](g^{-1}[F,g])^{2m}) = \frac{1}{2^{2m+1}}\,\mathrm{Tr}((g^{-1}[F,g])^{2m+1}) =$$

$$\frac{1}{2^{2m+1}}\,\mathrm{Tr}((g[F,g^{-1}])^{2m+1}).$$

□

6.3 Multilinear reformulation: cyclic cohomology (Connes)

Observe that if $T \in \mathcal{L}^1$ then

$$\mathrm{Tr}(\gamma T) = \frac{1}{2}\,\mathrm{Tr}(\gamma F[F,T]).$$

Indeed

$$\mathrm{Tr}(\gamma F[F,T]) = \mathrm{Tr}(\gamma(T - FTF)) = \mathrm{Tr}(\gamma T) + \mathrm{Tr}(\gamma T)$$

since $F\gamma + \gamma F = 0$.

Both formulas in proposition (6.12) can be obtained from multilinear forms $\tau \in \mathrm{Hom}(\mathfrak{A}^{\otimes n+1}, \mathbb{C})$.

$$\tau_F(a^0, a^1, \ldots, a^n) = \begin{cases} \mathrm{Tr}(\gamma F[F,a^0][F,a^1]\ldots[F,a^n]) & n \text{ even } > p-1, \\ \mathrm{Tr}(F[F,a^0][F,a^1]\ldots[F,a^n]) & n \text{ odd } > p-1. \end{cases}$$

The first comes from (using graded commutators)

$$\mathrm{Tr}(\gamma F[F,a^0[F,a^1]\ldots[F,a^n]]) = \mathrm{Tr}(\gamma F[F,a^0][F,a^1]\ldots[F,a^n]) +$$

$$+\sum_{i=1}^{n}\mathrm{Tr}(\gamma F a^0[F,a^1]\ldots[F,[F,a^i]]\ldots[F,a^n]),$$

where the terms in the sum are 0 because

$$[F,[F,a]] = F[F,a] + [F,a]F = a - FaF + FaF - a = 0.$$

For anti-commutation reasons, the first expression vanishes for n odd, while the second expression vanishes for n even.

Element $\phi \in \mathrm{Hom}(\mathfrak{A}^{\otimes n+1}, \mathbb{C})$ ic cyclic if

$$\phi(a^n, a^0, \dots, a^{n-1}) = (-1)^n \phi(a^0, a^1, \dots, a^n)$$

i. e. $\lambda_n \phi = \mathrm{Id}$ for cyclic operator $\lambda_n^{n+1} = \mathrm{Id}$. One has

$$b\tau_F(a^0, a^1, \dots, a^{n+1}) = \sum_{i=0}^{n} \tau_F(a^0, \dots, a^i a^{i+1}, \dots, a^{n+1}) +$$

$$+(-1)^{n+1}\tau_F(a^{n+1}a^0, a^1, \dots, a^n) =$$

$$= \sum_{i=1}^{n} (-1)^i \operatorname{Tr}(F[F, a^0] \dots [F, a^i a^{i+1}] \dots [F, a^n]) +$$

$$+(-1)^{n+1} \operatorname{Tr}(F[F, a^{n+1}a^0][F, a^1] \dots [F, a^n]).$$

Now

$$[F, a^i a^{i+1}] = [F, a^i]a^{i+1} + a^i[F, a^{i+1}].$$

Because of the alternating signs, terms cancel pairwise if $n+1$ is even

$$\operatorname{Tr}(F[F, a^0]a^1[F, a^2] \dots [F, a^{n+1}]) + \operatorname{Tr}(Fa^0[F, a^1][F, a^2] \dots [F, a^{n+1}])$$

$$- \operatorname{Tr}(F[F, a^0][F, a^1]a^2 \dots [F, a^{n+1}]) - \operatorname{Tr}(F[F, a^0]a^1[F, a^2] \dots [F, a^{n+1}]) + \dots$$

$$\dots + (-1)^{n+1} \operatorname{Tr}(F[F, a^{n+1}]a^0[F, a^1] \dots [F, a^{n+1}]) + (-1)^{n+1} \operatorname{Tr}(Fa^{n+1}[F, a^0][F, a^1] \dots [F, a^{n+1}]).$$

Hence for odd n

$$b\tau_F = 0.$$

For even n

$$\operatorname{Tr}(\gamma F[F, a^n][F, a^1] \dots [F, a^{n-1}]) = \operatorname{Tr}(F[F, a^n][F, a^0] \dots [F, a^{n-1}]) =$$

$$- \operatorname{Tr}(F[F, a^0] \dots [F, a^n]).$$

This leads to the definition of **cyclic cohomology**, a homology of complex

$$(C^\bullet_\lambda(\mathfrak{A}), b), \quad C^n_\lambda(\mathfrak{A}) = \mathrm{Hom}_{cont}(\mathfrak{A}^{\otimes n+1}, \mathbb{C})$$

for locally convex algebra $\mathfrak{A}$ (with continuous multiplication).

The fact that $n \mapsto n+2$ leaves formulas in proposition (6.12) unchanged is related to the periodicity operator

$$S\colon\ \mathrm{HC}^n_\lambda(\mathfrak{A}) \mapsto \mathrm{HC}^{n+2}_\lambda(\mathfrak{A})$$

which in turn is an arrow in Connes long exact sequence

$$\dots \xrightarrow{S} \mathrm{HC}^n_\lambda(\mathfrak{A}) \xrightarrow{I} \mathrm{HH}^n(\mathfrak{A}) \xrightarrow{B} \mathrm{HC}^{n-1}_\lambda(\mathfrak{A}) \xrightarrow{S} \mathrm{HC}^{n+1}(\mathfrak{A}) \xrightarrow{I} \dots$$

For $\mathfrak{A} = C^\infty(M)$, $\partial M = 0$

$$\tau(f^0, f^1, \dots, f^n) = \int_M f^0 df^1 \wedge \dots \wedge df^n$$

From Leibniz rule and Stokes theorem

$$b\tau = 0, \;\; \lambda(\tau) = \tau.$$

If $\omega \in \Omega^{n-k}(M)$ then

$$\tau_\omega(f^0, \dots, f^k) := \int_M f^0 df^1 \wedge \dots \wedge df^k \wedge \omega, \;\; d\omega = 0.$$

If C-k-current

$$\tau_C(f^0, \dots, f^k) = \langle C, f^0 df^1 \wedge \dots \wedge df^k \rangle, \;\; dC = 0.$$

Theorem 6.13 (Connes).

$$\begin{array}{ccccccc}
\mathrm{HC}^q_\lambda(\mathfrak{A}) & \simeq & \ker d_q^+ & \oplus \mathrm{H}^{dR}_{q-2}(M;\mathbb{C}) & \oplus \mathrm{H}^{dR}_{q-4}(M;\mathbb{C}) \oplus & & \dots \\
\Big\downarrow S & & & \searrow^{class} & \searrow^{\cong} & \searrow^{\cong} & \\
\mathrm{HC}^{q+2}_\lambda(\mathfrak{A}) & \simeq & \ker d_{q+2}^+ & \oplus \, \mathrm{H}^{dR}_{q}(M;\mathbb{C}) & \oplus \mathrm{H}^{dR}_{q-2}(M;\mathbb{C}) \oplus & & \dots
\end{array}$$

where the inclusion $\ker d_q^+ \hookrightarrow \mathrm{HC}^q_\lambda(\mathfrak{A})$ *is*

$$C \mapsto \phi_C(f^0, f^1, \dots, f^q) = \langle C, f^0 df^1 \wedge \dots \wedge df^q \rangle.$$

Compatibility considerations lead to the following normalization for the **Connes-Chern character** of a K-cycle F over $\mathfrak{A}$ of Schatten dimension p.

