Lectures on the geometry of flag varieties

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Introduction

In these notes, we present some fundamental results concerning flag varieties and their Schubert varieties. By a flag variety, we mean a complex projective algebraic variety X, homogeneous under a complex linear algebraic group. The orbits of a Borel subgroup form a stratification of X into Schubert cells. These are isomorphic to affine spaces; their closures in X are the Schubert varieties, generally singular.

The classes of the Schubert varieties form an additive basis of the cohomology ring $H^*(X)$, and one easily shows that the structure constants of $H^*(X)$ in this basis are all non-negative. Our main goal is to prove a related, but more hidden, statement in the Grothendieck ring $K(X)$ of coherent sheaves on X. The latter admits an additive basis formed of structure sheaves of Schubert varieties, and the corresponding structure constants turn out to have alternating signs.

These structure constants admit combinatorial expressions in the case of Grassmannians: those of $H^*(X)$ (the Littlewood-Richardson coefficients) have been known for many years, whereas those of $K(X)$ were only recently determined by Buch [10]. This displayed their alternation of signs, and Buch conjectured that this property extends to all the flag varieties. In this setting, the structure constants of the cohomology ring (a fortiori, those of the Grothendieck ring) are yet combinatorially elusive, and Buch's conjecture was proved in [6] by purely algebro-geometric methods.

Here we have endeavored to give a self-contained exposition of this proof. The main ingredients are geometric properties of Schubert varieties (e.g., their normality), and vanishing theorems for cohomology of line bundles on these varieties (these are deduced from the Kawamata-Viehweg theorem, a powerful generalization of the Kodaira vanishing theorem in complex geometry). Of importance are also the intersections of Schubert varieties with opposite Schubert varieties. These Richardson varieties are systematically used in these notes to provide geometric explanations for many formulae in the cohomology or Grothendieck ring of flag varieties.

These notes are organized as follows. The first section discusses Schubert cells and varieties, their classes in the cohomology ring, and the Picard group of flag varieties. In the second section, we obtain restrictions on the singularities of Schubert varieties, and also vanishing theorems for the higher cohomology groups of line bundles on these varieties. The third section is devoted to a degeneration of the diagonal of a flag variety into unions of products of Schubert varieties, with applications to the Grothendieck group. In the final fourth section, we obtain several "positivity" results in this group, including a solution of Buch's conjecture. Each section begins with a brief overview of its contents, and ends with bibliographical notes and open problems.

The prerequisites are familiarity with algebraic geometry (for example, the contents of the first three chapters of Hartshorne's book [23]); no knowledge of algebraic groups is assumed. In fact, we have presented all the notations and results in the case of the general linear group, but in such a way that they can be extended readily to arbitrary (connected, reductive) algebraic groups by readers familiar with their structure theory.

These notes grew out of courses at the Institut Fourier (Grenoble) in the spring of 2003, and at the mini-school "Schubert Varieties" (Banach Center, Warsaw) in May 2003. I am grateful to the organizers of this school, Piotr Pragacz and Andrzej Weber, for their invitation and encouragements. I also thank the auditors of both courses, especially Dima Timashev, for their attention and comments.

Conventions.
Throughout these notes, we consider algebraic varieties over the field $\mathbb{C}$ of complex numbers. We follow the notation and terminology of Hartshorne's book [23]; in particular, varieties are assumed to be irreducible. Unless otherwise stated, subvarieties are assumed to be closed.

1 Grassmannians and flag varieties

We begin this section by reviewing the definitions and fundamental properties of Schubert varieties in Grassmannians and in the variety of complete flags. Then we introduce the Schubert classes in the cohomology ring of flag varieties, and we study their multiplicative properties. Finally, we describe the Picard group of flag varieties, first in terms of Schubert divisors, and then in terms of homogeneous line bundles; we also sketch the relation of the latter to representation theory.

1.1 Grassmannians

The *Grassmannian* $\mathrm{Grass}(d,n)$ is the set of d-dimensional linear subspaces of $\mathbb{C}^n$. Given such a subspace E and a basis $(v_1,\ldots,v_d)$ of E, the exterior product $v_1\wedge\cdots\wedge v_d \in \bigwedge^d \mathbb{C}^n$ only depends on E up to a non-zero scalar multiple. In other words, the point

$$\iota(E) := [v_1 \wedge \cdots \wedge v_d]$$

of the projective space $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$ only depends on E. Further, $\iota(E)$ uniquely determines E, so that the map ι identifies $\mathrm{Grass}(d,n)$ with the image in $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$ of the cone of decomposable d-vectors in $\bigwedge^d \mathbb{C}^n$. It follows that $\mathrm{Grass}(d,n)$ is a subvariety of the projective space $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$; the map

$$\iota : \mathrm{Grass}(d,n) \to \mathbb{P}(\bigwedge^d \mathbb{C}^n)$$

is the *Plücker embedding.*

The general linear group

$$G := GL_n(\mathbb{C})$$

acts on the variety

$$X := \mathrm{Grass}(d,n)$$

via its natural action on $\mathbb{C}^n$. Clearly, X is a unique G-orbit, and the Plücker embedding is equivariant with respect to the action of G on $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$ arising from its linear action on $\bigwedge^d \mathbb{C}^n$. Let $(e_1,\ldots,e_n)$ denote the standard basis of $\mathbb{C}^n$, then the isotropy group of the subspace $\langle e_1,\ldots,e_d\rangle$ is

$$P := \left\{ \begin{pmatrix} a_{1,1} & \ldots & a_{1,d} & a_{1,d+1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{d,1} & \ldots & a_{d,d} & a_{d,d+1} & \ldots & a_{d,n} \\ 0 & \ldots & 0 & a_{d+1,d+1} & \ldots & a_{d+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & a_{n,d+1} & \ldots & a_{n,n} \end{pmatrix} \right\}$$

(this is a maximal parabolic subgroup of G.) Thus, X is the homogeneous space G/P. As a consequence, the algebraic variety X is nonsingular, of dimension $\dim(G) - \dim(P) = d(n-d)$.

For any multi-index $I := (i_1, \ldots, i_d)$, where $1 \leq i_1 < \ldots < i_d \leq n$, we denote by E_I the corresponding *coordinate subspace* of $\mathbb{C}^n$, i.e., $E_I = \langle e_{i_1}, \ldots, e_{i_d} \rangle \in X$. In particular, $E_{1,2,\ldots,d}$ is the standard coordinate subspace $\langle e_1, \ldots, e_d \rangle$. We may now state the following result, whose proof is straightforward.

1.1.1 Proposition. *(i) The E_I are precisely the T-fixed points in X, where*

$$T: = \left\{ \begin{pmatrix} a_{1,1} & 0 & \ldots & 0 \\ 0 & a_{2,2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{n,n} \end{pmatrix} \right\} \subseteq GL_n(\mathbb{C})$$

is the subgroup of diagonal matrices (this is a maximal torus of G).
(ii) X is the disjoint union of the orbits BE_I, where

$$B: = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ 0 & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{n,n} \end{pmatrix} \right\} \subseteq GL_n(\mathbb{C})$$

is the subgroup of upper triangular matrices (this is a Borel subgroup of G.)

1.1.2 Definition. The *Schubert cells* in the Grassmannian are the orbits $C_I := BE_I$, i.e., the B-orbits in X. The closure in X of the Schubert cell C_I (for the Zariski topology) is called the *Schubert variety* $X_I : = \overline{C_I}$.

Note that B is the semi-direct product of T with the normal subgroup

$$U: = \left\{ \begin{pmatrix} 1 & a_{1,2} & \ldots & a_{1,n} \\ 0 & 1 & \ldots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \right\}$$

(this is a maximal unipotent subgroup of G.) Thus, we also have $C_I = UE_I$: the Schubert cells are just the U-orbits in X.

Also, the isotropy group U_{E_I} is the subgroup of U where $a_{ij} = 0$ whenever $i \notin I$ and $j \in I$. Let U^I be the "complementary" subset of U, defined by $a_{ij} = 0$ if $i \in I$ or $j \notin I$. Then one checks that: U^I is a subgroup of U, and the map $U^I \to X$, $g \mapsto gE_I$ is a locally closed embedding with image C_I. It follows that C_I is a locally closed subvariety of X, isomorphic to the affine space $\mathbb{C}^{|I|}$, where $|I| := \sum_{j=1}^{d}(i_j - j)$. Thus, its closure X_I is a projective variety of dimension $|I|$.

Next we present a geometric characterization of Schubert cells and varieties (see e.g. [18] 9.4).

1.1.3 Proposition. *(i) C_I is the set of d-dimensional subspaces $E \subset \mathbb{C}^n$ such that:*

$$\dim(E \cap \langle e_1, \dots, e_j\rangle) = \#\{k \mid 1 \le k \le d,\ i_k < j\}, \quad \textit{for} \quad j = 1, \dots, n.$$

(ii) X_I is the set of d-dimensional subspaces $E \subset \mathbb{C}^n$ such that :

$$\dim(E \cap \langle e_1, \dots, e_j\rangle) \ge \#\{k \mid 1 \le k \le d,\ i_k < j\}, \quad \textit{for} \quad j = 1, \dots, n.$$

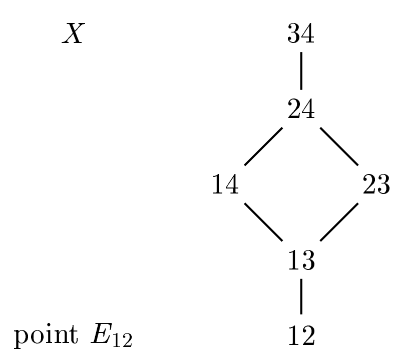
Thus, we have

$$X_I = \bigcup_{J \le I} C_J,$$

where $J \le I$ if and only if $j_k \le i_k$ for all k.

1.1.4 Examples. 1) For $d = 1$, the Grassmannian is just the projective space $\mathbb{P}^{n-1}$, and the Schubert varieties form a flag of linear subspaces $X_0 \subset X_1 \subset \cdots \subset X_n$, where $X_j \cong \mathbb{P}^{j-1}$.

2) For $d = 2$ and $n = 4$ one gets the following poset of Schubert varieties:



Further, the Schubert variety X_{24} is singular. Indeed, one checks that $X \subset \mathbb{P}(\bigwedge^2 \mathbb{C}^4) = \mathbb{P}^5$ is defined by one quadratic equation (the Plücker relation). Further, X_{24} is the intersection of X with its tangent space at the point E_{12}. Thus, X_{24} is a quadratic cone with vertex E_{12}, its unique singular point.

3) For arbitrary d and n, the Schubert variety $X_{1,2,\ldots,d}$ is just the point $E_{1,2,\ldots,d}$, whereas $X_{n-d+1,n-d+2,\ldots,n}$ is the whole Grassmannian. On the other hand, $X_{n-d,n-d+2,\ldots,n}$ consists of those d-dimensional subspaces E that meet $\langle e_1, \ldots, e_{n-d}\rangle$: it is the intersection of X with the hyperplane of $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$ where the coordinate on $e_{n-d+1} \wedge \cdots \wedge e_n$ vanishes.

Since X is the disjoint union of the open Schubert cell $C_{n-d+1,n-d+2,\ldots,n} \cong \mathbb{C}^{d(n-d)}$ with the irreducible divisor $D\colon = X_{n-d,n-d+2,\ldots,n}$, any divisor in X is linearly equivalent to a unique integer multiple of D. Equivalently, any line bundle on X is isomorphic to a unique tensor power of the line bundle $L\colon = \mathcal{O}_X(D)$, the pull-back of $\mathcal{O}(1)$ via the Plücker embedding. Thus, *the Picard group* $\mathrm{Pic}(X)$ *is freely generated by the class of the very ample line bundle* L.

We may re-index Schubert varieties in two ways:

1. By partitions: with any multi-index $I = (i_1, \ldots, i_d)$ we associate the partition $\lambda = (\lambda_1, \ldots, \lambda_d)$, where $\lambda_j := i_j - j$ for $j = 1, \ldots, d$. We then write X_λ instead of X_I.

This yields a bijection between the set of multi-indices $I = (i_1, \ldots, i_d)$ such that $1 \leq i_1 < \ldots < i_d \leq n$, and the set of tuples of integers $\lambda = (\lambda_1, \ldots, \lambda_d)$ satisfying $0 \leq \lambda_1 \leq \ldots \leq \lambda_d \leq n-d$. This is the set of *partitions with* $\leq d$ *parts of size* $\leq n-d$.

The *area* of the partition λ is the number $|\lambda| := \sum_{j=1}^{d} \lambda_j = |I|$. With this indexation, the dimension of X_λ is the area of λ; further, $X_\mu \subseteq X_\lambda$ if and only if $\mu \leq \lambda$, that is, $\mu_j \leq \lambda_j$ for all j.

2. By permutations: with a multi-index $I = (i_1, \ldots, i_d)$ we associate the permutation w of the set $\{1, 2, \ldots, n\}$, defined as follows: $w(\ell) = i_\ell$ for $l = 1, \ldots, d$, whereas $w(d+\ell)$ is the ℓ-th element of the ordered set $\{1, \ldots, n\} \setminus I$ for $l = 1, \ldots, n-d$. This sets up a bijection between the multi-indices and the permutations w such that : $w(1) < w(2) < \cdots < w(d)$ and $w(d+1) < \cdots < w(n)$. These permutations form a system of representatives of the coset space $S_n/(S_d \times S_{n-d})$, where S_n denotes the permutation group of the set $\{1, 2, \ldots, n\}$, and $S_d \times S_{n-d}$ is its subgroup stabilizing the subset $\{1, 2, \ldots, d\}$ (and $\{d+1, d+2, \ldots, n\}$). Thus, we may parametrize the T-fixed points of X, and hence the Schubert varieties, by the map $S_n/(S_d \times S_{n-d}) \to X$, $w(S_d \times S_{n-d}) \mapsto E_{w(1),\ldots,w(d)}$. This parametrization will be generalized to all flag varieties in the next subsection.

1.2 Flag varieties

Given a sequence $(d_1, \ldots, d_m)$ of positive integers with sum n, a *flag of type* $(d_1, \ldots, d_m)$ *in* $\mathbb{C}^n$ is an increasing sequence of linear subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_m = \mathbb{C}^n$$

such that $\dim(V_j/V_{j-1}) = d_j$ for $j = 1, \ldots, m$. The *coordinate flags* are those consisting of coordinate subspaces.

Let $X(d_1,\ldots,d_m)$ denote the set of flags of type $(d_1,\ldots,d_m)$. For example, $X(d,n-d)$ is just the Grassmannian $\mathrm{Grass}(d,n)$. More generally, $X(d_1,\ldots,d_m)$ is a subvariety of the product of the Grassmannians $\mathrm{Grass}(d_i,n)$, called the *partial flag variety* of type $(d_1,\ldots,d_m)$.

The group $G=\mathrm{GL}_n(\mathbb{C})$ acts transitively on $X(d_1,\ldots,d_m)$. Let $P=P(d_1,\ldots,d_m)$ be the isotropy group of the *standard flag* (consisting of the standard coordinate subspaces). Then $P(d_1,\ldots,d_m)$ consists of the block upper triangular invertible matrices with diagonal blocks of sizes $d_1,\ldots,d_m$. In particular, $P(d_1,\ldots,d_m)$ contains B; in fact, all subgroups of G containing B occur in this way. (These subgroups are the *standard parabolic subgroups of* G.) Since $X\cong G/P$, it follows that X is nonsingular of dimension $\sum_{1\le i<j\le m} d_i d_j$.