- For n odd $> p - 1$

$$\tau_n(a^0, a^1, \dots, a^n) = (-1)^{\frac{n-1}{2}} \frac{n}{2} \left(\frac{n}{2} - 1\right) \dots \frac{1}{2} \operatorname{Tr}(F[f, a^0][F, a^1] \dots [F, a^n]),$$
$$S\tau_n = \tau_{n+2}$$

- For n even $> p - 1$

$$\tau_n(a^0, a^1, \dots, a^n) = \left(\frac{n}{2}\right)! \frac{1}{2} \operatorname{Tr}(\gamma F[f, a^0][F, a^1] \dots [F, a^n]),$$
$$S\tau_n = \tau_{n+2}$$

Homological Chern character is a homomorphism

$$\mathrm{ch}_*\colon \; \mathrm{K}_*(M) \to \mathrm{H}^{dR}_*(M;\mathbb{C})$$

It is a special case of the Connes-Chern character for an algebra

$$\mathrm{ch}^* \, \mathrm{K}^*(\mathfrak{A}) \to \mathrm{HP}^*(\mathfrak{A})$$

if one takes $\mathfrak{A} = C^\infty(M)$. For a cocycle $(\mathfrak{A}, \mathcal{H}, F)$ representing an element in K-homology one has

$$\mathrm{ch}^*(\mathfrak{A}, \mathcal{H}, F) := [\phi^n],$$

where ϕ^n is the following cocycle

$$\phi^n(a^0, a^1, \dots, a^n) = \operatorname{Tr}(\gamma a^0[F, a^0] \dots [F, a^n])$$

for n even.

$$S[\phi^n] = [\phi^{n+2}]$$

For a Dirac operator D we can take $F = D|D|^{-1}$ and then

$$\mathrm{ch}_*(D) = \widehat{A}(M) = (\det)^{\frac{1}{2}} \left(\frac{\frac{R}{2}}{\sinh \frac{R}{2}} \right)$$

If γ is a gradation on $\mathcal{H}$ i.e.

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$$

then

$$\mathrm{Index}(D^+) = \mathrm{Tr}(\gamma e^{-tD^2}), \ t > 0$$

$$D^2 = \begin{pmatrix} D^- D^+ & 0 \\ 0 & D^+ D^- \end{pmatrix}.$$

For $t \to 0^+$ function $\mathrm{Tr}(\gamma e^{-tD^2})$ has an expansion

$$c_0 + c_1 t + c_2 t^2 + \ldots,$$

where

$$c_0 = \int_M \omega_\delta(D)$$

and $\omega_\delta(D)$ is called the local index formula.

6.4 Connes cyclic cohomology

$\mathrm{HC}^*(\mathfrak{A})$ is defined as the cohomology of a complex $(C_\lambda(\mathfrak{A}), b)$. A **cycle** representing an element in $\mathrm{HC}^*(\mathfrak{A})$ is a triple

$$(\Omega, d, \int),$$

where (Ω, d) is a differential graded algebra

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n, \quad d^2 = 0, \quad \text{(finite length)},$$

and $\int$ is a **closed graded trace** $\int \Omega^n \to \mathbb{C}$ i.e.

$$\int \omega_1 \omega_2 = (-1)^{|\omega_1||\omega_2|} \int \omega_2 \omega_1 \ \text{(graded trace)},$$

$$\int d\omega = 0 \ \text{(closed)}.$$

Using homomorphism $\rho \colon \mathfrak{A} \to \Omega^0$ we can write a character of $(\Omega, d, \int)$

$$\tau(a^0, a^1, \ldots, a^n) = \int a^0 da^1 \ldots da^n.$$

It is a cyclic cocycle.

Define a **chain** as a triple $(\Omega, \partial\Omega, \int)$, where $\partial\Omega \subset \Omega$, $\dim \Omega = n$, $\dim \partial\Omega = n-1$, and d preserves $\partial\Omega$. There is given a surjective homomorphism $r\colon \Omega \to \partial\Omega$ of degree 0 (restriction to the boundary) and

$$\int d\omega = 0, \ \ \forall \ \omega \quad \text{such that } r(\omega) = 0.$$

A **boundary** of such chain is a cycle $(\partial\Omega, d, \int')$, where for $\omega' \in \partial\Omega^{n-1}$

$$\int' \omega' := \int d\omega, \quad \text{for } r(\omega) = \omega'.$$

Two cycles Ω_1, Ω_2 are **cobordant**, $\Omega_1 \sim \Omega_2$ if and only if there exists a chain $(\Omega, \partial\Omega, \int)$ such that

$$\partial\Omega = \Omega_1 \oplus \widetilde{\Omega_2}$$

where $(\widetilde{\Omega_2}, d, \widetilde{\int})$ is a cycle in which $\widetilde{\int}\omega = -\int \omega$.

Theorem 6.14.

$$\Omega_1 \sim \Omega_2 \ \textit{iff.} \ \tau_2 - \tau_1 = B_0\phi \in \operatorname{im} B_0$$

where the operator B_0 is defined as follows.

$$B_0\phi(a^0, a^1, \ldots, a^n) = \phi(1, a^0, \ldots, a^n) - (-1)^{n+1}\phi(a^0, \ldots, a^n, 1).$$

The operator B is then equal to AB_0, where A is the cyclic antisymmetrization

$$(A\phi)(a^0, a^1, \ldots, a^n) := \sum_{i=0}^{n} (-1)^{ni}\phi(a^i, a^{i+1}, \ldots, a^{i-1}).$$

The Connes exact sequence

$$\ldots \xrightarrow{B} \mathrm{HC}_\lambda^{n-2}(\mathfrak{A}) \xrightarrow{S} \mathrm{HC}_\lambda^{n}(\mathfrak{A}) \xrightarrow{I} \mathrm{H}^n(\mathfrak{A}) \xrightarrow{B} \mathrm{HC}_\lambda^{n-1}(\mathfrak{A}) \xrightarrow{S}$$

starts with $\mathrm{HC}_\lambda^0(\mathfrak{A}) = \mathrm{H}^0(\mathfrak{A})$. Thus if there is an algebra homomorphism $\mathfrak{A} \to \mathfrak{A}'$ which induces isomorphism on Hochshild cohomology, then it also induces isomorphism on cyclic cohomology.

We can form a bicomplex $(C^{n,m}, b, B)$ with $b^2 = 0$, $B^2 = 0$, $bB + Bb = 0$, and $C^{n,m} = C^{n-m}(\mathfrak{A}) = \mathfrak{A}^{\otimes n-m+1}$. The homology of the total complex is then cyclic cohomology.

6.5 An alternate route, via the Families Index Theorem

Set up: $(\mathfrak{A}, \mathcal{H}, D)$, $D = D^*$ unbounded with

$$[D, \mathfrak{A}] \subset \mathcal{L}(\mathcal{H}), \ \ (1 + D^2) \in \mathcal{L}^p$$

In fact we shall assume that D is invertible with $D^{-1} \in \mathcal{L}^p$. The bounded version of this K-cycle is given by $(\mathfrak{A}, \mathcal{H}, F)$, where $F = D|D|^{-1}$ is a phase.

On $\mathfrak{A}$ one has a norm

$$|||a||| := ||a|| + ||[D, a]||, \text{ for } a \in \mathfrak{A}.$$

Let $\mathcal{V} = \mathcal{V}(\mathfrak{A})$ be the span of "vector potentials", that is

$$\mathcal{V} := \left\{ A = \sum_i a_i[D, b_i] \;\middle|\; a_i, b_i \in \mathfrak{A}, \ A = A^* \right\}.$$

Let $\mathcal{U} = \mathcal{U}(\mathfrak{A})$ be the gauge group, that is

$$\mathcal{U} = \mathcal{U}(\mathfrak{A}) := \left\{ u \in \mathrm{GL}_1(\mathfrak{A}) \,\middle|\, u^*u = uu^* = 1 \right\},$$

acting on $\mathcal{V}$ by (affine action)

$$u \cdot A := u[D, u^*] + uAu^* = u(D + A)u^* - D.$$

Denoting $D_A := D + A$ one has

$$D_{u \cdot A} = uD_A u^*.$$

Fact 6.15. *D_A has the same dimension as D and $D_A^* = D_A$. Also* $\ker D_A = \ker(\mathrm{Id} + D^{-1}A)$, *hence is finite dimensional.*

Let

$$\mathcal{V}_{inj} := \left\{ A \in \mathcal{V} \,\middle|\, D_A \text{ injective } \right\} \subset \mathcal{V}$$

It is an open subset with respect to $|||\cdot|||$. For $A \in \mathcal{V}_{inj}$ operator D_A is invertible with

$$D_A^{-1} = (1 + D^{-1}A)^{-1}D^{-1} \in \mathcal{L}^p.$$

Graded trivial vector bundle over $\mathcal{V}_{inj}$

$$\widetilde{\mathcal{H}}^{\pm} := \mathcal{V}_{inj} \times \mathcal{H}^{\pm}.$$

Superconnection is an operator $d + \widetilde{D}$, where

$$\widetilde{D} \colon \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}, \quad \text{is in the fiber } \widetilde{D}_A = D_A \colon \mathcal{H}^{\pm} \to \mathcal{H}^{\pm}.$$

Curvature

$$\mathcal{R} := (\gamma d + \widetilde{D})^2 = \gamma d\widetilde{D} + \widetilde{D}d + \widetilde{D}^2 = \underbrace{[\gamma d, \widetilde{D}]}_{=:\widetilde{D}'} + \widetilde{D}^2.$$

Explicit expression of $\widetilde{D}' = [d, \widetilde{D}] \in \Omega^1(\mathcal{V}_{inj}, \widetilde{\mathcal{H}})$:

$$d \colon \Omega^p(\mathcal{V}_{inj}, \widetilde{\mathcal{H}}) \to \Omega^{p+1}(\mathcal{V}_{inj}, \widetilde{\mathcal{H}})$$

$$(d\omega)(\widetilde{X}_0, \ldots, \widetilde{X}_{p+1}) = \sum_{i=0}^{p} \widetilde{X}_i \omega(\widetilde{X}_0, \ldots, \widehat{\widetilde{X}_i}, \ldots, \widetilde{X}_p)$$

(commutators vanish), where

$$\widetilde{X}_A f := \frac{d}{dt}\Big|_{t=0} f(A + tX), \quad X \in \mathcal{V}.$$

One has with $F \colon \mathcal{V}_{inj} \to \mathcal{L}(\mathcal{H})$, $F(A) := D + A$

$$\gamma d(\widetilde{D}\omega) = \gamma dF \wedge \omega,$$

Hence

$$\widetilde{D}'(\omega) = dF \wedge \omega, \quad dF_A(\widetilde{X}_A) = X,$$

$$\widetilde{D}'(\omega)_A(X_0, \ldots, X_{p+1}) = \sum_{i=0}^{r} (-1)^i \underbrace{X_i}_{\in \mathcal{L}(\mathcal{H})} \underbrace{\omega_A(X_0, \ldots, \widehat{X_i}, \ldots, X_p)}_{\in \mathcal{H}}$$

(Super) Chern form

$$\Omega_t^{(n)} := \mathrm{Tr}\left(\gamma e^{-(t\widetilde{D}'+t^2\widetilde{D}^2)}\right)^{(n)} = \mathrm{Tr}\left(\gamma e^{-\mathcal{R}_t^2}\right)^{(n)} =$$

$$= (-t)^n \int_{\Delta_n} \mathrm{Tr}\left(e^{-s_1 t^2 \widetilde{D}^2}\widetilde{D}' e^{-(s_1-s_2)t^2\widetilde{D}^2}\widetilde{D}' \dots e^{-(s_n-s_{n-1})t^2\widetilde{D}^2}\widetilde{D}' e^{-(1-s_n)t^2\widetilde{D}^2}\right) ds_1 ds_2 \dots ds_n,$$

and the integration is over a simplex

$$\Delta_n := \{0 \leqslant s_1 \leqslant s_2 \leqslant \dots \leqslant s_n \leqslant 1 \,\big|\, s_1 + s_2 + \dots + s_n = 1\}$$

One has

$$\frac{d}{ds}(e^{s(A+B)}e^{-sB}) = e^{s(A+B)}Ae^{-sB}$$

$$e^{u(A+B)} = e^{uB} + \int_0^u e^{s(A+B)}Ae^{(u-s)B}ds.$$

[TO BE CONTINUED ...]

6.6 Index theory for foliations

Let $(M^m, \mathcal{F})$ be a foliated manifold. To define an index in noncommutative geometry we have to complete definitions of the following tasks

1. transverse coordinates,
2. analog of elliptic operator,
3. index pairing between K-theory and K-homology.

Foliation can be described using 1-cocycle (V_i, f_i, g_{ij}), where

$$f_i \colon V_i \to U_i \subset \mathbb{R}^n, \;\; n = \operatorname{codim}\mathcal{F} \text{ are surjective submersions,}$$

$$\text{and } g_{ij} \colon f_j(V_i \cap V_j) \to f_i(V_i \cap V_j) \text{ are diffeomorphisms such that}$$

$$g_{ij} \circ g_{jk} = g_{ik}.$$

Above cocycle gives a grupoid $\Gamma = \{g_{ij}\}$ which leads to the algebra of foliation

$$\mathfrak{A}_\Gamma := C_c^\infty(FM) \rtimes \Gamma$$

$$fu_\phi \cdot gu_\psi = fg\phi^{-1}u_{\phi\psi}, \;\; \phi, \psi \in \Gamma.$$

where $FM = J^1(M)$ is a frame bundle. This gives a transverse coordinates. The advantage in working with frame bundle is that FM has a natural volume form. It is paralelizable (i.e. TFM is trivial). One has a principal bundle

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{R}) & \longrightarrow & FM \\ & & \big\downarrow \pi \\ & & M \end{array}$$

One has vertical vector fields Y_i^j coming from the $\mathrm{GL}_n(\mathbb{R})$ action, and when chooses a connection, also horizontal vector fields X_k. Let $\{\theta^k, \omega_j^i\}$ be the dual basis of differential forms. Then

$$\Lambda\omega_j^i \wedge \Lambda\theta^k$$

is an invariant volume form.

For our second task we have to give up ellipticity. Consider a quotient bundle

$$\begin{array}{rl} FM/\,\mathrm{SO}(n) =: & PM \\ & \Big\downarrow \pi \\ & M \end{array}$$

The fiber PM_x is the space of all Euclidean structures on T_xM

$$\langle \zeta, \eta \rangle = \langle a\zeta, a\eta \rangle, \quad a \in \mathrm{SO}(n).$$

Section of PM are all Riemannian metrics on TM. Let

$$\mathcal{V} \subset TPM = \ker \pi_*$$

be the vertical subbundle (vectors tangent to the fibers). On the quotient $\mathrm{GL}_n(\mathbb{R})/\,\mathrm{SO}(n)$ there is a metric, and determines a metric on $\mathcal{V}$.

$$\begin{array}{rl} TPM/\mathcal{V} =: & \mathcal{N} \\ & \Big\downarrow \\ & PM \end{array}$$

The horizontal bundle $\mathcal{N}$ has a tautological Riemannian structure. Indeed, $p \in PM$ is an Euclidean structure for $T_{\pi(p)}M$, and $\mathcal{N}_p$ is identified with $T_{\pi(p)}M$ by π_*.

The bundle TPM has a decomposition into vertical and horizontal part, $TPM = \mathcal{V} \oplus \mathcal{N}$. The Hilbert space

$$L^2(\Lambda T^*PM, \mathrm{vol}_P)$$

where vol_P is a volume form induced by canonical volume form on FM, decomposes also as a tensor product of corresponding Hilbert spaces

$$L^2(\Lambda T^*PM) = L^2(\Lambda\mathcal{V}^*) \otimes L^2(\Lambda\mathcal{N}^*).$$

On this two parts we have operators

- On $L^2(\Lambda\mathcal{V}^*)$ with vertical differential d_V

$$Q_V := i(d_V + d_V^*)(d_V - d_V^*) = -i(d_V d_V^* + d_V^* d_V)$$

- On $L^2(\Lambda\mathcal{N}^*)$ with horizontal differential d_H

$$Q_H := d_H + d_H^*$$

On the whole $L^2(\Lambda T^*PM)$ we put $Q = Q_V \oplus \gamma_V Q_H$, where γ_V is the grading of the vertical signature. Operator $Q = Q^*$ is called **hypoeliptic signature operator**. We have a spectral triple $(\mathfrak{A}_\Gamma, \mathcal{H}, D)$, where D is determined by the equation $Q = D|D|$.