In particular, we have the variety $X:=X(1,\ldots,1)$ of *complete flags*, also called the *full flag variety*; it is the homogeneous space G/B, of dimension $n(n-1)/2$. By sending any complete flag to the corresponding partial flag of a given type $(d_1,\ldots,d_m)$, we obtain a morphism

$$f: X=G/B\to G/P(d_1,\ldots,d_m)=X(d_1,\ldots,d_m).$$

Clearly, f is G-equivariant with fiber P/B at the base point B/B (the standard complete flag). Thus, f is a fibration with fiber the product of varieties of complete flags in $\mathbb{C}^{d_1}$, ..., $\mathbb{C}^{d_m}$. This allows to reduce many questions regarding flag varieties to the case of the variety of complete flags; see Example 1.2.3 below for details on this reduction. Therefore, we will mostly concentrate on the full flag variety.

We now introduce Schubert cells and varieties in G/B. Observe that the complete coordinate flags correspond to the permutations of the set $\{1,\ldots,n\}$, by assigning to the flag

$$0\subset\langle e_{i_1}\rangle\subset\cdots\subset\langle e_{i_1},e_{i_2},\ldots,e_{i_\ell}\rangle\subset\cdots$$

the permutation w such that $w(\ell)=i_\ell$ for all ℓ. We regard the permutation group S_n as a subgroup of $\mathrm{GL}_n(\mathbb{C})$ via its natural action on the standard basis $(e_1,\ldots,e_n)$. Then the (complete) coordinate flags are exactly the $F_w:=wF$, where F denotes the standard complete flag. Further, S_n identifies to the quotient $W:=N_G(T)/T$, where $N_G(T)$ denotes the normalizer of T in G. (In other words, S_n is the *Weyl group of G with respect to T*.)

We may now formulate an analogue of Proposition 1.1.1 (see e.g. [18] 10.2 for a proof).

1.2.1 Proposition. *(i) The fixed points of T in X are the coordinate flags F_w, $w\in W$.*
(ii) X is the disjoint union of the orbits $C_w:=BF_w=UF_w$, where $w\in W$.
(iii) Let $X_w:=\overline{C_w}$ (closure in the Zariski topology of X), then

$$X_w=\bigcup_{v\in W,\, v\le w} C_v$$

where $v\le w$ if and only if we have $(v(1),\ldots,v(d))_{\text{r.t.i.v.}}\le(w(1),\ldots,w(d))_{\text{r.t.i.v.}}$ for $d=1,\ldots,n-1$ (here r.t.i.v. stands for "reordered to increasing values").

1.2.2 Definition. $C_w := BF_w$ is a *Schubert cell*, and $X_w := \overline{C_w}$ is the corresponding *Schubert variety*. The partial ordering $\leq$ on W is the *Bruhat order*.

By the preceding proposition, we have $X_v \subseteq X_w$ if and only if this holds for the images of X_v and X_w in $\mathrm{Grass}(d,n)$, where $d = 1, \ldots, n-1$. Together with Proposition 1.1.3, this yields a geometric characterization of the Bruhat order on Schubert varieties. Also, note that the T-fixed points in X_w are the coordinate flags F_v, where $v \in W$ and $v \leq w$.

We now describe the Schubert cells UF_w. Note that the isotropy group

$$U_{F_w} = U \cap wUw^{-1} =: U_w$$

is defined by: $a_{i,j} = 0$ whenever $i < j$ and $w^{-1}(i) < w^{-1}(j)$. Let U^w be the "complementary" subset of U, defined by $a_{ij} = 0$ whenever $i < j$ and $w^{-1}(i) > w^{-1}(j)$. Then $U^w = U \cap wU^- w^{-1}$ is a subgroup, and one checks that the product map $U^w \times U_w \to U$ is an isomorphism of varieties. Hence the map $U^w \to C_w$, $g \mapsto gF_w$ is an isomorphism as well. It follows that each C_w is an affine space of dimension

$$\#\{(i,j) \mid 1 \leq i < j \leq n,\ w^{-1}(i) > w^{-1}(j)\} = \#\{(i,j) \mid 1 \leq i < j \leq n,\ w(i) > w(j)\}.$$

The latter set consists of the *inversions* of the permutation w; its cardinality is the *length* of w, denoted by $\ell(w)$. Thus, $C_w \cong \mathbb{C}^{\ell(w)}$.

More generally, we may define Schubert cells and varieties in any partial flag variety $X(d_1, \ldots, d_m) = G/P$, where $P = P(d_1, \ldots, P_m)$; these are parametrized by the coset space $S_n/(S_{d_1} \times \cdots \times S_{d_m}) =: W/W_P$.

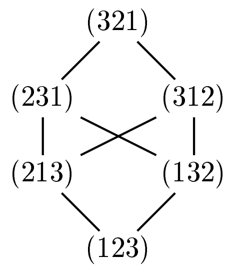
Specifically, each right coset mod W_P contains a unique permutation w such that we have $w(1) < \cdots < w(d_1)$, $w(d_1+1) < \cdots < w(d_1+d_2)$, ..., $w(d_1 + \cdots + d_{m-1} + 1) < \cdots < w(d_1 + \cdots + d_m) = w(n)$. Equivalently, $w \leq wv$ for all $v \in W_P$. This defines the set W^P of *minimal representatives* of W/W_P.

Now the Schubert cells in G/P are the orbits $C_{wP} := BwP/P = UwP/P \subset G/P$ ($w \in W^P$), and the Schubert varieties X_{wP} are their closures. One checks that the map $f: G/B \to G/P$ restricts to an isomorphism $C_w = BwB/B \to BwP/P = C_{wP}$, and hence to a birational morphism $X_w \to X_{wP}$, for any $w \in W^P$.

1.2.3 Examples. 1) The Bruhat order on S_2 is just

(21)
|
(12)

The picture of the Bruhat order on S_3 is



2) Let $w_0 := (n, n-1, \ldots, 1)$, the order-reversing permutation. Then $X = X_{w_0}$, i.e., w_0 is the unique maximal element of the Bruhat order on W. Note that $w_0^2 = \mathrm{id}$, and $\ell(w_0 w) = \ell(w_0) - \ell(w)$ for any $w \in W$.

3) The permutations of length 1 are exactly the *elementary transpositions* $s_1, \ldots, s_{n-1}$, where each s_i exchanges the indices i and $i+1$ and fixes all other indices. The corresponding Schubert varieties are the *Schubert curves* $X_{s_1}, \ldots, X_{s_{n-1}}$. In fact, X_{s_i} consists of the i-dimensional subspaces $E \subset \mathbb{C}^n$ such that $\langle e_1, \ldots, e_{i-1} \rangle \subset E \subset \langle e_1, \ldots, e_{i+1} \rangle$. Thus, X_{s_i} is the projectivization of the quotient space $\langle e_1, \ldots, e_{i+1} \rangle / \langle e_1, \ldots, e_{i-1} \rangle$, so that $X_{s_i} \cong \mathbb{P}^1$.

4) Likewise, the Schubert varieties of codimension 1 are $X_{w_0 s_1}, \ldots, X_{w_0 s_{n-1}}$, also called the *Schubert divisors.*

5) Apart from the Grassmannians, the simplest partial flag variety is the *incidence variety* $I = I_n$ consisting of the pairs (V_1, V_{n-1}), where $V_1 \subset \mathbb{C}^n$ is a line, and $V_{n-1} \subset \mathbb{C}^n$ is a hyperplane containing V_1. Denote by $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$ (resp. $\check{\mathbb{P}}^{n-1} = \mathbb{P}((\mathbb{C}^n)^*)$) the projective space of lines (resp. hyperplanes) in $\mathbb{C}^n$, then $I \subset \mathbb{P}^{n-1} \times \check{\mathbb{P}}^{n-1}$ is defined by the bi-homogeneous equation

$$x_1 y_1 + \cdots + x_n y_n = 0,$$

where $x_1, \ldots, x_n$ are the standard coordinates on $\mathbb{C}^n$, and $y_1, \ldots, y_n$ are the dual coordinates on $(\mathbb{C}^n)^*$.

One checks that the Schubert varieties in I are the

$$I_{i,j} := \{(V_1, V_{n-1}) \in I \quad | \quad V_1 \subseteq E_{1,\ldots,i} \quad \text{and} \quad E_{1,\ldots,j-1} \subseteq V_{n-1}\},$$

where $1 \leq i, j \leq n$ and $i \neq j$. Thus, $I_{i,j} \subseteq I$ is defined by the equations

$$x_{i+1} = \cdots = x_n = y_1 = \cdots = y_{j-1} = 0.$$

It follows that *$I_{i,j}$ is singular for $1 < j < i < n$ with singular locus $I_{j-1,i+1}$, and is nonsingular otherwise.*

6) For any partial flag variety G/P and any $w \in W^P$, the pull-back of the Schubert variety X_{wP} under $f : G/B \to G/P$ is easily seen to be the Schubert variety $X_{ww_{0,P}}$, where $w_{0,P}$ denotes the maximal element of W_P. Specifically, if $P = P(d_1, \ldots, d_m)$ so that $W_P = S_{d_1} \times \cdots \times S_{d_m}$, then $w_{0,P} = (w_{0,d_1}, \cdots, w_{0,d_m})$ with obvious notation. The products $ww_{0,P}$, $w \in W^P$, are the *maximal representatives* of the cosets modulo W_P. Thus, f restricts to a locally trivial fibration $X_{ww_{0,P}} \to X_{wP}$ with fiber P/B.

In particular, the preceding example yields many singular Schubert varieties in the variety of complete flags, by pull-back from the incidence variety.

1.2.4 Definition. The *opposite Schubert cell (resp. variety)* associated with $w \in W$ is $C^w := w_0 C_{w_0 w}$ (resp. $X^w := w_0 X_{w_0 w}$).

Observe that $C^w = B^- F_w$, where

$$B^- := \left\{ \begin{pmatrix} a_{1,1} & 0 & \dots & 0 \\ a_{2,1} & a_{2,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{pmatrix} \right\} = w_0 B w_0$$

(this is the opposite Borel subgroup to B containing the maximal torus T). Also, X^w has codimension $\ell(w)$ in X.

For example, $C^{\mathrm{id}} \cong U^-$ via the map $U^- \to X$, $g \mapsto gF$, where $U^- := w_0 U w_0$. Further, this map is an open immersion. Since $X = G/B$, this is equivalent to the fact that the product map $U^- \times B \to G$ is an open immersion (which, of course, may be checked directly). It follows that the quotient $q : G \to G/B$, $g \mapsto gB$, is a trivial fibration over C^{id}; thus, by G-equivariance, q is locally trivial for the Zariski topology. This also holds for any partial flag variety G/P, with the same proof. Likewise, the map $f : G/B \to G/P$ is a locally trivial fibration with fiber P/B.

1.3 Schubert classes

This subsection is devoted to the cohomology ring of the full flag variety. We begin by recalling some basic facts on the homology and cohomology of algebraic varieties, referring to [18] Appendix B for details. We will consider (co)homology groups with integer coefficients.

Let X be a projective nonsingular algebraic variety of dimension n. Then X (viewed as a compact differentiable manifold of dimension $2n$) admits a canonical orientation, hence a canonical generator of the homology group $H_{2n}(X)$: the *fundamental class* $[X]$. By Poincaré duality, the map $H^j(X) \to H_{2n-j}(X)$, $\alpha \mapsto \alpha \cap [X]$ is an isomorphism for all j.

Likewise, any nonsingular subvariety $Y \subseteq X$ of dimension p has a fundamental class in $H_{2p}(Y)$. Using Poincaré duality, the image of this class in $H_{2p}(X)$ yields the *fundamental class* $[Y] \in H^{2c}(X)$, where $c = n - p$ is the codimension of Y. In particular, we obtain the fundamental class of a point $[x]$, which is independent of x and generates the group $H^{2n}(X)$. More generally, one defines the fundamental class $[Y] \in H^{2c}(X)$ for any (possibly singular) subvariety Y, of codimension c.

Given α, β in the cohomology ring $H^*(X)$, let $\langle \alpha, \beta \rangle$ denote the coefficient of thc class $[x]$ in the cup product $\alpha \cup \beta$. Then $\langle , \rangle$ is a bilinear form on $H^*(X)$ called the *Poincaré duality pairing.* It is non-degenerate over the rationals, and even over the integers in the case where the group $H^*(X)$ is torsion-free.

For any two subvarieties Y, Z of X, each irreducible component C of $Y \cap Z$ satisfies $\dim(C) \geq \dim(Y) + \dim(Z)$, i.e., $\operatorname{codim}(C) \leq \operatorname{codim}(Y) + \operatorname{codim}(Z)$. We say that *$Y$ and Z meet properly in X*, if $\operatorname{codim}(C) = \operatorname{codim}(Y) + \operatorname{codim}(Z)$ for each C. Then we have in $H^*(X)$:

$$[Y] \cup [Z] = \sum_C m_C \, [C],$$

where the sum is over all irreducible components of $Y \cap Z$, and m_C is the *intersection multiplicity of Y and Z along C*, a positive integer. Further, $m_C = 1$ if and only if *Y and Z meet transversally along C*, i.e., there exists a point $x \in C$ such that: x is a nonsingular point of Y and Z, and the tangent spaces at x satisfy $T_xY + T_xZ = T_xX$. (Then x is a nonsingular point of C, and $T_xC = T_xY \cap T_xZ$).

In particular, if Y and Z are subvarieties such that $\dim(Y) + \dim(Z) = \dim(X)$, then Y meets Z properly if and only if their intersection is finite. In this case, we have $\langle [Y], [Z] \rangle = \sum_{x \in Y \cap Z} m_x$, where m_x denotes the intersection multiplicity at x. In the case of transversal intersection, this simplifies to $\langle [Y], [Z] \rangle = \#(Y \cap Z)$.

Returning to the case where X is a flag variety, we have the cohomology classes of the Schubert subvarieties, called the *Schubert classes.* Since X is the disjoint union of the Schubert cells, *the Schubert classes form an additive basis of $H^*(X)$*; in particular, this group is torsion-free.

To study the cup product of Schubert classes, we will need a version of Kleiman's transversality theorem, see [23] Theorem III.10.8.

1.3.1 Lemma. *Let Y, Z be subvarieties of a flag variety X and let $Y_0 \subseteq Y$ (resp. $Z_0 \subseteq Z$) be nonempty open subsets consisting of nonsingular points. Then there exists a nonempty open subset Ω of G such that: for any $g \in \Omega$, Y meets gZ properly, and $Y_0 \cap gZ_0$ is nonsingular and dense in $Y \cap gZ$. Thus, $[Y] \cup [Z] = [Y \cap gZ]$ for all $g \in \Omega$.*

In particular, if $\dim(Y) + \dim(Z) = \dim(X)$, then Y meets gZ transversally for all g in a nonempty open subset Ω of G. Thus, $Y \cap gZ$ is finite and $\langle [Y], [Z] \rangle = \#(Y \cap gZ)$, for all $g \in \Omega$.

Proof. Consider the map $m : G \times Z \to X$, $(g, z) \mapsto gz$. This is a surjective morphism, equivariant for the action of G on $G \times Z$ by left multiplication on the first factor. Since $X = G/P$, it follows that m is a locally trivial fibration for the Zariski topology. Thus, its scheme-theoretic fibers are varieties of dimension $\dim(G) + \dim(Z) - \dim(X)$.

Next consider the fibered product $V := (G \times Z) \times_X Y$ and the pull-back $\mu : V \to Y$ of m. Then μ is also a locally trivial fibration with fibers being varieties. It follows that the scheme V is a variety of dimension $\dim(G) + \dim(Z) - \dim(X) + \dim(Y)$.