For $a \in \mathfrak{A}$ $[D, a] \in \mathcal{L}(\mathcal{H})$ and $(1+D^2)^{-\frac{1}{2}} \in \mathcal{L}^p(\mathcal{H})$ for $p = \dim \mathcal{V} + 2n$, where $\dim M = n$. The K-cycle $(\mathfrak{A}, \mathcal{H}, D)$ gives an element in $\mathrm{K}^*_{\mathrm{Diff}_M}(\mathfrak{A})$ (Diff_M-equivariant K-cycle). Its character $\mathrm{ch}_*(D) \in \mathrm{HC}_*(\mathfrak{A}_\Gamma)$ can be expressed in terms of residues of spectrally defined zeta-functions, and is given by a cocycle $\{\phi_n\}$ in the (b, B)-bicomplex of $\mathfrak{A}_\Gamma$ whose components are of the following form

$$\mathrm{Res}_{s=0} \, \mathrm{Tr}(a^0 [a^1, D]^{(k_1)} \dots [a^n, D]^{(k_n)} |D|^{-n-2|k|-s})$$

which we denote by

$$\int\!\!\!\!\!\!- \mathrm{Tr}(a^0 [a^1, D]^{(k_1)} \dots [a^n, D]^{(k_n)} |D|^{-n-2|k|-s})$$

$$\phi_n(a^0, \dots, a^n) = \sum_{\mathbf{k}} c_{n,\mathbf{k}} \int\!\!\!\!\!\!- a^0 [Q, a^1]^{(k_1)} \dots [Q, a^n]^{(k_n)} |Q|^{-n-2|k|}$$

Chapter 7

Hopf cyclic cohomology

7.1 Preliminaries

Lecture given by **Piotr Hajac**

7.1.1 Cyclic cohomology in abelian category

Our task is to understand cup product for Hopf-cyclic cohomology with coefficients, that is mapping

$$\mathrm{HC}^m_H(C; M) \otimes \mathrm{HC}^n_H(A; M) \to HC^{m+n}(A; M).$$

Concider a category $\mathcal{C}$, with finite sets $[n] := \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$ as objects, and morphism which preserve order. To describe a cyclic structure we introduce following morphisms

- Face

  $$[n-1] \xrightarrow{\delta_i} [n], \ \ 0 \leqslant i \leqslant n,$$

  - injection which misses i.

- Degeneracy

  $$[n+1] \xrightarrow{\sigma_j} [n], \ \ 0 \leqslant j \leqslant n,$$

  - surjection which sends both j and $j+1$ to j.

- Cyclic operator

  $$[n] \xrightarrow{\tau_n} [n]$$

  - cyclic shift to the right.

The morphism above satisfy following identities, which we can group to obtain succesive complications of our category.

- Presimplicial category.

  $$\mathrm{Mor}(\mathcal{C}) := \{\delta_i^{(n)} \mid 0 \leqslant i \leqslant n, \ n \in \mathbb{N}\},$$

  with

  $$\delta_j \delta_i = \delta_i \delta_j, \ j > i.$$

- Simplicial category.

$$\mathrm{Mor}(\mathcal{C}) := \{\delta_i^{(n)},\ \sigma_j^{(m)} \bigm| 0 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant m,\ n, m \in \mathbb{N}\},$$

  with additional identities

$$\sigma_j\sigma_i = \sigma_i\sigma_{j+1},\ i \leqslant j,$$

$$\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1}, & i < j, \\ \mathrm{id}_{[n]}, & i \in \{j, j+1\}, \\ \delta_{i-1}\sigma_j, & i > j+1 \end{cases}$$

- Precyclic category.

$$\mathrm{Mor}(\mathcal{C}) := \{\delta_i^{(m)},\ \tau_n \bigm| 0 \leqslant i \leqslant m,\ m, n \in \mathbb{N}\},$$

  with the identities as for presimlicial category and

$$\tau_n^{n+1} = \mathrm{id}_{[n]},$$

$$\tau_n\delta_i = \delta_{i-1}\tau_{n-1},\ 1 \leqslant i \leqslant n.$$

- Cyclic Category.

$$\mathrm{Mor}(\mathcal{C}) := \{\delta_i^{(m)},\ \sigma_j^{(l)},\ \tau_n \bigm| 0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant l,\ m, l, n \in \mathbb{N}\},$$

  with all above identieties and

$$\tau_n\sigma_0 = \sigma_n\tau_{n+1}^2,$$

$$\tau_n\sigma_j = \sigma_{j-1}\tau_{n+1},\ 1 \leqslant j \leqslant n.$$

Now, let $\mathcal{A}$ be an abelian category, and $F\colon \mathbb{C} \to \mathcal{A}$ a functor. It means that we have a sequence of objects, and morphisms

$$A_n \xrightarrow{\delta_i} A_n \xrightarrow{\tau_n} A_n \xleftarrow{\sigma_i} A_{n+1}.$$

Define

$$b_n := \sum_{i=0}^{n}(-1)^i\delta_i,\ \ b'_n := \sum_{i=0}^{n-1}(-1)^i\delta_i,$$

$$\lambda_n := (-1)^n\tau_n,\ \ n \in \mathbb{N}.$$

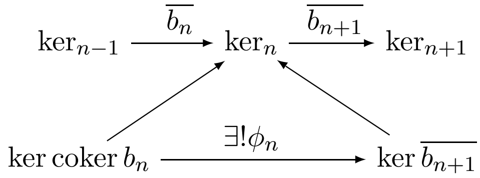
These morphisms satisfy the following identities

$$b_{n+1}b_n = 0,\ \ (1-\lambda_n)b_n = b'_n(1-\lambda_{n-1}).$$

Consider a diagram

$$\begin{array}{ccccc}
\ker_{n+1} & \longrightarrow & A_{n+1} & \xrightarrow{1-\lambda_{n+1}} & A_{n+1} \\
\uparrow{\scriptstyle \overline{b_{n+1}}} & & \uparrow{\scriptstyle b_{n+1}} & & \uparrow{\scriptstyle b'_{n+1}} \\
\ker_n & \longrightarrow & A_n & \xrightarrow{1-\lambda_n} & A_n \\
\uparrow{\scriptstyle \overline{b_n}} & & \uparrow{\scriptstyle b_n} & & \uparrow{\scriptstyle b'_n} \\
\ker_{n-1} & \longrightarrow & A_{n-1} & \xrightarrow{1-\lambda_{n-1}} & A_{n-1}
\end{array}$$

The composition $\overline{b_{n+1}b_n} = 0$, so we have a complex



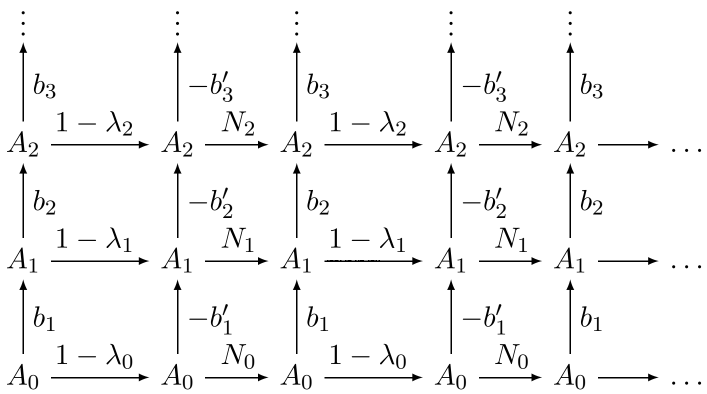
Define the cyclic cohomology of the complex $(A_\bullet, b_n)$ as the cokernel of the unique map ϕ_n

$$\mathrm{HC}^n(F) := \mathrm{HC}^n(A_\bullet) := \operatorname{coker} \phi_n.$$

Define another operator

$$N_n := \sum_{i=0}^{n} (\lambda_n)^i,\ n \in \mathbb{N}.$$

Now one can form a bicomplex



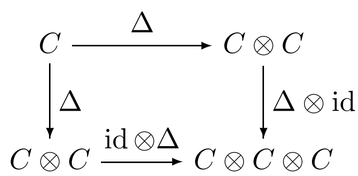
Then the cohomology of the total complex is the cyclic cohomology of the functor $F\colon \mathcal{C} \to \mathcal{A}$

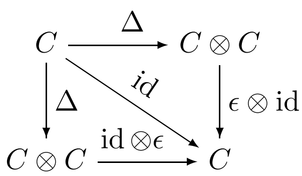
$$\mathrm{HC}^n(F) = \mathrm{H}^n(\operatorname{Tot} A_{\bullet\bullet}).$$

7.1.2 Hopf algebras

Summary of notations.