Denote by $\pi : V \to G$ the composition of the projections $(G \times Z) \times_X Y \to G \times Z \to G$. Then the fiber of π at any $g \in G$ identifies to the scheme-theoretic intersection $Y \cap gZ$. Further, there exists a nonempty open subset Ω of G such that the fibers of π at points of Ω are either empty or equidimensional of dimension $\dim(Y) + \dim(Z) - \dim(X)$, i.e., of codimension $\mathrm{codim}(Y) + \mathrm{codim}(Z)$. This shows that Y meets gZ properly for any $g \in \Omega$.

Likewise, the restriction $m_0 : G \times Z_0 \to X$ is locally trivial with nonsingular fibers, so that the fibered product $V_0 := (G \times Z_0) \times_X Y_0$ is a nonempty open subset of V, consisting of nonsingular points. By generic smoothness, it follows that $Y_0 \cap gZ_0$ is nonsingular and dense in $Y \cap gZ$, for all g in a (possibly smaller) nonempty open subset of G. This implies, in turn, that all intersection multiplicities of $Y \cap Z$ are 1.

Thus, we have $[Y] \cup [gZ] = [Y \cap gZ]$ for any $g \in \Omega$. Further, $[Z] = [gZ]$ as G is connected, so that $[Y] \cup [Z] = [Y \cap gZ]$. $\square$

As a consequence, in the full flag variety X, any Schubert variety X_w meets properly any opposite Schubert variety X^v. (Indeed, the open subset Ω meets the open subset $BB^- = BU^- \cong B \times U^-$ of G; further, X_w is B-invariant, and X^v is B^--invariant). Thus, $X_w \cap X^v$ is equidimensional of dimension $\dim(X_w) + \dim(X^v) - \dim(X) = \ell(w) - \ell(v)$. Moreover, the intersection $C_w \cap C^v$ is nonsingular and dense in $X_w \cap X^v$. In fact, we have the following more precise result (which may be proved by the argument of Lemma 1.3.1, see [9] for details).

1.3.2 Proposition. *Given $v, w \in W$, the intersection $X_w \cap X^v$ is non-empty if and only if $v \leq w$; then $X_w \cap X^v$ is irreducible.*

1.3.3 Definition. Given v, w in W such that $v \leq w$, the corresponding *Richardson variety* is $X_w^v := X_w \cap X^v$.

Note that X_w^v is T-invariant with fixed points being the coordinate flags $F_x = xB/B$, where $x \in W$ satisfies $v \leq x \leq w$. It follows that $X_w^v \subseteq X_{w'}^{v'}$ if and only if $v' \leq v \leq w \leq w'$. Thus, the Richardson varieties may be viewed as geometric analogues of intervals for the Bruhat order.

1.3.4 Examples. 1) As special cases of Richardson varieties, we have the Schubert varieties $X_w = X_w^{\mathrm{id}}$ and the opposite Schubert varieties $X^v = X_{w_0}^v$. Also, note that the Richardson variety X_w^w is just the T-fixed point F_w, the transversal intersection of X_w and X^w.

2) Let X_w^v be a Richardson variety of dimension 1, that is, $v \leq w$ and $\ell(v) = \ell(w) - 1$. Then X_w^v is isomorphic to the projective line, and we have $v = ws$ for some transposition $s = s_{ij}$ (exchanging i and j, and fixing all the other indices). More generally, any T-invariant curve

$Y \subset X$ is isomorphic to $\mathbb{P}^1$ and contains exactly two T-fixed points v, w, where $v = ws$ for some transposition s. (Indeed, after multiplication by an element of W, we may assume that Y contains the standard flag F. Then $Y \cap C^{\mathrm{id}}$ is a T-invariant neighborhood of F in Y, and is also a T-invariant curve in $C^{\mathrm{id}} \cong U^-$. Now any such curve is a "coordinate line" given by $a_{i,j} = 0$ for all $(i,j) \neq (i_0, j_0)$, for some (i_0, j_0) such that $1 \leq j_0 < i_0 \leq n$. The closure of this line in X has fixed points F and $s_{i_0,j_0}F$.)

Richardson varieties may be used to describe the local structure of Schubert varieties along Schubert subvarieties, as follows.

1.3.5 Proposition. *Let v, $w \in W$ such that $v \leq w$. Then $X_w \cap vC^{\mathrm{id}}$ is an open T-invariant neighborhood of the point F_v in X_w, which meets X_w^v along $X_w \cap C^v$. Further, the map*

$$(U \cap vU^-v^{-1}) \times (X_w \cap C^v) \to X_w, \quad ,(g,x) \mapsto gx$$

is an open immersion with image $X_w \cap vC^{\mathrm{id}}$. (Recall that $U \cap vU^-v^{-1}$ is isomorphic to $\mathbb{C}^{\ell(v)}$, and that the map $U \cap vU^-v^{-1} \to X$, $g \mapsto gF_v$ is an isomorphism onto C_v.)

If, in addition, $\ell(v) = \ell(w) - 1$, then $X_w \cap C^v$ is isomorphic to the affine line. As a consequence, X_w is nonsingular along its divisor X_v.

Proof. Note that vC^{id} is an open T-invariant neighborhood of F_v in X, isomorphic to the group vU^-v^{-1}. In turn, the latter is isomorphic to $(U \cap vU^-v^{-1}) \times (U^- \times vU^-v^{-1})$ via the product map; and the map $U^- \times vU^-v^{-1} \to X$, $g \mapsto gF_v$ is a locally closed immersion with image C^v. It follows that the map

$$(U \cap vU^-v^{-1}) \times C^v \to X, \quad ,(g,x) \mapsto gx$$

is an open immersion with image vF^{id}, and that $vF^{\mathrm{id}} \cap X^v = C^v$. Intersecting with the subvariety X_w (invariant under $U \cap vU^-v^{-1}$) completes the proof of the first assertion. The second assertion follows from the preceding example. □

Richardson varieties also appear when multiplying Schubert classes. Indeed, by Proposition 1.3.2, we have in $H^*(X)$:

$$[X_w] \cup [X^v] = [X_w^v].$$

Since $\dim(X_w^v) = \ell(w) - \ell(v)$, it follows that $\langle [X_w], [X^v] \rangle$ equals 1 if $w = v$, and 0 otherwise. This implies easily the following result.

1.3.6 Proposition. *(i) The bases $\{[X_w]\}$ and $\{[X^w]\} = \{[X_{w_0 w}]\}$ of $H^*(X)$ are dual for the Poincaré duality pairing.*
(ii) For any subvariety $Y \subseteq X$, we have

$$[Y] = \sum_{w \in W} a^w(Y)\ [X_w],$$

where $a^w(Y) = \langle [Y], [X^w] \rangle = \#(Y \cap gX^w)$ *for all* g *in a non-empty Zariski open subset of* G*. In particular, the coefficients of* $[Y]$ *in the basis of Schubert classes are non-negative.*
(iii) Let

$$[X_v] \cup [X_w] = \sum_{x \in W} a_{vw}^x \, [X_x] \quad \textit{in} \quad H^*(X),$$

then the integers a_{vw}^x *are non-negative.*

Note finally that all these results adapt readily to any partial flag variety G/P. In fact, the map $f : G/B \to G/P$ induces a ring homomorphism $f^* : H^*(G/P) \to H^*(G/B)$ which sends any Schubert class $[X_{wP}]$ to the Schubert class $[X_{ww_{0,P}}]$, where $w \in W^P$. In particular, f^* is injective.

1.4 The Picard group

In this subsection, we study the Picard group of the full flag variety $X = G/B$. We first give a very simple presentation of this group, viewed as the group of divisors modulo linear equivalence.

1.4.1 Proposition. *The group* $\mathrm{Pic}(X)$ *is freely generated by the classes of the Schubert divisors* $X_{w_0 s_i}$ *where* $i = 1, \ldots, n-1$*. Any ample (resp. generated by its global sections) divisor on* X *is linearly equivalent to a positive (resp. non-negative) combination of these divisors. Further, any ample divisor is very ample.*

Proof. The open Schubert cell C_{w_0} has complement the union of the Schubert divisors. Since C_{w_0} is isomorphic to an affine space, its Picard group is trivial. Thus, the classes of $X_{w_0 s_1}, \ldots, X_{w_0 s_{n-1}}$ generate the group $\mathrm{Pic}(X)$.

If we have a relation $\sum_{i=1}^{n-1} a_i X_{w_0 s_i} = 0$ in $\mathrm{Pic}(X)$, then there exists a rational function f on X having a zero or pole of order a_i along each $X_{w_0 s_i}$, and no other zero or pole. In particular, f is a regular, nowhere vanishing function on the affine space C_{w_0}. Hence f is constant, and $a_i = 0$ for all i.

Each Schubert divisor $X_{w_0 s_d}$ is the pull-back under the projection $X \to \mathrm{Grass}(d, n)$ of the unique Schubert divisor in $\mathrm{Grass}(d, n)$. Since the latter divisor is a hyperplane section in the Plücker embedding, it follows that $X_{w_0 s_d}$ is generated by its global sections. As a consequence, any non-negative combination of Schubert divisors is generated by its global sections. Further, the divisor $\sum_{d=1}^{n-1} X_{w_0 s_d}$ is very ample, as the product map $X \to \prod_{d=1}^{n-1} \mathrm{Grass}(d, n)$ is a closed immersion. Thus, any positive combination of Schubert divisors is very ample.

Conversely, let $D = \sum_{i=1}^{n-1} a_i X_{w_0 s_i}$ be a globally generated (resp. ample) divisor on X. Then for any curve Y on X, the intersection number $\langle [D], [Y] \rangle$ is non-negative (resp. positive). Now take for Y a Schubert curve X_{s_j}, then

$$\langle [D],[Y]\rangle = \langle \sum_{i=1}^{n-1} a_i[X_{w_0 s_i}],[X_{s_j}]\rangle = \sum_{i=1}^{n-1} a_i \langle [X^{s_i}],[X_{s_j}]\rangle = a_j.$$

This completes the proof. □

1.4.2 Remark. We may assign to each divisor D on X, its cohomology class $[D] \in H^2(X)$. Since linearly equivalent divisors are homologically equivalent, this defines the cycle map $\mathrm{Pic}(X) \to H^2(X)$, which is an isomorphism by Proposition 1.4.1.

More generally, assigning to each subvariety of X its cohomology class yields the *cycle map* $A^*(X) \to H^{2*}(X)$, where $A^*(X)$ denotes the *Chow ring* of rational equivalence classes of algebraic cycles on X (graded by the codimension; in particular, $A^1(X) = \mathrm{Pic}(X)$). Since X has a "cellular decomposition" by Schubert cells, *the cycle map is a ring isomorphism* by [19] Example 19.1.11.

We will see in Section 4 that *the ring* $H^*(X)$ *is generated by* $H^2(X) \cong \mathrm{Pic}(X)$*, over the rationals.* (In fact, this holds over the integers for the variety of complete flags, as follows easily from its structure of iterated projective space bundle.)

Next we obtain an alternative description of $\mathrm{Pic}(X)$ in terms of homogeneous line bundles on X; these can be defined as follows. Let λ be a *character* of B, i.e., a homomorphism of algebraic groups $B \to \mathbb{C}^*$. Let B act on the product $G \times \mathbb{C}$ by $b(g,t) := (gb^{-1}, \lambda(b)t)$. This action is free, and the quotient

$$L_\lambda = G \times^B \mathbb{C} := (G \times \mathbb{C})/B$$

maps to G/B via $(g,t)B \mapsto gB$. This makes L_λ the total space of a line bundle over G/B, the *homogeneous line bundle associated to the weight* λ.

Note that G acts on L_λ via $g(h,t)B := (gh,t)B$, and that the projection $f : L_\lambda \to G/B$ is G-equivariant; further, any $g \in G$ induces a linear map from the fiber $f^{-1}(x)$ to $f^{-1}(gx)$. In other words, L_λ is a *G-linearized line bundle* on X.

We now describe the characters of B. Note that any such character λ is uniquely determined by its restriction to T (since $B = TU$ and U is isomorphic to an affine space, so that any regular invertible function on U is constant). Further, one easily sees that the characters of the group T of diagonal invertible matrices are the maps

$$\mathrm{diag}(t_1,\ldots,t_n) \mapsto t_1^{\lambda_1} \cdots t_n^{\lambda_n},$$

where $\lambda_1,\ldots,\lambda_n$ are integers. This identifies the multiplicative group of characters of B (also called *weights*) to the additive group $\mathbb{Z}^n$.

Next we express the *Chern classes* $c_1(L_\lambda) \in H^2(X) \cong \mathrm{Pic}(X)$ in the basis of Schubert divisors. More generally, we obtain the *Chevalley formula* which decomposes the products $c_1(L_\lambda) \cup [X_w]$ in this basis.

1.4.3 Proposition. *For any weight* λ *and any* $w \in W$*, we have*

$$c_1(L_\lambda) \cup [X_w] = \sum (\lambda_i - \lambda_j)\ [X_{ws_{ij}}],$$

where the sum is over the pairs (i,j) *such that* $1 \leq i < j \leq n$*,* $ws_{ij} < w$*, and* $\ell(ws_{ij}) = \ell(w) - 1$*. In particular,*

$$c_1(L_\lambda) = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\ [X_{w_0 s_i}] = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\ [X^{s_i}].$$

Thus, the map $\mathbb{Z}^n \to \mathrm{Pic}(X)$*,* $\lambda \mapsto c_1(L_\lambda)$ *is a surjective group homomorphism, and its kernel is generated by* $(1,\ldots,1)$*.*

Proof. We may write

$$c_1(L_\lambda) \cup [X_w] = \sum_{v \in W} a_v\ [X_v],$$

where the coefficients a_v are given by

$$a_v = \langle c_1(L_\lambda) \cup [X_w], [X^v] \rangle = \langle c_1(L_\lambda), [X_w] \cup [X^v] \rangle = \langle c_1(L_\lambda), [X_w^v] \rangle.$$

Thus, a_v is the degree of the restriction of L_λ to X_w^v if $\dim(X_w^v) = 1$, and is 0 otherwise. Now $\dim(X_w^v) = 1$ if and only if : $v < w$ and $\ell(v) = \ell(w) - 1$. Then $v = ws_{ij}$ for some transposition s_{ij}, and X_w^v is isomorphic to $\mathbb{P}^1$, by Example 1.3.4.2. Further, one checks that the restriction of L_λ to X_w^v is isomorphic to the line bundle $\mathcal{O}_{\mathbb{P}^1}(\lambda_i - \lambda_j)$, of degree $\lambda_i - \lambda_j$. □

This relation between weights and line bundles motivates the following

1.4.4 Definition. We say that the weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ is *dominant* (resp. *regular dominant*), if $\lambda_1 \geq \cdots \geq \lambda_n$ (resp. $\lambda_1 > \cdots > \lambda_n$).

The *fundamental weights* are the weights $\chi_1, \ldots, \chi_{n-1}$ such that

$$\chi_j := (1, \ldots, 1\ (j \text{ times}), 0, \ldots, 0\ (n-j \text{ times})).$$

The *determinant* is the weight $\chi_n := (1, \ldots, 1)$. We put

$$\rho := \chi_1 + \cdots + \chi_{n-1} = (n-1, n-2, \ldots, 1, 0).$$

By Propositions 1.4.1 and 1.4.3, the line bundle L_λ is globally generated (resp. ample) if and only if the weight λ is dominant (resp. regular dominant). Further, the dominant weights are the combinations $a_1\chi_1 + \cdots + a_{n-1}\chi_{n-1} + a_n\chi_n$, where $a_1, \ldots, a_{n-1}$ are non-negative integers, and a_n is an arbitrary integer; χ_n is the restriction to T of the determinant function on G. For $1 \leq d \leq n-1$, the line bundle $L(\chi_d)$ is the pull-back of $\mathcal{O}(1)$ under the composition $X \to \mathrm{Grass}(d,n) \to \bigwedge^d \mathbb{C}^n$. Further, we have by Proposition 1.4.3:

$$c_1(L_{\chi_d}) \cup [X_w] = [X_{w_0 s_d}] \cup [X_w] = \sum_v [X_v],$$

the sum over the $v \in W$ such that $v \leq w$, $\ell(v) = \ell(w) - 1$, and $v = ws_{ij}$ with $i < d < j$.

We now consider the spaces of global sections of homogeneous line bundles. For any weight λ, we put

$$H^0(\lambda) := H^0(X, L_\lambda).$$

This is a finite-dimensional vector space, as X is projective. Further, since the line bundle L_λ is G-linearized, the space $H^0(\lambda)$ is a *rational G-module*, i.e., G acts linearly on this space and the corresponding homomorphism $G \to \mathrm{GL}(H^0(\lambda))$ is algebraic. Further properties of this space, and a refinement of Proposition 1.4.3, are given by the following

1.4.5 Proposition. *The space $H^0(\lambda)$ is non-zero if and only if λ is dominant. Then $H^0(\lambda)$ contains a unique line of eigenvectors of the subgroup B^-, and the corresponding character of B^- is $-\lambda$. The divisor of any such eigenvector p_λ satisfies*

$$\mathrm{div}(p_\lambda) = \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})\, X^{s_i}.$$

More generally, for any $w \in W$, the G-module $H^0(\lambda)$ contains a unique line of eigenvectors of the subgroup wB^-w^{-1}, and the corresponding weight is $-w\lambda$. Any such eigenvector $p_{w\lambda}$ has a non-zero restriction to X_w, with divisor

$$\mathrm{div}(p_{w\lambda}|_{X_w}) = \sum (\lambda_i - \lambda_j) X_{ws_{ij}},$$

the sum over the pairs $i < j$ such that $X_{ws_{ij}}$ is a divisor in X_w. (This makes sense as X_w is nonsingular in codimension 1, see Proposition 1.3.5.)

In particular, taking $\lambda = \rho$, the zero locus of $p_{w\rho}|_{X_w}$ is exactly the union of all the Schubert subvarieties of codimension 1 in X_w.

Proof. If λ is dominant, then we know that L_λ is generated by its global sections, and hence admits a non-zero section. Conversely, if $H^0(\lambda) \neq 0$ then L_λ has a section σ which does not vanish at some point of X. Since X is homogeneous, the G-translates of σ generate L_λ. Thus, L_λ is dominant.

Now choose a dominant weight λ and put $D := \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1}) X^{s_i}$. By Proposition 1.4.3, we have $L_\lambda \cong \mathcal{O}_X(D)$, so that L_λ admits a section σ with divisor D. Since D is B^--invariant, σ is a B^--eigenvector; in particular, a T-eigenvector. And since D does not contain the standard flag F, it follows that $\sigma(F) \neq 0$. But T acts on the fiber of L_λ at F

by the weight λ, so that σ has weight $-\lambda$. If σ' is another B-eigenvector in $H^0(\lambda)$, then the quotient σ'/σ is a rational function on X, which is U^--invariant as σ and σ' are. Since U^-F is open in X, it follows that σ' is a scalar multiple of σ.

By G-equivariance, it follows that $H^0(\lambda)$ contains a unique line of eigenvectors of the subgroup wB^-w^{-1}, with weight $-w\lambda$. Let $p_{w\lambda}$ be such an eigenvector, then $p_{w\lambda}$ does not vanish at F_w, hence (by T-equivariance) it has no zero on C_w. So the zero locus of the restriction $p_{w\lambda}|_{X_w}$ has support in $X_w \setminus C_w$ and hence is B-invariant. The desired formula follows by the above argument together with Proposition 1.4.3. □

1.4.6 Remark. For any dominant weight λ, the G-module $H^0(\lambda)$ contains a unique line of eigenvectors for $B = w_0B^-w_0$, of weight $-w_0\lambda$. On the other hand, the evaluation of sections at the base point B/B yields a non-zero linear map $H^0(\lambda) \to \mathbb{C}$ which is a B-eigenvector of weight λ. In other words, the dual G-module

$$V(\lambda) := H^0(\lambda)^*$$

contains a canonical B-eigenvector of weight λ.

One can show that both G-modules $H^0(\lambda)$ and $V(\lambda)$ are *simple*, i.e., they admit no non-trivial proper submodules. Further, any simple rational G-module V is isomorphic to $V(\lambda)$ for a unique dominant weight λ, the *highest weight* of V. The T-module $V(\lambda)$ is the sum of its weight subspaces, and the corresponding weights lie in the convex hull of the orbit $W\lambda$. For these results, see e.g. [18] 8.2 and 9.3.

1.4.7 Example. For $d = 1, \ldots, n-1$, the space $\bigwedge^d \mathbb{C}^n$ has a basis consisting of the vectors

$$e_I := e_{i_1} \wedge \cdots \wedge e_{i_d},$$

where $I = (i_1, \ldots, i_d)$ and $1 \le i_1 < \cdots < i_d \le n$. These vectors are T-eigenvectors with pairwise distinct weights, and they form a unique orbit of W. It follows easily that the G-module $\bigwedge^d \mathbb{C}^n$ is simple, with highest weight χ_d (the weight of the unique B-eigenvector $e_{1\ldots d}$). In other words, we have $V(\chi_d) = \wedge^d \mathbb{C}^n$, so that $H^0(\chi_d) = (\bigwedge^d \mathbb{C}^n)^*$.

Denote by $p_I \in (\bigwedge^d \mathbb{C}^n)^*$ the elements of the dual basis of the basis $\{e_I\}$ of $\bigwedge^d \mathbb{C}^n$. The p_I are homogeneous coordinates on $\mathrm{Grass}(d, n)$, the *Plücker coordinates*. From the previous remark, one readily obtains that

$$\mathrm{div}(p_I|_{X_I}) = \sum_{J<I,\, |J|=|I|-1} X_J.$$

This is a refined version of the *Pieri formula*

$$c_1(L) \cup [X_I] = \sum_{J<I,\, |J|=|I|-1} [X_J]$$

in $H^*(\mathrm{Grass}(d,n))$, where L denotes the pull-back of $\mathcal{O}(1)$ via the Plücker embedding. Note that $c_1(L)$ is the class of the unique Schubert divisor.

Notes. The results of this lecture are classical; they may be found with more details in [18] and [35]. We also refer to [47] Chapter 8 for an exposition of the theory of reductive algebraic groups, with some fundamental results on their Schubert varieties. Further references are the survey [47] of Schubert varieties and their generalizations in this setting, and the book [28] regarding the general framework of Kac-Moody groups.

The irreducibility of the intersections $X_w \cap X^v$ is due to Richardson [45], whereas the intersections $C_w \cap C^v$ have been studied by Deodhar [15]. The Richardson varieties play an important role in several recent works, in relation to standard monomial theory; see [33], [30], [29], [9].

The decomposition of the products $c_1(L_\lambda) \cup [X_w]$ in the basis of Schubert classes is due to Monk [41] for the variety of complete flags, and to Chevalley [11] in general. The Chevalley formula is equivalent to the decomposition into Schubert classes of the products of classes of Schubert divisors by arbitrary Schubert classes. This yields a closed formula for certain structure constants a_{vw}^x of $H^*(X)$; specifically, those where $v = w_0 s_d$ for some elemntary transposition s_d. However, the only known proof of the positivity of the general structure constants is geometric. In fact, a central problem in Schubert calculus is to find a combinatorial expression for these constants, which makes their positivity evident.

2 Singularities of Schubert varieties

As seen in Examples 1.1.4 and 1.2.3, Schubert varieties are generally singular. In this section, we show that their singularities are rather mild. We begin by showing that they are normal. Then we introduce the Bott-Samelson desingularizations, and we establish the rationality of singularities of Schubert varieties. In particular, these are Cohen-Macaulay; we also describe their dualizing sheaves. Finally, we obtain the vanishing of all higher cohomology groups $H^j(X_w, L_\lambda)$ where λ is any dominant weight, and the surjectivity of the restriction map $H^0(\lambda) \to H^0(X_w, L_\lambda)$.

2.1 Normality

First we review an inductive construction of Schubert cells and varieties. Given $w \in W$ and an elementary transposition s_i, we have either $\ell(s_i w) = \ell(w) - 1$ (and then $s_i w < w$), or $\ell(s_i w) = \ell(w) + 1$ (and then $s_i w > w$). In the first case, we have $Bs_iC_w = C_w \cup C_{s_i w}$, whereas $Bs_iC_w = C_{s_i w}$ in the second case. Further, if $w \neq \mathrm{id}$ (resp. $w \neq w_0$), then there exists an index i such that the first (resp. second) case occurs. (These properties of the Bruhat decomposition are easily checked in the case of the general linear group; for arbitrary reductive groups, see e.g. [47].)

Next let P_i be the subgroup of $G = \mathrm{GL}_n(\mathbb{C})$ generated by B and s_i. (This is a minimal parabolic subgroup of G.) Then P_i is the stabilizer of the partial flag consisting of all the standard coordinate subspaces, except $\langle e_1, \ldots, e_i \rangle$. Further, P_i/B is the Schubert curve X_{s_i} (isomorphic to $\mathbb{P}^1$), and $P_i = B \cup Bs_iB$ is the closure in G of Bs_iB.

The group B acts on the product $P_i \times X_w$ by $b(g, x) := (gb^{-1}, bx)$. This action is free; denote the quotient by $P_i \times^B X_w$. Then the map

$$P_i \times X_w \to P_i \times X, \quad (g, x) \mapsto (g, gx)$$

yields a map

$$\iota : P_i \times^B X_w \to P_i/B \times X, \quad (g, x)B \mapsto (gB, gx).$$

Clearly, ι is injective and its image consists of those pairs $(gB, x) \in P_i/B \times X$ such that $g^{-1}x \in X_w$; this defines a closed subset of $P_i/B \times X$. It follows that $P_i \times^B X_w$ is a projective variety, equipped with a proper morphism

$$\pi : P_i \times^B X_w \to X$$

with image $P_i X_w$, and with a morphism

$$f : P_i \times^B X_w \to P_i/B \cong \mathbb{P}^1.$$

The action of P_i by left multiplication on itself yields an action on $P_i \times^B X_w$; the maps π and f are P_i-equivariant. Further, f is a locally trivial fibration with fiber $B \times^B X_w \cong X_w$.

In particular, P_iX_w is closed in X, and hence is the closure of Bs_iC_w. If $s_iw < w$, then $P_iX_w = X_w$. Then one checks that $P_i \times^B X_w$ identifies to $P_i/B \times X_w$, so that π becomes the second projection. On the other hand, if $s_iw > w$, then $P_iX_w = X_{s_iw}$. Then one checks that π restricts to an isomorphism

$$Bs_iB \times^B C_w \to Bs_iC_w = C_{s_iw},$$

so that π is birational onto its image X_{s_iw}.

We are now in a position to prove

2.1.1 Theorem. *Any Schubert variety X_w is normal.*

Proof. We argue by decreasing induction on $\dim(X_w) =: \ell$. In the case where $\ell = \dim(X)$, the variety $X_w = X$ is nonsingular and hence normal. So we may assume that $\ell < \dim(X)$ and that all Schubert varieties of dimension $> \ell$ are normal. Then we may choose an elementary transposition s_i such that $s_iw > w$. We divide the argument into three steps.

Step 1. We show that the morphism $\pi : P_i \times^B X_w \to X_{s_iw}$ satisfies $R^j\pi_*\mathcal{O}_{P_i \times^B X_w} = 0$ for all $j \geq 1$.

Indeed, π factors as the closed immersion

$$\iota : P_i \times^B X_w \to P_i/B \times X_{s_iw} \cong \mathbb{P}^1 \times X_{s_iw}, \quad (g,x)B \mapsto (gB, gx)$$

followed by the projection

$$p : \mathbb{P}^1 \times X_{s_iw} \to X_{s_iw}, \quad (z,x) \mapsto x.$$

Thus, the fibers of π are closed subschemes of $\mathbb{P}^1$ and it follows that $R^j\pi_*\mathcal{O}_{P/B \times X_w} = 0$ for $j > 1 = \dim \mathbb{P}^1$.

It remains to check the vanishing of $R^1\pi_*\mathcal{O}_{P_i \times^B X_w}$. For this, we consider the following short exact sequence of sheaves:

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^1 \times X_{s_iw}} \to \iota_*\mathcal{O}_{P_i \times^B X_w} \to 0,$$

where $\mathcal{I}$ denotes the ideal sheaf of $P_i \times^B X_w$ in $\mathbb{P}^1 \times X_{s_iw}$. The derived long exact sequence for p yields an exact sequence

$$R^1p_*\mathcal{O}_{\mathbb{P}^1 \times X_{s_iw}} \to R^1p_*(\iota_*\mathcal{O}_{P_i \times^B X_w}) \to R^2p_*\mathcal{I}.$$

Further, $R^1p_*\mathcal{O}_{\mathbb{P}^1 \times X_{s_iw}} = 0$ as $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$; $R^1p_*(\iota_*\mathcal{O}_{P_i \times^B X_w}) = R^1\pi_*\mathcal{O}_{P_i \times^B X_w}$ as ι is a closed immersion; and $R^2p_*\mathcal{I} = 0$ as all the fibers of p have dimension 1. This yields the desired vanishing.

Step 2. We now analyze the normalization map

$$\nu : \tilde{X}_w \to X_w.$$

We have an exact sequence of sheaves

$$0 \to \mathcal{O}_{X_w} \to \nu_* \mathcal{O}_{\tilde{X}_w} \to \mathcal{C} \to 0,$$

where $\mathcal{C}$ is a coherent sheaf with support the non-normal locus of X_w. Further, the action of B on X_w lifts to an action on $\tilde{X}_w$ so that ν is equivariant. Thus, both sheaves $\mathcal{O}_{X_w}$ and $\nu_*\mathcal{O}_{\tilde{X}_w}$ are B-linearized, and $\mathcal{C}$ is B-linearized as well. (See [8] §2 for details on linearized sheaves.)

Now any B-linearized coherent sheaf $\mathcal{F}$ on X_w yields an "induced " P_i-linearized sheaf $P_i \times^B \mathcal{F}$ on $P_i \times^B X_w$ (namely, the unique P_i-linearized sheaf which pulls back to the B-linearized sheaf $\mathcal{F}$ under the inclusion $X_w \cong B \times^B X_w \to P_i \times^B X_w$). Therefore, one obtains a short exact sequence of P_i-linearized sheaves on $P_i \times^B X_w$:

$$0 \to \mathcal{O}_{P_i \times^B X_w} \to (P_i \times^B \nu)_* \mathcal{O}_{P_i \times^B \tilde{X}_w} \to P_i \times^B \mathcal{C} \to 0.$$

Apply π_*, we obtain an exact sequence of sheaves on $X_{s_i w}$:

$$0 \to \pi_*\mathcal{O}_{P_i \times^B X_w} \to \pi_*(P_i \times^B \nu)_* \mathcal{O}_{P_i \times^B \tilde{X}_w} \to \pi_*(P_i \times^B \mathcal{C}) \to R^1\pi_*\mathcal{O}_{P_i \times^B X_w}.$$

Now $\pi_*\mathcal{O}_{P_i \times^B X_w} = \mathcal{O}_{X_{s_i w}}$ by Zariski's main theorem, since $\pi : P_i \times^B X_w \to X_{s_i w}$ is a proper birational morphism, and $X_{s_i w}$ is normal by the induction assumption. Likewise, $\pi_*(P_i \times^B \nu)_* \mathcal{O}_{P_i \times^B \tilde{X}_w} = \mathcal{O}_{X_{s_i w}}$. Further, $R^1\pi_*\mathcal{O}_{P_i \times^B X_w} = 0$ by Step 1. It follows that $\pi_*(P_i \times^B \mathcal{C}) = 0$.