- Coalgebra (C, Δ, ϵ)





- Comodule (M, Δ_R)

$$\begin{array}{ccc} M & \xrightarrow{\Delta_R} & M \otimes C \\ \downarrow{\scriptstyle \Delta_R} & & \downarrow{\scriptstyle \Delta_R \otimes \mathrm{id}} \\ M \otimes C & \xrightarrow{\mathrm{id} \otimes \Delta_R} & M \otimes C \otimes C \end{array}$$

$$\begin{array}{ccc} M & \xrightarrow{\Delta_R} & M \otimes C \\ & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \epsilon \otimes \mathrm{id}} \\ & & M \end{array}$$

- Bicomodule (M, Δ_L, Δ_R)

$$\begin{array}{ccc} M & \xrightarrow{\Delta_R} & M \otimes C \\ \downarrow{\scriptstyle \Delta_L} & & \downarrow{\scriptstyle \Delta_L \otimes \mathrm{id}} \\ M \otimes C & \xrightarrow{\mathrm{id} \otimes \Delta_R} & C \otimes M \otimes C \end{array}$$

- Hopf algebra $(H, m, 1, \Delta, \epsilon, S)$, where
  - $(H, m, 1)$ algebra,
  - (H, Δ, ϵ) coalgebra,
  - Δ, ϵ are algebra homomorphisms,
  - Convoloution product $f * g$

$$f * g \colon H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{m} H,$$

  - Antipode S

$$S * \mathrm{id} = 1\epsilon = \mathrm{id} * S.$$

Properties of S:

- if exists, it is unique,
- it is an antialgebra map: $S(ab) = S(b)S(a)$,
- it is an anticoalgebra map: $\Delta \circ S = (S \otimes S) \circ \Delta^{op}$,
- if there exists S^{-1}, it has the above properties and satisfies

$$S^{-1} *_{cop} \mathrm{id} = 1\epsilon = \mathrm{id} *_{cop} S^{-1}.$$

Sweedler notation:

$$\Delta h = \sum_i a_i \otimes b_i =: h^{(1)} \otimes h^{(2)}.$$

If we treat multiple tensor products as trees, then we can forget how the tree was constructed.

$$\Delta^2 h = h^{(1)(1)} \otimes h^{(1)(2)} \otimes h^{(2)} = h^{(1)} \otimes h^{(2)(1)} \otimes h^{(2)(2)} = h^{(1)} \otimes h^{(2)} \otimes h^{(3)}.$$

$$\Delta_R m = m^{(0)} \otimes m^{(1)}, \;\; \Delta_L m = m^{(-1)} \otimes m^{(0)}.$$

7.1.3 Motivation for Hopf-cyclic cohomology

If D is a Dirac operator, E idempotent, then there exists an index pairing

$$\langle \mathrm{ch}^*(D), ch_*(E)\rangle =: \mathrm{Index}(D_E).$$

For the transverse geometry of a codim $= n$ foliation

$$\mathrm{ch}^*(D)(a_0,\dots,a_m) = \mathrm{tr}_\delta(a_0 h_1(a_1)\dots h_m(a_m)),$$

where $h_i \in \mathcal{H}_n$ - the universal Hopf algebra for codim $= n$ foliations, $\delta\colon H \to k$- character, tr_δ - δ-invariant trace.

$$\mathcal{H}_n \otimes A \to A$$

$$h(ab) = h^{(1)}(a)h^{(2)}(b), \quad 1_H(a) = a.$$

In particular

$$\Delta(g) = g\otimes g \text{ (group-like element)} \implies g(ab) = g(a)g(b),$$

$$\Delta x = x\otimes 1 + 1\otimes x \text{ (primitive element)} \implies x(ab) = x(a)b + ax(b).$$

One has

$$\begin{aligned}\mathrm{tr}_\delta(a_0h_1(a_1)\dots h_m(a_m)) &= (-1)^m \mathrm{tr}_\delta(a_mh_1(a_0)\dots h_m(a_{m-1}))\\ &= (-1)^m \mathrm{tr}_\delta(h_1(a_0)\dots h_m(a_{m-1})a_m).\end{aligned}$$

In particular

$$\mathrm{tr}_\delta(h(a)) = \delta(h)\,\mathrm{tr}_\delta(a),$$

$$\begin{aligned}\mathrm{tr}_\delta(h(a)b) &= \mathrm{tr}_\delta(h^{(1)}(a)(h^{(2)}S(h^{(3)}))(b)) = \mathrm{tr}_\delta(h^{(1)}(a)h^{(2)}(S(h^{(3)}))(b))) =\\ &= \mathrm{tr}_\delta(h^{(1)}(aS(h^{(2)})(b))) = \delta(h^{(1)})\,\mathrm{tr}_\delta(aS(h^{(2)})(b)) =\\ &= \mathrm{tr}_\delta(a(\delta * S)(h)(b)).\end{aligned}$$

Hence

$$\mathrm{tr}_\delta(a_0h_1(a_1)\dots h_m(a_m)) = (-1)^m \mathrm{tr}_\delta(a_0(\delta*S)(h_1)(h_2(a_1)\dots h_m(a_{m-1})a_m))$$

Denote

$$h_1\otimes\dots\otimes h_m = (-1)^m(\delta*S)(h_1)(h_2\otimes\dots h_m\otimes 1) =: (-1)^m\tau_m(h_1\otimes\dots\otimes h_m).$$

For an element $\sigma\in\mathcal{H}_n$ such that $\Delta\sigma = \sigma\otimes\sigma$, $\delta(\sigma) = 1$

$$\mathrm{tr}_\delta^\sigma(ab) = \mathrm{tr}_\delta^\sigma(b\sigma(a))$$

which implies

$$\tau_m(h_1\otimes\dots\otimes h_m) = (\delta*S)(h_1)(h_2\otimes\dots\otimes h_m\otimes\sigma).$$

$$(-1)^m \mathrm{tr}_\delta(h_1(a_0)\underbrace{h_2(a_1)\dots h_m(a_{m-1})a_m}_{b}) = (-1)^m \mathrm{tr}_\delta(a_0\underbrace{(\delta*S)(h_1)}_{\tilde h}(h_2(a_1)\dots h_m(a_{m-1})a_m)) =$$

$$= (-1)^m \mathrm{tr}_\delta(a_0\tilde h(b)).$$

$$(-1)^m(\delta*S)(h_1)(h_2\otimes\dots\otimes h_m\otimes 1) = \lambda_m(h_1\otimes\dots\otimes h_m).$$

Now one has to check that $\tau_m^{m+1} = \mathrm{id}$. For $m = 1$

$$\tau_1^2(h) = \tau_1((\delta * S)(h)\sigma) = \delta(h^{(1)})(\delta * S)(S(h^{(2)})\sigma)\sigma =$$

$$\delta(h^{(1)})\delta(S(h^{(3)}))\sigma^{-1}S^2(h^{(2)})\sigma = \sigma^{-1}(\delta * S^2 * \delta^{-1})(h)\sigma = h$$

Denote

$$S_\delta^\sigma(h) := (\delta * S)(h)\sigma.$$

Now from $(\tau_1)^2 = (S_\delta^\sigma)^2 = \mathrm{id}$ one can deduce after computation that for all m $\tau_m^{m+1} = \mathrm{id}$ (Connes-Moscovici). This yields a new cyclic complex

$$(H^{\otimes m}, \delta_i, \sigma_j, \tau_m)_{m\in\mathbb{N}}$$

for any Hopf algebra H equipped with modular pair in involution (MPII) (δ, σ). For example, if $S^2 = \mathrm{id}$, then $(\epsilon, 1)$ is a modular pair in involution.