Step 3. Finally, we assume that X_w is non-normal and we derive a contradiction.

Recall that the support of $\mathcal{C}$ is the non-normal locus of X_w. By assumption, this is a non-empty B-invariant closed subset of X. Thus, the irreducible components of supp($\mathcal{C}$) are certain Schubert varieties X_v. Choose such a v and let $\mathcal{C}_v$ denote the subsheaf of $\mathcal{C}$ consisting of sections killed by the ideal sheaf of X_v in X_w. Then supp($\mathcal{C}_v$) $= X_v$, since X_v is an irreducible component of supp($\mathcal{C}$). Further, $\pi_*(P_i \times^B \mathcal{C}_v) = 0$, since $\mathcal{C}_v$ is a subsheaf of $\mathcal{C}$.

Now choose the elementary transposition s_i such that $v < s_i v$. Then $w < s_i w$ (otherwise, $P_i X_w = X_w$, so that P_i stabilizes the non-normal locus of X_w; in particular, P_i stabilizes X_v, whence $s_i v < v$). Thus, the morphism $\pi : P_i \times^B X_v \to X_{s_i v}$ restricts to an isomorphism above $C_{s_i v}$. Since supp($P_i \times^B \mathcal{C}_v$) $= P_i \times^B X_v$, it follows that the support of $\pi_*(P_i \times^B \mathcal{C}_v)$ contains $C_{s_i v}$, i.e., this support is the whole $X_{s_i v}$. In particular, $\pi_*(P_i \times^B \mathcal{C}_v)$ is non-zero, which yields the desired contradiction. □

2.2 Rationality of singularities

Let $w \in W$. If $w \neq \text{id}$ then there exists a simple transposition s_{i_1} such that $\ell(s_{i_1} w) = \ell(w) - 1$. Applying this to $s_{i_1} w$ and iterating this process, we obtain a decomposition

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}, \quad \text{where} \quad \ell = \ell(w).$$

We then say that the sequence of simple transpositions

$$\underline{w} := (s_{i_1}, s_{i_2}, \ldots, s_{i_\ell})$$

is a *reduced decomposition* of w.

For such a decomposition, we have $X_w = P_{i_1} X_{s_{i_1} w} = P_{i_1} P_{i_2} \cdots P_{i_\ell}/B$. We put $v := s_{i_1} w$ and $\underline{v} := (s_{i_2}, \ldots, s_{i_\ell})$, so that $\underline{w} = (s_{i_1}, \underline{v})$ and $X_w = P_{i_1} X_v$. We define inductively the *Bott-Samelson variety* $Z_{\underline{w}}$ by

$$Z_{\underline{w}} := P_{i_1} \times^B Z_{\underline{v}}.$$

Thus, $Z_{\underline{w}}$ is equipped with an equivariant fibration to $P_{i_1}/B \cong \mathbb{P}^1$, with fiber $Z_{\underline{v}}$ at the base point. Further, $Z_{\underline{w}}$ is the quotient of the product $P_{i_1} \times \cdots \times P_{i_\ell}$ by the action of B^ℓ via

$$(b_1, \ldots, b_\ell)(g_1, g_2, \ldots, g_\ell) = (g_1 b_1^{-1}, b_1 g_2 b_2^{-1}, \ldots, b_{\ell-1} g_\ell b_\ell^{-1}).$$

The following statement is easily checked.

2.2.1 Proposition. *(i) The space $Z_{\underline{w}}$ is a nonsingular projective B-variety of dimension ℓ, where B acts via $g(g_1, \ldots, g_\ell)B^\ell := (gg_1, \ldots, g_\ell)B^\ell$. For any subsequence $\underline{v}$ of $\underline{w}$, we have a closed B-equivariant immersion $Z_{\underline{v}} \to Z_{\underline{w}}$.*
(ii) The map

$$Z_{\underline{w}} \to (G/B)^\ell = X^\ell, \quad (g_1, g_2, \ldots, g_\ell)B^\ell \mapsto (g_1 B, g_1 g_2 B, \ldots, g_1 \cdots g_\ell B)$$

is a closed B-equivariant embedding.
(iii) The map

$$\varphi : Z_{\underline{w}} = Z_{s_{i_1}, \ldots, s_{i_\ell}} \to Z_{s_{i_1}, \ldots, s_{i_{\ell-1}}}, \quad (g_1, \ldots, g_\ell)B^\ell \mapsto (g_1, \ldots, g_{\ell-1})B^{\ell-1}$$

is a B-equivariant locally trivial fibration with fiber $P_{i_\ell}/B \cong \mathbb{P}^1$.
(iv) The map

$$\pi = \pi_{\underline{w}} : Z_{\underline{w}} \to P_{i_1} \cdots P_{i_\ell}/B = X_w, \quad (g_1, \ldots, g_\ell)B^\ell \mapsto g_1 \cdots g_\ell B,$$

is a proper B-equivariant morphism, and restricts to an isomorphism over C_w. In particular, π is birational.

An interesting combinatorial consequence of this Proposition is the following description of the Bruhat order (which may also be proved directly).

2.2.2 Corollary. *Let $v, w \in W$. Then $v \leq w$ if and only if there exist a reduced decomposition $\underline{w} = (s_{i_1}, \ldots, s_{i_\ell})$, and a subsequence $\underline{v} = (s_{j_1}, \ldots, s_{j_m})$ with product v. Then there exists a reduced subsequence $\underline{v}$ with product v.*

As a consequence, $v < w$ if and only if there exists a sequence $(v_1, \ldots, v_k)$ in W such that: $v = v_1 < \cdots < v_k = w$, and $\ell(v_{j+1}) = \ell(v_j) + 1$ for all j.

Proof. Since π is a proper T-equivariant morphism, any fiber at a T-fixed point contains a fixed point (by Borel's fixed point theorem, see e.g. [47] Theorem 6.2.6). But the fixed points in X_w (resp. $Z_{\underline{w}}$) correspond to the $v \in W$ such that $v \leq w$ (resp. to the subsequences of $\underline{w}$). This proves the first assertion.

If $v = s_{j_1} \cdots s_{j_m}$, then the product $Bs_{j_1}B \cdots Bs_{j_m}B/B$ is open in X_v. By induction on m, it follows that there exists a reduced subsequence $(s_{k_1}, \ldots, s_{k_n})$ of $(s_{j_1}, \ldots, s_{j_m})$ such that $Bs_{k_1}B \cdots Bs_{k_n}B/B$ is open in X_v; then $v = s_{k_1} \cdots s_{k_n}$. This proves the second assertion.

The third assertion follows from the second one. Alternatively, one may observe that the complement $X_w \setminus C_w$ has pure codimension one in X_w, since C_w is an affine open subset of X_w. Thus, for any $v < w$ there exists $x \in W$ such that $v \leq x < w$ and $\ell(x) = \ell(w) - 1$. Now induction on $\ell(w) - \ell(v)$ completes the proof. □

2.2.3 Theorem. *The morphism $\pi : Z_{\underline{w}} \to X_w$ satisfies $\pi_* \mathcal{O}_{Z_{\underline{w}}} = \mathcal{O}_{X_w}$, and $R^j \pi_* \mathcal{O}_{Z_{\underline{w}}} = 0$ for all $j \geq 1$.*

Proof. We argue by induction on $\ell = \ell(w)$, the case where $\ell = 0$ being trivial. For $\ell \geq 1$, we may factor $\pi = \pi_{\underline{w}}$ as

$$P_{i_1} \times^B \pi_{\underline{v}} : P_{i_1} \times^B Z_{\underline{v}} \to P_{i_1} \times^B X_v, \quad (g, z)B \mapsto (g, \pi_v(z))B$$

followed by the map

$$\pi_1 : P_{i_1} \times^B X_v \to X_w, \quad (g, x)B \mapsto gx.$$

By the induction assumption, the morphism $\pi_{\underline{v}}$ satisfies the conclusions of the theorem. It follows easily that so does the induced morphism $P_{i_1} \times^B \pi_{\underline{v}}$. But the same holds for the morphism π_1, by the first step in the proof of Theorem 2.1.1. Now the Grothendieck spectral sequence for the composition $\pi_1 \circ (P_{i_1} \times^B \pi_{\underline{v}}) = \pi_{\underline{w}}$ yields the desired statements. □

Thus, π is a desingularization of the Schubert variety X_w, and the latter has rational singularities, in the sense of the following

2.2.4 Definition. A *desingularization* of an algebraic variety Y consists of a nonsingular algebraic variety Z together with a proper birational morphism $\pi : Z \to Y$. We say that Y has *rational singularities*, if there exists a desingularization $\pi : Z \to Y$ satisfying $\pi_* \mathcal{O}_Z = \mathcal{O}_Y$ and $R^j \pi_* \mathcal{O}_Z = 0$ for all $j \geq 1$.

Note that the equality $\pi_* \mathcal{O}_Z = \mathcal{O}_Y$ is equivalent to the normality of Y, by Zariski's main theorem. Also, one can show that Y has rational singularities if and only if $\pi_* \mathcal{O}_Z = \mathcal{O}_Y$ and $R^j \pi_* \mathcal{O}_Z = 0$ for all $j \geq 1$, where $\pi : Z \to Y$ is *any* desigularization.

Next we recall the definition of the *canonical sheaf* ω_Y of a normal variety Y. Let $\iota : Y^{\mathrm{reg}} \to Y$ denote the inclusion of the nonsingular locus, then $\omega_Y := \iota_* \omega_{Y^{\mathrm{reg}}}$, where $\omega_{Y^{\mathrm{reg}}}$ denotes the sheaf of differential forms of maximal degree on the nonsingular variety

Y^{reg}. Since the sheaf $\omega_{Y^{\mathrm{reg}}}$ is invertible and $\mathrm{codim}(Y - Y^{\mathrm{reg}}) \geq 2$, the canonical sheaf is the sheaf of local sections of a Weil divisor K_Y: the *canonical divisor*, defined up to linear equivalence. If, in addition, Y is Cohen-Macaulay, then ω_Y is its *dualizing sheaf.*

For any desingularization $\pi : Z \to Y$ where Y is normal, we have an injective trace map $\pi_* \omega_Z \to \omega_Y$. Further, $R^j \pi_* \omega_Z = 0$ for any $j \geq 1$, by the Grauert-Riemenschneider theorem. We may now formulate the following characterization of rational singularities, proved e.g. in [25] p. 50.

2.2.5 Proposition. *Let Y be a normal variety. Then Y has rational singularities if and only if: Y is Cohen-Macaulay and $\pi_* \omega_Z = \omega_Y$ for any desingularization $\pi : Z \to Y$.*

In particular, any Schubert variety X_w is Cohen-Macaulay, and its dualizing sheaf may be determined from that of a Bott-Samelson desingularization $Z_{\underline{w}}$. To describe the latter, put $Z := Z_{\underline{w}}$ and for $1 \leq j \leq \ell$, let $Z^j \subset Z$ be the Bott-Samelson subvariety associated with the subsequence $\underline{w}^j := (s_{i_1} \ldots, \widehat{s_{i_j}}, \ldots, s_{i_\ell})$ obtained by suppressing s_{i_j}.

2.2.6 Proposition. *(i) With the preceding notation, $Z^1, \ldots, Z^\ell$ identify to nonsingular irreducible divisors in Z, which meet transversally at a unique point (the class of B^ℓ).*
(ii) The complement in Z of the boundary

$$\partial Z := Z^1 \cup \cdots \cup Z^\ell$$

equals $\pi^{-1}(C_w) \cong C_w$.
(iii) The classes $[Z^j]$, $j = 1, \ldots, \ell$, form a basis of the Picard group of Z.

Indeed, (i) follows readily from the construction of Z; (ii) is a consequence of Proposition 2.2.1, and (iii) is checked by the argument of Proposition 1.4.1.

Next put

$$\partial X_w := X_w \setminus C_w = \bigcup_{v \in W,\, v < w} X_v,$$

this is the *boundary* of X_w. By Corollary 2.2.2, we have

$$\partial X_w = \bigcup_{v < w,\, \ell(v) = \ell(w) - 1} X_v.$$

In particular, ∂X_w has pure codimension 1 in X_w. Further, $\pi^{-1}(\partial X_w) = \partial Z$ (as sets).

We may now describe the dualizing sheaves of Bott-Samelson and Schubert varieties.

2.2.7 Proposition. *(i) $\omega_Z \cong (\pi^* L_{-\rho})(-\partial Z)$.*
(ii) $\omega_{X_w} \cong L_{-\rho}|_{X_w}(-\partial X_w)$. In particular, $\omega_X \cong L_{-2\rho}$.
(iii) The reduced subscheme ∂X_w is Cohen-Macaulay.

Proof. (i) Consider the curves $C_j := Z_{s_j} = P_j/B$ for $j = 1, \ldots, \ell$. We may regard each C_j as a subvariety of Z, namely, the transversal intersection of the Z^k for $k \neq j$. Thus, we have $\langle [Z^j], [C_j] \rangle = 1$ for all j.

On the other hand, we claim that $\langle [Z^j], [C_k] \rangle = 0$ for $j < k$. Indeed, we have a natural projection $\varphi_j : Z = Z_{s_{i_1},\ldots,s_{i_\ell}} \to Z_{s_{i_1},\ldots,s_{i_j}}$ such that Z^j is the pull-back of the corresponding divisor $Z^j_{s_{i_1},\ldots,s_{i_j}}$. Further, φ_j maps C_k to a point whenever $k > j$.

From this claim, it follows that it suffices to check the equality of the degrees of $\omega_Z(\partial Z)$ and $\pi^* L_{-\rho}$ when restricted to each curve C_j. Now we obtain

$$\omega_Z(\partial Z)|_{C_j} \cong \omega_{C_j}(\partial C_j)$$

by the adjunction formula. Further, $C_j \cong \mathbb{P}^1$, and ∂C_j is one point, so that $\omega_{C_j}(\partial C_j) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. On the other hand, π maps C_j isomorphically to the Schubert curve X_{s_j}, and $\mathcal{L}_\rho|_{X_{s_j}} \cong \mathcal{O}_{\mathbb{P}^1}(1)$, so that $\pi^* L_{-\rho}|C_j \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. This shows the desired equality.

(ii) Since X_w has rational singularities, we have $\omega_{X_w} = \pi_* \omega_Z$. Further, the projection formula yields $\omega_{X_w} \cong L_{-\rho} \otimes \pi_* \mathcal{O}(-\partial Z)$, and $\pi_* \mathcal{O}(-\partial Z) \cong \mathcal{O}_{X_w}(-\partial X_w)$ as $\pi^{-1}(\partial X_w) = \partial Z$.

(iii) By (ii), the ideal sheaf of ∂X_w in X_w is locally isomorphic to the dualizing sheaf ω_{X_w}. Therefore, this ideal sheaf is Cohen-Macaulay of depth $\dim(X_w)$. Now the exact sequence

$$0 \to \mathcal{O}_{X_w}(-\partial X_w) \to \mathcal{O}_{X_w} \to \mathcal{O}_{\partial X_w} \to 0$$

yields that the sheaf $\mathcal{O}_{\partial X_w}$ is Cohen-Macaulay of depth $\dim(X_w) - 1 = \dim(\partial X_w)$. $\square$

2.3 Cohomology of line bundles

The aim of this subsection is to prove the following

2.3.1 Theorem. *Let λ be a dominant weight and let $w \in W$. Then the restriction map $H^0(\lambda) \to H^0(X_w, L_\lambda)$ is surjective. Further, $H^j(X_w, L_\lambda) = 0$ for any $j \geq 1$.*

Proof. We first prove the second assertion in the case where $X_w = X$ is the full flag variety. Then $\omega_X \cong L_{-2\rho}$, so that $\omega_X^{-1} \otimes L_\lambda \cong L_{\lambda+2\rho}$ is ample. Thus, the assertion follows from the *Kodaira vanishing theorem* : $H^j(X, \omega_X \otimes \mathcal{L}) = 0$ for $j \geq 1$, where $\mathcal{L}$ is any ample line bundle on any projective nonsingular variety X.

Next we prove the second assertion for arbitrary X_w. For this, we will apply a generalization of the Kodaira vanishing theorem to a Bott-Samelson desingularization of X_w. Specifically, choose a reduced decomposition $\underline{w}$ and let $\pi : Z_{\underline{w}} \to X_w$ be the corresponding morphism. Then the projection formula yields isomorphisms

$$R^i \pi_*(\pi^* L_\lambda) \cong L_\lambda \otimes R^i \pi_* \mathcal{O}_{Z_{\underline{w}}}$$

for all $i \geq 0$. Together with Theorem 2.2.3 and the Leray spectral sequence for π, this yields isomorphisms

$$H^j(Z_{\underline{w}}, \pi^* L_\lambda) \cong H^j(X, L_\lambda)$$

for all $j \geq 0$.

We now recall a version of the *Kawamata-Viehweg vanishing theorem*, see [16] §5. Consider a nonsingular projective variety Z, a line bundle $\mathcal{L}$ on Z, and a family $(D_1, \ldots, D_\ell)$ of nonsingular divisors on Z intersecting transversally. Put $D := \sum_i \alpha_i D_i$, where $\alpha_1, \ldots, \alpha_\ell$ are positive integers. Let N be an integer such that $N > \alpha_i$ for all i, and put $\mathcal{M} := \mathcal{L}^N(-D)$. Assume that some positive tensor power of the line bundle $\mathcal{M}$ is globally generated, and that the corresponding morphism to a projective space is generically finite over its image (e.g., $\mathcal{M}$ is ample). Then $H^j(Z, \omega_Z \otimes \mathcal{L}) = 0$ for all $j \geq 1$.

We apply this result to the variety $Z := Z_{\underline{w}}$, the line bundle $\mathcal{L} := (\pi^* L_{\lambda+\rho})(\partial Z)$, and the divisor $D := \sum_i (N - b_i) Z^i$ where $b_1, \ldots, b_\ell$ are positive integers such that $\sum_i b_i Z^i$ is ample (these exist by Lemma 2.3.2 below). Then $\mathcal{L}^N(-D) = (\pi^* L_{N(\lambda+\rho)})(b_1 Z^1 + \cdots + b_\ell Z^\ell)$ is ample, and $\omega_Z \otimes \mathcal{L} = \pi^* L_\lambda$. This yields the second assertion.

To complete the proof, it suffices to show that the restriction map $H^0(X_w, L_\lambda) \to H^0(X_v, L_\lambda)$ is surjective whenever $w = s_i v > v$ for some elementary transposition s_i. As above, this reduces to checking the surjectivity of the restriction map $H^0(Z, \pi^* L_\lambda) \to H^0(Z^1, \pi^* L_\lambda)$. For this, by the long exact sequence

$$0 \to H^0(Z, (\pi^* L_\lambda)(-Z^1)) \to H^0(Z, \pi^* L_\lambda) \to H^0(Z^1, \pi^* L_\lambda) \to H^1(Z, (\pi^* L_\lambda)(-Z^1)),$$

it suffices in turn to show the vanishing of $H^1(Z, (\pi^* L_\lambda)(-Z^1))$.

We will deduce this again from the Kawamata-Viehweg vanishing theorem. Let $a_1, \ldots, a_\ell$ be positive integers such that the line bundle $(\pi^* L_{\lambda + a_1 \rho})(a_2 Z^2 + \cdots + a_\ell Z^\ell)$ is ample (again, these exist by Lemma 2.3.2 below). Put $\mathcal{L} := (\pi^* L_{\lambda+\rho})(Z^2 + \cdots + Z^\ell)$ and $D := \sum_{i=2}^\ell (N - a_i) Z^i$, where $N > a_1, a_2, \ldots, a_\ell$. Then $\mathcal{L}^N(-D) = (\pi^* L_{N(\lambda+\rho)})(a_2 Z^2 + \cdots + a_\ell Z^\ell)$ is ample, and $\omega_Z \otimes \mathcal{L} = (\pi^* L_\lambda)(-Z^1)$. Thus, we obtain $H^j(Z, (\pi^* L_\lambda)(-Z^1)) = 0$ for all $j \geq 1$. □

2.3.2 Lemma. *Let $Z = Z_{\underline{w}}$ with boundary divisors $Z^1, \ldots, Z^\ell$. Then there exist positive integers $a_1, \ldots, a_\ell$ such that the line bundle $(\pi^* L_{a_1 \rho})(a_2 Z^2 + \cdots + a_\ell Z^\ell)$ is ample. Further, there exist positive integers $b_1, \ldots, b_\ell$ such that the divisor $b_1 Z^1 + \cdots + b_\ell Z^\ell$ is ample.*

Proof. We prove the first assertion by induction on ℓ. If $\ell = 1$, then π embeds Z into X, so that $\pi^* L_{a_1 \rho}$ is ample for any $a_1 > 0$. In the general case, the map

$$\varphi : Z \to Z^\ell = (P_{i_1} \times \cdots \times P_{i_{\ell-1}})/B^{\ell-1}, \quad (g_1, \ldots, g_\ell) B^\ell \mapsto (g_1, \ldots, g_{\ell-1}) B^{\ell-1}$$

fits into a cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & Z^\ell \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\psi}\downarrow \\ G/B & \xrightarrow{f} & G/P_{i_\ell}, \end{array}$$

where $\psi((g_1,\ldots,g_{\ell-1})B^{\ell-1}) = g_1\cdots g_{\ell-1}P_{i_\ell}$. Further, the boundary divisors $Z^{1,\ell}$, …, $Z^{\ell-1,\ell}$ of Z^ℓ satisfy $\varphi^* Z^{i,\ell} = Z^i$. Denote by

$$\pi_\ell : Z^\ell = Z_{(s_{i_1},\ldots,s_{i_{\ell-1}})} \to X_{s_{i_1}\cdots s_{i_{\ell-1}}} = X_{ws_{i_\ell}}$$

the natural map. By the induction assumption, there exist positive integers $a_1, a_2, \ldots, a_{\ell-1}$ such that the line bundle $(\pi_\ell^* L_{a_1\rho})(a_2 Z^{1,\ell} + \cdots + a_{\ell-1}Z^{\ell-1,\ell})$ is very ample on Z^ℓ. Hence its pull-back

$$\varphi^*((\pi_\ell^* L_{a_1\rho})(a_2 Z^{2,\ell} + \cdots + a_{\ell-1} Z^{\ell-1,\ell})) = (\varphi^*\pi_\ell^* L_{a_1\rho})(a_2 Z^2 + \cdots + a_{\ell-1} Z^{\ell-1})$$

is a globally generated line bundle on Z. Thus, it suffices to show that the line bundle $\pi^* L_{b\rho} \otimes (\varphi^*\pi_\ell^* L_{-a_1\rho})(a_1 Z^\ell)$ is globally generated and φ-ample for $b \gg a_1$. (Indeed, if $\mathcal{M}$ is a globally generated, φ-ample line bundle on Z, and $\mathcal{N}$ is an ample line bundle on Z^ℓ, then $\mathcal{M}\otimes\varphi^*\mathcal{N}$ is ample on Z). Equivalently, it suffices to show that $\pi^* L_{c\rho}\otimes\pi^* L_\rho \otimes (\varphi^*\pi^* L_{-\rho})(Z^\ell)$ is globally generated and φ-ample for $c \gg 0$. But we have by Proposition 2.2.6:

$$\pi^* L_\rho \otimes (\varphi^* \pi_\ell^* L_{-\rho})(Z^\ell) = \omega_Z^{-1}(-\partial Z) \otimes \varphi^*(\omega_{Z^\ell}(\partial Z^\ell))(Z^\ell) = \omega_Z^{-1} \otimes \varphi^*\omega_{Z^\ell} = \omega_\varphi^{-1} = \pi^*\omega_f^{-1},$$

where ω_φ (resp. ω_f) denotes the relative dualizing sheaf of the morphism φ (resp. f). Further, $L_{c\rho}\otimes\omega_f^{-1}$ is very ample on G/B for $c \gg 0$, as L_ρ is ample. Thus, $\pi^*(L_{c\rho}\otimes\omega_f^{-1})$ is globally generated and φ-ample. This completes the proof of the first assertion.

The second assertion follows by recalling that the restriction of L_ρ to X_w admits a section vanishing exactly on ∂X_w (Remark 1.4.6 2). Thus, $\pi^* L_\rho$ admits a section vanishing exactly on $\partial Z = Z^1 \cup \cdots \cup Z^\ell$. □

Next we consider a regular dominant weight λ and the corresponding very ample homogeneous line bundle L_λ. This defines a projective embedding

$$X \to \mathbb{P}(H^0(X, L_\lambda)^*) = \mathbb{P}(V(\lambda))$$

and, in turn, a subvariety $\tilde{X} \subseteq V(\lambda)$, invariant under the action of $G \times \mathbb{C}^*$ where $\mathbb{C}^*$ acts by scalar multiplication. We say that $\tilde{X}$ is the *affine cone over* X associated with this projective embedding. Likewise, we have the affine cones $\tilde{X}_w$ over Schubert varieties.

2.3.3 Corollary. *For any regular dominant weight λ, the affine cone over X_w in $V(\lambda)$ has rational singularities. In particular, X_w is projectively normal in its embedding into $\mathbb{P}(V(\lambda))$.*

Proof. Consider the total space Y_w of the line bundle $L_\lambda^{-1}|_{X_w}$. We have a proper morphism

$$\pi : Y_w \to \tilde{X}_w$$

which maps the zero section to the origin, and restricts to an ismorphism from the complement of the zero section to the complement of the origin. In particular, π is birational. Further, Y_w has rational singularities, since it is locally isomorphic to $X_w \times \mathbb{C}$. Thus, it suffices to show that the natural map $\mathcal{O}_{\tilde{X}_w} \to \pi_* \mathcal{O}_{Y_w}$ is surjective, and that $R^j \pi_* \mathcal{O}_{Y_w} = 0$ for any $j \geq 1$. Since $\tilde{X}_w$ is affine, this amounts to: the algebra $H^0(Y_w, \mathcal{O}_{Y_w})$ is generated by the image of $H^0(\lambda)$, and $H^j(Y_w, \mathcal{O}_{Y_w}) = 0$ for $j \geq 1$. Further, since the projection $f : Y_w \to X_w$ is affine and satisfies

$$f_* \mathcal{O}_{Y_w} = \bigoplus_{n=0}^{\infty} L_\lambda^{\otimes n} = \bigoplus_{n=0}^{\infty} L_{n\lambda},$$

we obtain

$$H^j(Y_w, \mathcal{O}_{Y_w}) = \bigoplus_{n=0}^{\infty} H^j(X_w, L_{n\lambda}).$$

So $H^j(Y_w, \mathcal{O}_{Y_w}) = 0$ for $j \geq 1$, by Theorem 2.3.1. To complete the proof, it suffices to show that the algebra $\bigoplus_{n=0}^{\infty} H^0(X_w, L_{n\lambda})$ is generated by the image of $H^0(\lambda)$. Using the surjectivity of the restriction maps $H^0(L_{n\lambda}) \to H^0(X_w, L_{n\lambda})$ (Theorem 2.3.1 again), it is enough to consider the case where $X_w = X$. Now the multiplication map

$$H^0(\lambda)^{\otimes n} \to H^0(n\lambda), \quad \sigma_1 \otimes \cdots \otimes \sigma_n \mapsto \sigma_1 \cdots \sigma_n$$

is a non-zero morphism of G-modules. Since $H^0(n\lambda)$ is simple, this morphism is surjective, which completes the proof. ∎

Notes. In their full generality, the results of this section were obtained by many mathematicians, during the mid-eighties. Their most elegant proofs use reduction to positive characteristics and the techniques of Frobenius splitting, see [40], [43], [44].

Here we have presented alternative proofs: for normality and rationality of singularities, we rely on an argument of Seshadri [46] simplified in [8], which is also valid in arbitrary characteristics. For cohomology of line bundles, our approach (based on the Kawamata-Viehweg vanishing theorem) is a variant of that of Kumar; see [28].

The construction of the Bott-Samelson varieties is due to ... Bott and Samelson [4] in the framework of compact Lie groups, and to Hansen [22] and Demazure [13] in our algebro-geometric setting. The line bundles on Bott-Samelson varieties have been studied by Lauritzen and Thomsen in [34]; in particular, they determined the globally generated (resp. ample) line bundles.

For Schubert varieties, the Picard group (consisting of linear equivalence classes of Cartier divisors) is generally smaller than the divisor class group (of linear equivalence classes of Weil divisors). In fact, for any Schubert variety Y in any flag variety $X = G/P$, one easily shows that the divisor class group of Y is freely generated by the classes of the Schubert subvarieties of codimension 1. Further, any line bundle on Y extends to a homogeneous line bundle on X, and the globally generated (resp. ample) line bundles on Y extend to globally generated (resp. ample) line bundles on X. One may also show that the boundary of Y is Cohen-Macaulay, see [8] Lemma 4. But a simple formula for the dualizing sheaf of Y is only known in the case where X is the full flag variety.

An important open question is the explicit determination of the singular locus of a Schubert variety, and of the corresponding generic singularities (i.e., the singularities along each irreducible components of the singular locus). The book [1] by Billey and Lakshmibai is a survey of this question, which was recently solved (independently and simultaneously) by several mathematicians in the case of the general linear group; see [2], [12], [24], [36], [37]. The generic singularities of Richardson varieties are also worth investigating.

3 The diagonal of a flag variety

Let $X = G/B$ be the full flag variety and denote by $\operatorname{diag} X$ the diagonal in $X \times X$. In this section, we construct a degeneration of $\operatorname{diag} X$ in $X \times X$ to the union of all the products $X_w \times X^w$, where the X_w (resp. X^w) are the Schubert (resp. opposite Schubert) varieties.

Specifically, we construct a subvariety $\mathcal{X} \subseteq X \times X \times \mathbb{P}^1$ such that the fiber of the projection $\pi : \mathcal{X} \to \mathbb{P}^1$ at any $t \neq 0$ is isomorphic to $\operatorname{diag} X$, and we show that the fiber at 0 (resp. ∞) is the union of all the $X_w \times X^w$ (resp. $X^w \times X_w$). For this, we use the normality of $\mathcal{X}$, which is deduced from a general normality criterion for varieties with group actions, obtained in turn by adapting the argument for the normality of Schubert varieties.

Then we turn to applications to the Grothendieck ring $K(X)$. After a brief presentation of the definition and main properties of Grothendieck rings, we obtain two additive bases of $K(X)$ which are dual for the bilinear pairing given by the Euler characteristic of the product. Further applications will be given in Section 4.