Example 7.1. Let $H = \mathcal{H}_1$ be an universal algebra for codim $= 1$ foliations. First take a Lie algebra $\mathfrak{h}_1$ with generators $X, Y, \lambda_n,\ n \in \mathbb{N}$ satisfying

$$[Y, X] = X,$$

$$[X, \lambda_n] = \lambda_{n+1},$$

$$[Y, \lambda_n] = n\lambda_n,$$

$$[\lambda_n, \lambda_m] = 0 \ \ \forall\, n, m \geqslant 1.$$

Then form an universal enveloping algebra $\mathcal{H}_1 := U(\mathfrak{h}_1)$. The coproduct on $\mathcal{H}_1$ id uniquely determined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X + \lambda_1 \otimes Y,$$

$$\Delta(Y) = Y \otimes 1 + 1 \otimes Y,$$

$$\Delta(\lambda_1) = \lambda_1 \otimes 1 + 1 \otimes \lambda_1.$$

The counit

$$\epsilon(X) = \epsilon(Y) = \epsilon(\lambda_1) = 0.$$

The antipode

$$S(Y) = -Y, \quad S(\lambda_1) = -\lambda_1,$$

$$S(X) = -X + \lambda_1 Y.$$

Now take $\sigma = 1$,

$$\delta(X) = 0, \quad \delta(\lambda_1) = 0, \quad \delta(Y) = -1.$$

One has to check that

$$\delta(h^{(1)})S^2(h^{(2)})\delta(S(h^{(3)})) = h.$$

On generators

$$Y^{(1)} \otimes Y^{(2)} \otimes Y^{(3)} = Y \otimes 1 \otimes 1 + 1 \otimes Y \otimes 1 + 1 \otimes 1 \otimes Y,$$

$$\delta(Y) + S^2(Y) - \delta(Y) = Y.$$

Similarly for λ_1.

$$X^{(1)} \otimes X^{(2)} \otimes X^{(3)} =$$

$$= X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X + 1 \otimes \lambda_1 \otimes Y + \lambda_1 \otimes Y \otimes 1 + \lambda_1 \otimes 1 \otimes Y,$$

$$S^2(X) + \underbrace{\delta(S(X))}_{=0} - S^2(\lambda_1)\delta(Y) = S(-X + \lambda_1 Y) + \lambda_1 =$$
$$= X \underbrace{-\lambda_1 Y + S(Y)S(\lambda_1)}_{=[Y,\lambda_1]=\lambda_1} + \lambda_1 =$$
$$X + \lambda_1 - \lambda_1 = X.$$

Thus $(\delta, 1)$ is a modular pair in involution.

7.1.4 Hopf-cyclic cohomology with coefficients

Motivation:

- Short proof of
$$\tau_1^2 = \mathrm{id} \implies \tau_n^{n+1} = \mathrm{id}\,.$$
- Constructive common denominator for all known cyclic theories.
- Non-trivial coefficients are geometrically desired and occur in "real life" in the number theory work of Connes-Moscovici.

Simplicial structure in coalgebra case:

$$\mathcal{C}^n(C, M) := M \otimes C \otimes C^{\otimes n}, \;\; n \in \mathbb{N},$$

C is an H-module coalgebra

$$\Delta(hc) = h^{(1)}c^{(1)} \otimes h^{(2)}c^{(2)}, \;\; \epsilon(hc) = \epsilon(h)\epsilon(c).$$

M is a C-bimodule

$$\Delta_R(m \otimes c) = (m \otimes c^{(1)}) \otimes c^{(2)},$$
$$\Delta_L(m \otimes c) = m^{(-1)}c^{(1)} \otimes (m^{(0)} \otimes c^{(2)}).$$

The standard example yields

$$\delta_i(m \otimes c_0 \otimes \ldots \otimes c_{n-1}) = m \otimes c_0 \ldots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes \ldots \otimes c_{n-1},$$
$$\delta_n(m \otimes c_0 \otimes \ldots \otimes c_{n-1}) = m^{(0)} \otimes c_0^{(2)} \otimes c_1 \otimes \ldots \otimes c_{n-1} \otimes m^{(-1)}c_0^{(1)},$$
$$\sigma_i(m \otimes c_0 \otimes \ldots \otimes c_{n+1}) = m \otimes c_0 \otimes \ldots \otimes \epsilon(c_{i+1}) \otimes \ldots \otimes c_{n+1}.$$

Simplicial structure in algebra case:

$$\mathcal{C}^n(A, M) := \mathrm{Hom}(M \otimes A \otimes A^{\otimes n}, k), \;\; n \in \mathbb{N}.$$

A is an H-module algebra

$$h(ab) = (h^{(1)}a)(h^{(2)}b), \quad h1 = \epsilon(h).$$

M is aleft H-comodule

$$\mathrm{Hom}(M \otimes A \otimes A^{\otimes n}, k) \simeq \mathrm{Hom}(A^{\otimes n}, \mathrm{Hom}(M \otimes A, k)).$$

$M \otimes A$ is an A-bimodule

$$(m \otimes a)b = m \otimes ab, \quad b(m \otimes a) = m^{(0)} \otimes (S^{-1}(m^{(-1)})b)a$$

The standard example yields

$$(\delta_i f)(m \otimes a_0 \otimes \ldots \otimes a_n) = f(m \otimes a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n),$$

$$(\delta_n f)(m \otimes a_0 \otimes \ldots \otimes a_n) = f(m^{(0)}(S^{-1}(m^{(-1)})a_n)a_0 \otimes \ldots \otimes a_{n-1}),$$

$$(\sigma_i f)(m \otimes a_0 \otimes \ldots \otimes a_n) = f(m \otimes a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n).$$

Paracyclic structures:

For $\{\mathcal{C}^n(A, M)\}_{n \in \mathbb{N}}$

$$(\tau_n f)(m \otimes a_0 \otimes \ldots \otimes a_n) = f(m^{(0)}(S^{-1}(m^{(-1)})a_n) \otimes a_0 \otimes \ldots \otimes a_{n-1}).$$

For $\{\mathcal{C}^n(C, M)\}_{n \in \mathbb{N}}$

$$\tau_n(m \otimes c_0 \otimes \ldots \otimes c_n) = m^{(0)} \otimes c_1 \otimes \ldots \otimes c_n \otimes m^{(-1)} c_0.$$

Invariant complexes:

$$\mathcal{C}^n_H(A, M) := \mathrm{Hom}_H(M \otimes A^{\otimes n+1}, k),$$

$$M \in^H \mathcal{M}_H, \quad (m \otimes \tilde{a})h = mh^{(1)} \otimes S(h^{(2)})\tilde{a}, \quad k = k_\epsilon$$

$$\mathcal{C}^n_H(C, M) := M \otimes_H C^{\otimes n+1},$$

$$M \in^H \mathcal{M}_H, \quad h(c_0 \otimes \ldots c_n) = h^{(1)} c_0 \otimes \ldots \otimes h^{(n+1)} c_n.$$

Cyclic structures:

We say that a bimodule $M \in^H \mathcal{M}_H$ is **stable** iff.

$$\forall\, m \in M \;\; m^{(0)} m^{(-1)} = m.$$

It is **anti-Yetter-Drinfeld** iff.

$$\Delta_L(mh) = S(h^{(3)}) m^{(-1)} h^{(1)} \otimes m^{(0)} h^{(2)}, \;\; \forall\, m, h.$$

Theorem 7.2. *If M is a stable anti-Yetter-Drinfeld module (SAYD), then the formulas for δ_i, σ_i and τ_n define cyclic structures on $\mathcal{C}^n_H(A, M)$ and $\mathcal{C}^n_H(C, M)$.*

Shortly

- anti-Yetter-Drinfeld $\implies$ τ_n is well defined,
- stability $\implies$ $\tau_n^{n+1} = \mathrm{id}$.

Proof. First we check that τ_n is well defined, that is

$$\tau_n(mh \otimes c_0 \otimes \ldots \otimes c_n) = \tau_n(m \otimes h(c_0 \otimes \ldots \otimes c_n)),$$

$$(mh)^{(0)} \otimes_H (c_1 \otimes \ldots \otimes c_n \otimes (mh)^{-1} c_0) = m^{(0)} \otimes_H (h^{(2)}(c_1 \otimes \ldots \otimes c_n) \otimes m^{(-1)} h^{(1)} c_0),$$

hence it suffices to prove the following identity

$$(mh)^{(0)} \otimes_H (1 \otimes (mh)^{(-1)}) = m^{(0)} \otimes_H (h^{(2)} \otimes m^{(-1)} h^{(1)}).$$

Take

$$M \otimes_H (H. \otimes H.) \text{ (diagonal structure)}$$

and morphism

$$H_\cdot \otimes H_\cdot \xrightarrow{\Phi} H_\cdot \otimes H \text{ (multiplication on the first term)}$$
$$\Phi(h \otimes k) = h^{(1)} \otimes S(h^{(2)})k,$$
$$\Phi^{-1}(h \otimes k) = h^{(1)} \otimes h^{(2)}k.$$

Now

$$\Phi^{(-1)}(l(h \otimes k)) = \Phi^{-1}(lh \otimes k) = l\Phi^{-1}(h \otimes k).$$

Consider

$$M \otimes_H (H_\cdot \otimes H_\cdot) \xrightarrow{\mathrm{id} \otimes_H \Phi} M \otimes_H (H_\cdot \otimes H) \simeq M \otimes H.$$
$$(mh)^{(0)} \otimes (mh)^{(-1)} = m^{(0)}h^{(2)} \otimes S(h^{(3)})m^{(-1)}h^{(1)}.$$