3.1 A degeneration of the diagonal

Let X be the full flag variety, and $\operatorname{diag} X$ the diagonal in $X \times X$. We begin by determining its cohomology class.

3.1.1 Lemma. *We have* $[\operatorname{diag} X] = \sum_{w \in W} [X_w \times X^w]$ *in* $H^*(X \times X)$.

Proof. By the results in Subsection 1.3 and the Künneth isomorphism, a basis for $H^*(X \times X)$ consists of the classes $[X_w \times X^v]$, where $v, w \in W$. Further, the dual basis (with respect to the Poincaré duality pairing) consists of the $[X^w \times X_v]$. Thus, we may write

$$[\operatorname{diag} X] = \sum_{v,w \in W} a_{wv} [X_w \times X^v],$$

where the coefficients a_{wv} are given by

$$a_{wv} = \langle [\operatorname{diag} X], [X^w \times X_v] \rangle.$$

Further, since X^w meets X_v properly along X_v^w, with intersection multiplicity 1, it follows that $\operatorname{diag} X$ meets $X^w \times X_v$ properly along $\operatorname{diag} X_v^w$ in $X \times X$, with intersection multiplicity 1. This yields

$$[\operatorname{diag} X] \cup [X^w \times X_v] = [\operatorname{diag} X_w^v].$$

And since $\dim(X_v^w) = 0$ if and only if $v = w$, we see that a_{wv} equals 1 if $v = w$, and 0 otherwise. □

This formula suggests the existence of a degeneration of $\operatorname{diag} X$ to $\bigcup_{w \in W} X_w \times X^w$. We now constuct such a degeneration. The idea is to move $\operatorname{diag} X$ in $X \times X$ by a general one-parameter subgroup of the torus T acting on $X \times X$ via its action on the second copy, and to take limits.

Specifically, let

$$\lambda : \mathbb{C}^* \to T, \quad t \mapsto (t^{a_1}, \ldots, t^{a_n})$$

where $a_1, \ldots, a_n$ are integers satisfying $a_1 > \cdots > a_n$. Define $\mathcal{X}$ to be the closure in $X \times X \times \mathbb{P}^1$ of the subset

$$\{(x, \lambda(t)x, t) \mid x \in X,\ t \in \mathbb{C}^*\} \subseteq X \times X \times \mathbb{C}^*.$$

Then $\mathcal{X}$ is a projective variety, and the fibers of the projection $\pi : \mathcal{X} \to \mathbb{P}^1$ identify with closed subschemes of $X \times X$. Further, the fiber $\pi^{-1}(1)$ equals $\operatorname{diag} X$. In fact, $\pi^{-1}(\mathbb{C}^*)$ identifies to $(\operatorname{diag} X) \times \mathbb{C}^*$ via $(x, y, t) \mapsto (x, \lambda(t^{-1})x, t)$, and this identifies the restriction of π to the projection $(\operatorname{diag} X) \times \mathbb{C}^* \to \mathbb{C}^*$.

3.1.2 Theorem. *We have equalities of subschemes of $X \times X$:*

$$\pi^{-1}(0) = \bigcup_{w \in W} X_w \times X^w \quad \textit{and} \quad \pi^{-1}(\infty) = \bigcup_{w \in W} X^w \times X_w.$$

Proof. By symmetry, it suffices to prove the first equality. We begin by showing the inclusion $\bigcup_{w \in W} X_w \times X^w \subseteq \pi^{-1}(0)$. Equivalently, we claim that $C_w \times C^w \subset \pi^{-1}(0)$ for all $w \in W$.

For this, we analyze the structure of $X \times X$ in a neighborhood of the base point (F_w, F_w) of $C_w \times C^w$ (recall that F_w denotes the image under w of the standard flag F). Recall from Proposition 1.3.5 that wC^{id} is a T-invariant open neighborhood of F_w in X, isomorphic to wU^-w^{-1}. Further, $C_w = UF_w \cong (wU^-w^{-1} \cap U)F_w$ identifies via this isomorphism to the subgroup $wU^-w^{-1} \cap U$. Likewise, C^w identifies to the subgroup $wU^-w^{-1} \cap U^-$, and the product map in the group wU^-w^{-1}

$$(wU^-w^{-1} \cap U) \times (wU^-w^{-1} \cap U^-) \to wU^-w^{-1}$$

is an isomorphism.

The group $\mathbb{C}^*$ acts on wU^-w^{-1} via its homomorphism $t \mapsto (t^{a_1}, \ldots, t^{a_n})$ to T and the action of T on wU^-w^{-1} by conjugation. In fact, this action is linear, and hence decomposes into a direct sum of weight subspaces. Using the assumption that $a_1 > \cdots > a_n$, one checks that the sum of all the positive weight subspaces is $wU^-w^{-1} \cap U = C_w$; likewise, the sum of all the negative weight subspaces is C^w. In other words,

$$C_w = \{x \in wU^-w^{-1} \mid \lim_{t \to 0} \lambda(t)x = \mathrm{id}\}, \quad C^w = \{y \in wU^-w^{-1} \mid \lim_{t \to \infty} \lambda(t)y = \mathrm{id}\}.$$

Now identify our neighborhood $wC^{\mathrm{id}} \times wC^{\mathrm{id}}$ with $C_w \times C^w \times C_w \times C^w$. Take arbitrary $x \in C_w$ and $y \in C^w$, then

$$(x, \lambda(t)^{-1}y, \lambda(t)x, y) \to (x, \mathrm{id}, \mathrm{id}, y) \quad \text{as} \quad t \to 0.$$

By the definition of $\mathcal{X}$, it follows that $\pi^{-1}(0)$ contains the point $(x, \mathrm{id}, \mathrm{id}, y)$, identified to $(x, y) \in X \times X$. This proves the claim.

From this claim, it follows that $\pi^{-1}(0)$ contains $\bigcup_{w \in W} X_w \times X^w$ (as schemes). On the other hand, the cohomology class of $\pi^{-1}(0)$ equals that of $\pi^{-1}(1)$, i.e., $\sum_{w \in W}[X_w \times X^w]$ by Lemma 3.1.1. Further, the cohomology class of any non-empty subvariety of $X \times X$ is a positive integer combination of classes $[X_w \times X^v]$ by Proposition 1.3.6. It follows that the irreducible components of $\pi^{-1}(0)$ are exactly the $X_w \times X^w$, and that the corresponding multiplicities are all 1. Thus, the scheme $\pi^{-1}(0)$ is generically reduced.

To complete the proof, it suffices to show that $\pi^{-1}(0)$ is reduced. Since π may be regarded as a regular function on $\mathcal{X}$, it suffices in turn to show that $\mathcal{X}$ is normal. In the next subsection, this will be deduced from a general normality criterion for varieties with group action. $\square$

To apply Theorem 3.1.2, we will also need to analyze the structure sheaf of the special fiber $\pi^{-1}(0)$. This is the content of the following statement.

3.1.3 Proposition. *The sheaf $\mathcal{O}_{\pi^{-1}(0)}$ admits a filtration with associated graded*

$$\bigoplus_{w \in W} \mathcal{O}_{X_w} \otimes \mathcal{O}_{X^w}(-\partial X^w).$$

Proof. We may index the finite partially ordered set $W = \{w_1, \ldots, w_N\}$ so that $i \leq j$ whenever $w_i \leq w_j$ (then $w_N = w_0$). Put

$$Z_i := X_{w_i} \times X^{w_i} \quad \text{and} \quad Z_{\geq i} := \bigcup_{j \geq i} Z_j,$$

for $1 \leq j \leq N$. Then $Z_{\geq 1} = \pi^{-1}(0)$ and $Z_{\geq N} = X_{w_0} \times X^{w_0} = X \times \{w_0 F\}$. Further, the $Z_{\geq i}$ form a decreasing filtration of $\pi^{-1}(0)$. This yields exact sequences

$$0 \to \mathcal{I}_i \to \mathcal{O}_{Z_{\geq i}} \to \mathcal{O}_{Z_{\geq i+1}} \to 0,$$

where $\mathcal{I}_i$ denotes the ideal sheaf of $Z_{\geq i+1}$ in $Z_{\geq i}$. In turn, these exact sequences yield an increasing filtration of the sheaf $\mathcal{O}_{\pi^{-1}(0)}$, with associated graded $\bigoplus_i \mathcal{I}_i$. Since $Z_{\geq i} = Z_{\geq i+1} \cup Z_i$, we may identify $\mathcal{I}_i$ with the ideal sheaf of $Z_i \cap Z_{\geq i+1}$ in $Z_i = X_{w_i} \times X^{w_i}$. To complete the proof, it suffices to show that

$$Z_i \cap Z_{\geq i+1} = X_{w_i} \times \partial X^{w_i}.$$

We first check the inclusion "$\subseteq$". Note that $Z_i \cap Z_{\geq i+1}$ is invariant under $B \times B^-$, and hence is a union of products $X_u \times X^v$ for certain $u, v \in W$. We must have $u \leq w_i \leq v$ (since $X_u \times X^v \subseteq Z_i$) and $w_i \neq v$ (since $X_u \times X^v \subseteq Z_{\geq i+1}$). Thus, $X_u \times X^v \subseteq X_{w_i} \times \partial X^{w_i}$. To check the opposite inclusion, note that if $X^v \subseteq \partial X^{w_i}$ then $v > w_i$, so that $v = w_j$ with $j > i$. Thus, $X_{w_i} \times X^v \subset X_{w_j} \times X^{w_j} \subseteq Z_{\geq i+1}$. $\square$

3.2 A normality criterion

Let G be a connected linear algebraic group acting on an algebraic variety Z. Let $Y \subset Z$ be a subvariety, invariant under the action of a Borel subgroup $B \subseteq G$, and let $P \supset B$ be a parabolic subgroup of G. Then, as in Subsection 2.1, we may define the "induced" variety $P \times^B Y$. It is equipped with a P-action and with P-equivariant maps $\pi : P \times^B Y \to Z$ (a proper morphism with image PY), and $f : P \times^B Y \to P/B$ (a locally trivial fibration with fiber Y). If, in addition, P is a minimal parabolic subgroup (i.e., $P/B \cong \mathbb{P}^1$), and if $PY \neq Y$, then $\dim(PY) = \dim(Y) + 1$, and the morphism π is generically finite over its image PY.

We say that Y is *multiplicity-free* if it satisfies the following conditions:
(i) $GY = Z$.
(ii) either $Y = Z$, or Z contains no G-orbit.
(iii) For all minimal parabolic subgroups $P \supset B$ such that $PY \neq Y$, the morphism $\pi : P \times^B Y \to PY$ is birational, and the variety PY is multiplicity-free.

(This defines indeed the class of multiplicity-free subvarieties by induction on the codimension, starting with Z).

For example, Schubert varieties are multiplicity-free. Further, the proof of their normality given in Subsection 2.1 readily adapts to show the following

3.2.1 Theorem. *Let Y be a B-invariant subvariety of a G-variety Z. If Z is normal and Y is multiplicity-free, then Y is normal.*

Next we obtain a criterion for multiplicity-freeness of any B-stable subvariety of $Z := G$, where G acts by left multiplication. Note that the B-stable subvarieties $Y \subseteq G$ correspond to the subvarieties $V \subseteq G/B$, where $V = \{g^{-1}B \mid g \in Y\}$.

3.2.2 Lemma. *With the preceding notation, Y is multiplicity-free if and only if $[V]$ is a multiplicity-free combination of Schubert classes, i.e., the coefficients of $[V]$ in the basis $\{[X_w]\}$ are either 0 or 1. Equivalently, $\langle [V], [X^w] \rangle \leq 1$ for all w.*

Proof. Clearly, Y satisfies conditions (i) and (ii) of multiplicity-freeness. For condition (iii), consider a minimal parabolic subgroup $P \supset B$ and the natural map $f : G/B \to G/P$. Then the subvariety of G associated with $f^{-1}f(V)$ is PY. As a consequence, $PY \neq Y$ if and only if the restriction $f|_V : V \to f(V)$ is generically finite. Further, the fibers of $f|_V$ identify to those of the natural map $\pi : P \times^B Y \to PY$; in particular, both morphisms have the same degree d. Note that $d = 1$ if and only if π (or, equivalently, $f|_V$) is birational.

Let $X_w \subseteq G/B$ be a Schubert variety of positive dimension. We may write $X_w = P_1 \cdots P_\ell/B$, where $(P_1, \ldots, P_\ell)$ is a sequence of minimal parabolic subgroups, and $\ell = \dim(X_w)$. Put $P := P_\ell$ and $X_v := P_1 \cdots P_{\ell-1}/B$. Then $X_w = f^{-1}f(X_w)$, and the restriction $X_v \to f(X_v) = f(X_w)$ is birational. We thus obtain the equalities of intersection numbers

$$\langle [V],[X_w]\rangle_{G/B} = \langle [V], f^{-1}[f(X_w)]\rangle_{G/B} = \langle f_*[V],[f(X_w)]\rangle_{G/P}$$
$$= d\,\langle [f(V)],[f(X_w)]\rangle_{G/P} = d\,\langle [f^{-1}f(V),[X_v]\rangle_{G/B},$$

as follows from the projection formula and from the equalities $f_*[V] = d\,[f(V)]$, $f_*[X_v] = [f(X_v)] = [f(X_w)]$. From these equalities, it follows that $[V]$ is a multiplicity-free combination of Schubert classes if and only if: $d = 1$ and $[f^{-1}f(V)]$ is a multiplicity-free combination of Schubert classes, for any minimal parabolic subgroup P such that $PY \neq Y$. Now the proof is completed by induction on $\operatorname{codim}_{G/B}(V) = \operatorname{codim}_G(Y)$. □

We may now complete the proof of Theorem 3.1.2 by showing that $\mathcal{X}$ is normal. Consider first the group $G \times G$, the Borel subgroup $B \times B$, and the variety $Z := G \times G$ where $G \times G$ acts by left multiplication. Then the subvariety $Y := (B \times B)\operatorname{diag} G$ is multiplicity-free. (Indeed, Y corresponds to the diagonal $\operatorname{diag} X \subset X \times X$, where $X = G/B$. By Lemma 3.1.1, the coefficients of $[\operatorname{diag} X]$ in the basis of Schubert classes are either 0 or 1, so that Lemma 3.2.2 applies.)

Next consider the same group $G \times G$ and take $Z := G \times G \times \mathbb{P}^1$, where $G \times G$ acts by left multiplication on the factor $G \times G$. Let Y be the preimage in Z of the subvariety $\mathcal{X} \subset X \times X \times \mathbb{P}^1$, under the natural map $G \times G \times \mathbb{C} \to X \times X \times \mathbb{P}^1$ (a locally trivial fibration). Clearly, Y satisfies conditions (i), (ii) of multiplicity-freeness. Further, condition (iii) follows from the fact that Y contains an open subset isomorphic to $(B \times B)\operatorname{diag} G \times \mathbb{C}^*$, together with the multiplicity-freeness of $(B \times B)\operatorname{diag} G$. Since Z is nonsingular, it follows that Y is normal by Theorem 3.2.1. Hence, $\mathcal{X}$ is normal as well.

3.3 The Grothendieck group

For any scheme X, the *Grothendieck group of coherent sheaves on* X is the abelian group $K_0(X)$ generated by symbols $[\mathcal{F}]$ where $\mathcal{F}$ is a coherent sheaf on X, subject to the relations: $[\mathcal{F}] = [\mathcal{F}_1] + [\mathcal{F}_2]$ whenever there exists an exact sequence of sheaves $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$. (In particular, $[\mathcal{F}]$ only depends on the isomorphism class of $\mathcal{F}$.)