-anti-Yetter-Drinfeld condition.

$$\tau_n^{n+1}(m \otimes_H c_0 \otimes \ldots \otimes c_n) = \tau_n^n(m^{(0)} \otimes_H c_1 \otimes \ldots \otimes c_n \otimes m^{(-1)}c_0) =$$
$$= m^{(0)} \otimes m^{(-1)}(c_0 \otimes \ldots \otimes c_n) = m^{(0)}m^{(-1)} \otimes c_0 \otimes \ldots \otimes c_n =$$
$$= m \otimes_H c_0 \otimes \ldots \otimes c_n,$$

where in the last equality we used stability of M. □

7.1.5 Special cases

1. Connes-Moscovici construction.

$$C = H, \quad M =^\sigma k_\delta$$

   Then ${}^\sigma k_\delta$ is SAYD iff. (δ, σ) is MPII. Let F be the isomorphism

$$F\colon k \otimes_H (H_\cdot \otimes H_\cdot^{\otimes n}) \xrightarrow{\simeq} H^{\otimes n}.$$

   Then for $\tilde{f} \in H^{\otimes n}$

$$\tau_n(h_1 \otimes \ldots h_n) = (F \circ \widetilde{\tau_n} \circ F^{-1})(\tilde{h}) = (F \circ \widetilde{\tau_n})(1 \otimes_H \widetilde{\Phi^{-1}}(1 \otimes \tilde{h})) =$$
$$F(1 \otimes_H (\tilde{h} \otimes \sigma)) = 1 \otimes_H \widetilde{\Phi}(h_1 \otimes \ldots \otimes h_n \otimes \sigma) =$$
$$= 1 \otimes_H h_1^{(1)} \otimes S(h_1^{(2)})(h_2 \otimes \ldots \otimes h_n \otimes \sigma) = \delta(h_1^{(1)})S(h_1^2)(h_2 \otimes \ldots \otimes h_n \otimes \sigma).$$

2. $$\mathrm{tr}_\delta^\sigma \in \mathrm{HC}_H^0(A;^\sigma k_\delta)$$

3. Characteristic map of Connes-Moscovici

$$\mathrm{HC}_H^m(H;^\sigma k_\delta) \otimes \mathrm{HC}_H^0(A;^\sigma k_\delta) \to \mathrm{HC}^m(A),$$
$$h_1 \otimes \ldots \otimes h_m \mapsto ((a_0 \otimes \ldots \otimes a_m) \mapsto \mathrm{tr}_\delta^\sigma(a_0 h_1(a_1) \otimes h_m(a_m)))$$

4. The $n > 0$ and $\dim M > 1$ already applied in Connes-Moscovici work on number theory.

5. $$\mathrm{HC}_k^m(A;k) = \mathrm{HC}^m(A)$$

6. Twisted cyclic cohomology
$$\mathrm{HC}^*_{k[\sigma,\sigma^{-1}]}(A;{}^\sigma k_\epsilon).$$

Lemma 7.3.
$${}^\sigma k_\delta \textit{ is SAYD} \Longleftrightarrow (\delta,\sigma) \textit{ is MPII.}$$

Proof.
$$m^{(0)}m^{(-1)} = m \Leftrightarrow 1\cdot\sigma = \delta(\sigma) = 1,$$
$$(mh)^{(-1)} \otimes (mh)^{(0)} = S(h^{(3)})m^{(-1)}h^{(1)} \otimes m^{(0)}h^{(2)}$$
$$\sigma\delta(h) = S(h^{(3)})\sigma h^{(1)}\delta(h^{(2)})$$
$$L(h) = R(h) \Leftrightarrow (L *_{op} S^{-1})(h) = (R *_{op} S^{-1})(h)$$
$$L(h^{(2)})S^{(-1)}(h^{(1)}) = R(h^{(2)})S^{(-1)}(h^{(1)})$$
$$\tilde{S}^\sigma_\delta(h) = \sigma\delta(h^{(2)})S^{(-1)}(h^{(1)}) = S(h^{(2)})\sigma\delta(h^{(1)}) =: S^\sigma_\delta(h)$$
By direct computation
$$\tilde{S}^\sigma_\delta \circ S^\sigma_\delta = \mathrm{id} = S^\sigma_\delta \circ \tilde{S}^\sigma_\delta, \text{ i.e.}$$
$$\tilde{S}^\sigma_\delta = (S^\sigma_\delta)^{-1}.$$
Therefore
$$\mathrm{AYD} \Leftrightarrow (S^\sigma_\delta)^{-1} = S^\sigma_\delta$$
$$(S^\sigma_\delta)^2 = \mathrm{id} \text{ (involution condition).}$$
□

7.2 The Hopf algebra $\mathcal{H}_n$

Let the manifold M^n be affine flat (the $\mathbb{R}^n$ or the disjoint union of $\mathbb{R}^n$). The frame bundle is then trivial with $FM \simeq M \times \mathrm{GL}_n(\mathbb{R})$. In local coordinates (x^μ) for $x \in U \subset M$, we can view the frame coordinates x^μ, y^μ_j as a 1-jet of a map $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$
$$\phi(t) = x + yt, \;\; x,t \in \mathbb{R}^n, \;\; y \in \mathrm{GL}_n(\mathbb{R}),$$
where $(yt)^\mu = \sum_i y^\mu_i t^i$ for $t = (t^i) \in \mathbb{R}^n$.

We endow it with the trivial connection, given by the matrix-valued 1-form $\omega = (\omega^i_j)$, where
$$\omega^i_j := \sum_\mu (y^{-1})^i_\mu dy^\mu_j = (y^{-1}dy)^i_j$$
The corresponding basic horizontal fields on FM are
$$X_k = \sum_\mu y^\mu_k \partial_\mu, \;\; k = 1,\dots,n, \;\; \partial_\mu = \frac{\partial}{\partial x^\mu}.$$
Denote by θ^k be the canonical form of the frame bundle
$$\theta^k := \sum_\mu (y^{-1})^k_\mu dx^\mu = (y^{-1}dx)^k, \;\; k = 1,\dots,n.$$

Then let

$$Y_i^j = \sum_\mu y_i^\mu \partial_\mu^j, \ \ i,j = 1,\dots,n, \ \ \partial_\mu^j := \frac{\partial}{\partial y_j^\mu}$$

be the fundamental vertical vector fields associated to the standard basis of $\mathfrak{gl}_n(\mathbb{R})$ and generating the canonical right action of $\mathrm{GL}_n(\mathbb{R})$ on FM. At each point of FM, $\{X_k, Y_i^j\}$ and $\{\theta^k, \omega_j^i\}$ form bases of the tangent and cotangent space, dual to each other

$$\langle \omega_j^i, Y_k^l \rangle = \delta_k^i \delta_j^l, \ \ \langle \omega_j^i, X_k \rangle = 0,$$

$$\langle \theta^i, Y_k^l \rangle = 0, \ \ \langle \theta^i, X_j \rangle = \delta_j^i.$$

The group of diffeomorphism $\mathrm{Diff}_M = \mathrm{Diff}_{\mathbb{R}^n}$ acts on FM by the natural lift of the tautological action to the frame level

$$\widetilde{\varphi}(x,y) := (\varphi(x), \varphi'(x)y)$$

where $\varphi'(x)$ is Jacobi matrix $\varphi'(x)_j^i = \frac{\partial \varphi^i}{\partial x^j}$.

Viewing Diff_M as a discrete group we form the crossed product algebra

$$\mathfrak{A}_M := C_c^\infty(FM) \rtimes \mathrm{Diff}_M$$

As a vector space, it is spanned by monomials of the form fu_φ^*, where $f \in C^\infty(FM)$ and u_φ^* stands for φ^{-1}. The product is given by

$$f_1 u_{\varphi_1}^* \cdot f_2 u_{\varphi_2}^* = f_1 (f_2 \circ \widetilde{\varphi}_1) u_{\varphi_2 \varphi_1}^*.$$

Since the right action of $\mathrm{GL}_n(\mathbb{R})$ on FM commutes with the action of Diff_M, at the Lie algebra level one has

$$u_\varphi Y_i^j u_\varphi^* = Y_i^j.$$

This allows to promote the vertical vector fields to derivations of $\mathfrak{A}_M$. Indeed, setting

$$Y_i^j(fu_\varphi^*) = Y_i^j(f) u_\varphi^*$$

the extended operators satisfy the derivation rule

$$Y_i^j(ab) = Y_i^j(a)b + aY_i^j(b), \ \ a,b \in \mathfrak{A}_M.$$

We shall also prolong the horizontal vector fields to linear transformations $X_k \in \mathcal{L}(\mathfrak{A}_M)$ in similar fashion

$$X_k(fu_\varphi^*) = X_k(f) u_\varphi^*.$$

The resulting operators are no longer Diff_M-invariant. They satisfy

$$u_\varphi X_k u_\varphi^* = X_k - \gamma_{jk}^i(\varphi^{-1}) Y_i^j,$$

where $\varphi \mapsto \gamma_{jk}^i(\varphi)$ is a group 1-cocycle on Diff_M with values in $C^\infty(FM)$. Specifically

$$\gamma_{jk}^i(\varphi)(x,y) = \sum_\mu (y^{-1} \cdots \varphi'(x)^{-1} \cdot \partial_\mu \cdot y)_j^i y_k^\mu$$

The above expression comes from the pull-back formula for the connection

$$\widetilde{\varphi}^*(\omega_j^i) = \omega_j^i + \gamma_{jk}^i(\varphi)\theta^k.$$

Now one uses the fact that $\{\theta^k, (\widetilde{\varphi}^{-1})^*(\omega^i_j)\}$ is the dual basis to $\{u_\varphi X_k u^*_\varphi, Y^j_i\}$.

As a consequence, the operators $X_k \in \mathcal{L}(\mathfrak{A}_M)$ are no longer derivations of $\mathfrak{A}_M$, but satisfy a non-symmetric Leibniz rule

$$X_k(a,b) = X_k(a)b + aX_k(b) + \delta^i_{jk}(a)Y^j_i(b), \quad a, b \in \mathfrak{A}_M,$$

where the linear operators $\delta^i_{jk} \in \mathcal{L}(\mathfrak{A}_M)$ are defined by

$$\delta^i_{jk}(fu^*_\varphi) = \gamma^i_{jk} f u^*_\varphi.$$

These are derivations, i.e.

$$\delta^i_{jk}(ab) = \delta^i_{jk}(a)b + a\delta^i_{jk}(b).$$

The operators $\{X_k, Y^i_j\}$ satisfy the commutation relations of the group of affine transformations of $\mathbb{R}^n$

$$[Y^j_i, Y^l_k] = \delta^j_k Y^l_i - \delta^l_i Y^j_k,$$
$$[Y^j_i, X_k] = \delta^j_k X_i,$$
$$[X_k, X_l] = 0.$$

The succesive commutators of the operators δ^i_{jk} with the X_l's yield new generations of

$$\delta^i_{jk|l_1\dots l_r} := [X_{l_r}, \dots [X_{l_1}, \delta^i_{jk}] \dots],$$

which involve multiplication by higher order jets of diffeomorphisms

$$\delta^i_{jk|l_1\dots l_r}(fu^*_\varphi) = \gamma^i_{jk|l_1\dots l_r} f u^*_\varphi, \text{ where}$$

$$\delta^i_{jk|l_1\dots l_r} := X_{l_r} \dots X_{l_1}(\gamma^i_{jk}).$$

They commute among themselves

$$[\delta^i_{jk|l_1\dots l_r}, \delta^{i'}_{j'k'|l'_1\dots l'_r}] = 0.$$

It can be checked that the order of $\{j,k\}$ and $\{l_1, \dots, l_r\}$ does not matter - in any case we get the same operator.

The commutators between Y^λ_μ's and δ^i_{jk}'s can be obtained from explicit expression of the cocycle γ, by computing its derivatives in the direction of the vertical vector fields. One obtains

$$[Y^\lambda_\mu, \delta^i_{jk}] = \delta^\lambda_j \delta^i_{\mu k} + \delta^\lambda_k \delta^i_{j\mu} - \delta^i_\mu \delta^\lambda_{jk}$$

By induction

$$[Y^\lambda_\mu, \delta^i_{j_1 j_2|j_3\dots j_r}] = \sum_{s=0}^{r} \delta^\lambda_{j_s} \delta^i_{j_1 j_2|j_3\dots j_{s-i}\mu j_{s+1}\dots j_r} - \delta^i_\mu \delta^\lambda_{j_1 j_2|j_3\dots j_r}.$$

Definition 7.4. *Let $\mathcal{H}_n$ be the universal enveloping algebra of the Lie algebra $\mathfrak{h}_n$ with basis*

$$\{X_\lambda, Y^\mu_\nu, \delta^i_{jk|l_1\dots l_r} \mid 1 \leqslant \lambda, \mu, \nu, i \leqslant n,\ 1 \leqslant j \leqslant k \leqslant n,\ 1 \leqslant l_1 \leqslant \dots \leqslant l_r \leqslant n\}$$

and the following presentation

$$[X_k, X_l] = 0,$$
$$[Y^j_i, Y^l_k] = \delta^j_k Y^l_i - \delta^l_i Y^j_k,$$

$$[Y_i^j, X_k] = \delta_k^j X_i,$$
$$[X_{l_r}, \delta^i_{jk|l_1\dots l_{r-1}}] = \delta^i_{jk|l_1\dots l_r},$$
$$[Y_\nu^\lambda, \delta^i_{j_1j_2|j_3\dots j_r}] = \sum_{s=0}^{r} \delta^\lambda_{j_s} \delta^i_{j_1j_2|j_3\dots j_{s-i}\nu j_{s+1}\dots j_r} - \delta^i_\nu \delta^\lambda_{j_1j_2|j_3\dots j_r},$$
$$[\delta^i_{jk|l_1\dots l_r}, \delta^{i'}_{j'k'|l'_1\dots l'_r}] = 0.$$

We shall endow $\mathcal{H}_n := U(\mathfrak{h}_n)$ with a canonical Hopf structure, which is noncommutative, and therefore different from the standard structure of a universal enveloping algebra.

Proposition 7.5. *1. The formulae*

$$\Delta X_k = X_k \otimes 1 + 1 \otimes X_k + \delta^i_{jk} \otimes Y_i^j,$$
$$\Delta Y_i^j = Y_i^j \otimes 1 + 1 \otimes Y_i^j,$$
$$\Delta \delta^i_{jk} = \delta^i_{jk} \otimes 1 + 1 \otimes \delta^i_{jk},$$

uniquely determine a coproduct $\Delta\colon \mathcal{H}_n \to \mathcal{H}_n \otimes \mathcal{H}_n$*, which makes* $\mathcal{H}_n$ *a bialgebra with respect to the product* $m\colon \mathcal{H}_n \otimes \mathcal{H}_n \to \mathcal{H}_n$ *and the counit* $\varepsilon\colon \mathcal{H}_n \to \mathbb{C}$ *inherited from* $U(\mathfrak{h}_n)$*.*

2. The formulae

$$S(X_k) = -X_k + \delta^i_{jk} Y_i^j,$$
$$S(Y_i^j) = -Y_i^j,$$
$$S(\delta^i_{jk}) = -\delta^i_{jk},$$

uniquely determine an anti-homomorphism $S\colon \mathcal{H}_n \to \mathcal{H}_n$*, which provides the antipode that turns* $\mathcal{H}_n$ *into a Hopf algebra.*

The notation is justified while one proves that the subalgebra of $\mathcal{L}(\mathfrak{A}_M)$ generated by the linear operators $\{X_k, Y_j^i, \delta^i_{jk} \mid i, j, k = 1, \dots, n\}$ is isomorphic to the algebra $\mathcal{H}_n$. The action of $\mathcal{H}_n$ turns $\mathfrak{A}_n$ into a left $\mathcal{H}_n$-module algebra. Morover to any element $h^1 \otimes \dots \otimes h^p \in \mathcal{H}_n^p$ we can associate a multilinear differential operator T acting on $\mathfrak{A}_M$ as follows

$$T(h^1 \otimes \dots \otimes h^p)(a^1, \dots, a^p) = h^1(a_1) \dots h^p(a_p).$$

The linearization $T\colon T\mathcal{H}_n^p \to \mathcal{L}(\mathfrak{A}_M^{\otimes p}, \mathfrak{A}_M)$ of this assignment is injective for each $p \in \mathbb{N}$.