Likewise, we have the *Grothendieck group* $K^0(X)$ *of vector bundles on* X, generated by symbols $[E]$ where E is a vector bundle on X, subject to the relations: $[E] = [E_1] + [E_2]$ whenever there exists an exact sequence of vector bundles $0 \to E_1 \to E \to E_2 \to 0$. The tensor product of vector bundles yields a commutative, associative multiplication law on $K^0(X)$ denoted by $(\alpha, \beta) \mapsto \alpha \cdot \beta$. With this multiplication, $K^0(X)$ is a commutative ring, the identity element being the class of the trivial bundle of rank 1.

The duality of vector bundles $E \mapsto E^\vee$ is compatible with the defining relations of $K^0(X)$. Thus, it yields a map $K^0(X) \to K^0(X)$, $\alpha \mapsto \alpha^\vee$, which is an involution of the ring $K^0(X)$: the *duality involution*.

By associating with each vector bundle E its (locally free) sheaf of sections $\mathcal{E}$, we obtain a map

$$\varphi : K^0(X) \to K_0(X).$$

More generally, since tensoring with a locally free sheaf is exact, the ring $K^0(X)$ acts on $K_0(X)$ via

$$[E] \cdot [\mathcal{F}] := [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}],$$

where E is a vector bundle on X with sheaf of sections $\mathcal{E}$, and $\mathcal{F}$ is a coherent sheaf on X. This makes $K_0(X)$ a module over $K^0(X)$; further, $\varphi(\alpha) = \alpha \cdot [\mathcal{O}_X]$ for any $\alpha \in K^0(X)$. If Y is another scheme, then the external tensor product of sheaves (resp. vector bundles) yields product maps $K_0(X) \times K_0(Y) \to K_0(X \times Y)$, $K^0(X) \times K^0(Y) \to K^0(X \times Y)$, compatible with the corresponding maps φ. We will denote both product maps by $(\alpha, \beta) \mapsto \alpha \times \beta$.

If X is a nonsingular variety, then φ is an isomorphism. In this case, we identify $K_0(X)$ with $K^0(X)$, and we denote this ring by $K(X)$, the *Grothendieck ring of* X. For any coherent sheaves $\mathcal{F}$, $\mathcal{G}$ on X, we have

$$[\mathcal{F}] \cdot [\mathcal{G}] = \sum_j (-1)^j \; [Tor_i^X(\mathcal{F}, \mathcal{G})].$$

(This formula makes sense because the sheaves $Tor_j^X(\mathcal{F}, \mathcal{G})$ are coherent and vanish for $j > \dim(X)$). In particular, $[\mathcal{F}] \cdot [\mathcal{G}] = 0$ if the sheaves $\mathcal{F}$ and $\mathcal{G}$ have disjoint supports. Further, we have

$$[\mathcal{F}]^\vee = \sum_j (-1)^j \; [Ext_X^j(\mathcal{F}, \mathcal{O}_X)].$$

In particular, if Y is an equidimensional Cohen-Macaulay subscheme of X, then

$$\begin{aligned}[\mathcal{O}_Y]^\vee = (-1)^{\operatorname{codim}(Y)} \; &[Ext_X^{\operatorname{codim}(Y)}(\mathcal{O}_Y, \mathcal{O}_X)] \\ &= (-1)^{\operatorname{codim}(Y)} \; [\omega_{Y/X}] = (-1)^{\operatorname{codim}(Y)} \; [\omega_Y] \cdot [\omega_X]^\vee,\end{aligned}$$

where $\omega_{Y/X} := \omega_Y \otimes \omega_X^{-1}$ denotes the relative dualizing sheaf of Y in X.

Returning to an arbitrary scheme X, any morphism of schemes $f : X \to Y$ yields a pull-back map

$$f^* : K^0(Y) \to K^0(X), \quad [E] \mapsto [f^*E].$$

If, in addition, f is *flat*, then it defines similarly a pull-back map $f^* : K_0(Y) \to K_0(X)$.

On the other hand, any *proper* morphism $f : X \to Y$ yields a push-forward map

$$f_* : K_0(X) \to K_0(Y), \quad [\mathcal{F}] \mapsto \sum_j (-1)^j \; [R^j f_*(\mathcal{F})].$$

As above, this formula makes sense since the higer direct images $R^j f_*(\mathcal{F})$ are coherent sheaves on Y, which vanish for $j > \dim(X)$. Moreover, we have the *projection formula*

$$f_*((f^*\alpha)\cdot\beta) = \alpha \cdot f_*\beta$$

for all $\alpha \in K^0(Y)$ and $\beta \in K_0(X)$.

In particular, if X is complete then we obtain a map

$$\chi : K_0(X) \to \mathbb{Z},\ [\mathcal{F}] \mapsto \chi(\mathcal{F}) = \sum_j (-1)^j h^j(\mathcal{F}),$$

where $h^j(\mathcal{F})$ denotes the dimension of the j-th cohomology group of $\mathcal{F}$, and χ stands for the *Euler-Poincaré characteristic.*

We will repeatedly use the following result of "homotopy invariance" in the Grothendieck group.

3.3.1 Lemma. *Let X be a variety and let $\mathcal{X}$ be a subvariety of $X \times \mathbb{P}^1$, with projections $\pi : \mathcal{X} \to \mathbb{P}^1$ and $p : \mathcal{X} \to X$. Then the class $[\mathcal{O}_{p(\pi^{-1}(z))}] \in K_0(X)$ is independent of $z \in \mathbb{P}^1$.*

Proof. The exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_z \to 0$ of sheaves on $\mathbb{P}^1$ shows that the class $[\mathcal{O}_z] \in K_0(\mathbb{P}^1)$ is independent of z. Since π is flat, it follows that the class $\pi^*[\mathcal{O}_z] = [\mathcal{O}_{\pi^{-1}(z)}] \in K_0(\mathcal{X})$ is also independent of z, and the same holds for $p_*[\mathcal{O}_{\pi^{-1}(z)}] \in K_0(X)$ since p is proper. But $p_*[\mathcal{O}_{\pi^{-1}(z)}] = [\mathcal{O}_{p(\pi^{-1}(z))}]$, since p restricts to an isomorphism $\pi^{-1}(z) \to p(\pi^{-1}(z))$. □

Finally, we present a relation of $K_0(X)$ to the Chow group $A_*(X)$ of rational equivalence classes of algebraic cycles on X (graded by the dimension), see [19] Example 15.1.5. Define the *topological filtration* on $K_0(X)$ by letting $F_j K_0(X)$ to be the subgroup generated by coherent sheaves whose support has dimension at most j. Let $\operatorname{Gr} K_0(X)$ be the associated graded group. Then assigining to any subvariety $Y \subseteq X$ the class of its structure sheaf $[\mathcal{O}_Y]$ passes to rational equivalence (as follows from Lemma 3.3.1) and hence defines a morphism $A_*(X) \to \operatorname{Gr} K_0(X)$ of graded abelian groups. This morphism is surjective ; it is an isomorphism over the rationals if, in addition, X is nonsingular (see [F98] Example 15.2.16).

3.4 The Grothendieck group of the flag variety

The Chow group of the full flag variety X is isomorphic to its cohomology group and, in particular, is torsion-free. It follows that the associated graded of the Grothendieck group (for the topological filtration) is isomorphic to the cohomology group; this isomorphism maps the image of the structure sheaf $\mathcal{O}_Y$ of any subvariety, to the cohomology class $[Y]$. Thus, the following result may be viewed as a refinement in $K(X \times X)$ of the equality $[\operatorname{diag}(X)] = \sum_{w\in W}[X_w \times X^w]$ in $H^*(X \times X)$.

3.4.1 Theorem. *(i) In $K(X \times X)$ holds*

$$[\mathcal{O}_{\mathrm{diag}\,X}] = \sum_{w \in W} [\mathcal{O}_{X_w}] \times [\mathcal{O}_{X^w}(-\partial X^w)].$$

(ii) The bilinear map

$$K(X) \times K(X) \to \mathbb{Z}, \quad (\alpha, \beta) \mapsto \chi(\alpha \cdot \beta)$$

is a nondegenerate pairing. Further, $\{[\mathcal{O}_{X_w}]\}$, $\{[\mathcal{O}_{X^w}(-\partial X^w)]\}$ are bases of the abelian group $K(X)$, dual for this pairing.

Proof. (i) By Theorem 3.1.2 and Lemma 3.3.1, we have $[\mathcal{O}_{\mathrm{diag}\,X}] = [\mathcal{O}_{\bigcup_{w \in W} X_w \times X^w}]$. Further, $[\mathcal{O}_{\bigcup_{w \in W} X_w \times X^w}] = \sum_{w \in W} [\mathcal{O}_{X_w}] \times [\mathcal{O}_{X^w}(-\partial X^w)]$ by Proposition 3.1.3.

(ii) Let $p_1, p_2 : X \times X \to X$ be the projections. Let $\mathcal{E}$ be a locally free sheaf on X. Then we have by (i):

$$\begin{aligned}[\mathcal{E}|_{\mathrm{diag}\,X}] &= [p_2^*\mathcal{E}] \cdot [\mathcal{O}_{\mathrm{diag}(X)}] \\ &= \sum_{w \in W} [p_2^*\mathcal{E}] \cdot [p_1^*\mathcal{O}_{X_w} \otimes p_2^*\mathcal{O}_{X^w}(-\partial X^w)] = \sum_{w \in W} [p_1^*\mathcal{O}_{X_w} \otimes p_2^*\mathcal{E}|_{X^w}(-\partial X^w)].\end{aligned}$$

Applying $p_{1,*}$ to both sides and using the projection formula yields

$$[\mathcal{E}] = \sum_{w \in W} \chi(\mathcal{E}|_{X^w}(-\partial X^w)\ [\mathcal{O}_{X_w}].$$

Since the group $K(X)$ is generated by classes of locally free sheaves, it follows that

$$\alpha = \sum_{w \in W} \chi(\alpha \cdot [\mathcal{O}_{X^w}(-\partial X^w)])\ [\mathcal{O}_{X_w}]$$

for all $\alpha \in K(X)$. Thus, the classes $[\mathcal{O}_{X_w}]$ generate the group $K(X)$.

To complete the proof, it suffices to show that these classes are linearly independent. If $\sum_{w \in W} n_w\ [\mathcal{O}_{X_w}] = 0$ is a non-trivial relation in $K(X)$, then we may choose $v \in W$ maximal such that $n_v \neq 0$. Now a product $[\mathcal{O}_{X_w}] \cdot [\mathcal{O}_{X^v}]$ is non-zero only if $X_w \cap X^v$ is non-empty, i.e., $v \leq w$. Thus, we have by maximality of v:

$$0 = \sum_{w \in W} n_w [\mathcal{O}_{X_w}] \cdot [\mathcal{O}_{X^v}] = n_v [\mathcal{O}_{X_v}] \cdot [\mathcal{O}_{X^v}].$$

Further, we have $[\mathcal{O}_{X_v}] \cdot [\mathcal{O}_{X^v}] = [\mathcal{O}_{vF}]$. (Indeed, X_v and X^v meet transversally at the unique point vF ; see Lemma 4.1.1 below for a more general result). Further, $[\mathcal{O}_{vF}]$ is non-zero since $\chi(\mathcal{O}_{vF}) = 1$; a contradiction. □

We put for simplicity

$$\mathcal{O}_w := [\mathcal{O}_{X_w}] \quad \text{and} \quad \mathcal{I}^w := [\mathcal{O}_{X^w}(-\partial X^w)],$$

then the $\mathcal{O}_w$ are the *Schubert classes* in $K(X)$. Now Theorem 3.4.1(ii) yields the equalities

$$\alpha = \sum_{w \in W} \chi(\alpha \cdot \mathcal{I}^w)\, \mathcal{O}_w = \sum_{w \in W} \chi(\alpha \cdot \mathcal{O}_w)\, \mathcal{I}^w,$$

for any $\alpha \in K(X)$.

3.4.2 Remarks. 1) We have $[\mathcal{O}_{gY}] = [\mathcal{O}_Y]$ for any $g \in G$ and any subvariety $Y \subseteq X$. Indeed, this follows from Lemma 3.3.1 together with the fact that there is a connected chain of rational curves in G joining g to id (since the group G is generated by images of algebraic group homomorphisms $\mathbb{C} \to G$ and $\mathbb{C}^* \to G$). Thus, *G acts trivially on $K(X)$.*

2) Theorem 3.4.1 and the isomorphism $\operatorname{Gr} K(X) \cong H^*(X)$ imply that the classes $\mathcal{O}_w$ ($w \in W$, $\ell(w) \leq j$) form a basis of $F_j K(X)$; another basis of this group consists of the $\mathcal{I}^w$ ($w \in W$, $\ell(w) \leq j$).

Put $\mathcal{I}_w := [\mathcal{O}_{X_w}(-\partial X_w)]$, then we have $\mathcal{I}^w = \mathcal{I}_{w_0 w}$ by the preceding remark. Further, $\mathcal{I}_w = [\mathcal{O}_{X_w}] - [\mathcal{O}_{\partial X_w}]$ by the exact sequence $0 \to \mathcal{O}_{X_w}(-\partial X_w) \to \mathcal{O}_{X_w} \to \mathcal{O}_{\partial X_w} \to 0$. We will express the $\mathcal{I}_w$ in terms of the $\mathcal{O}_w$, and vice versa, in Proposition 4.3.2 below.

3) All the results of this section extend to an arbitrary flag variety G/P by replacing W with the set W^P of minimal representatives.

3.4.3 Examples. 1) Consider the case where X is the projective space $\mathbb{P}^n$. Then the Schubert varieties are the linear subspaces $\mathbb{P}^j$, $0 \leq j \leq n$, and the corresponding opposite Schubert varieties are the $\mathbb{P}^{n-j}$. Further, $\partial \mathbb{P}^j = \mathbb{P}^{j-1}$ so that

$$[\mathcal{O}_{\mathbb{P}^j}(-\partial \mathbb{P}^j)] = [\mathcal{O}_{\mathbb{P}^j}] - [\mathcal{O}_{\mathbb{P}^{j-1}}] = [\mathcal{O}_{\mathbb{P}^j}(-1)].$$

Thus, $\{[\mathcal{O}_{\mathbb{P}^j}]\}$ is a basis of $K(\mathbb{P}^n)$, with dual basis $\{[\mathcal{O}_{\mathbb{P}^{n-j}}(-1)]\}$.

The group $K(\mathbb{P}^n)$ may be described more concretely in terms of polynomials, as follows. For each coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$, the function $\mathbb{Z} \to \mathbb{Z}$, $k \mapsto \chi(\mathcal{F}(k))$ is polynomial of degree equal to the dimension of the support of $\mathcal{F}$; this defines the *Hilbert polynomial* $P_{\mathcal{F}}(t) \in \mathbb{Q}[t]$. Clearly, $P_{\mathcal{F}}(t) = P_{\mathcal{F}_1}(t) + P_{\mathcal{F}_2}(t)$ for any exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$. Thus, the Hilbert polynomial yields an additive map

$$P : K(\mathbb{P}^{n-1}) \to \mathbb{Q}[t], \quad [\mathcal{F}] \mapsto P_{\mathcal{F}}(t).$$

Since $\chi(\mathcal{O}_{\mathbb{P}^j}(k)) = \binom{k+j}{j}$, it follows that P maps the basis $\{[\mathcal{O}_{\mathbb{P}^j}]\}$ to the linearly independent polynomials $\{\binom{t+j}{j}\}$. Thus, P identifies $K(\mathbb{P}^{n-1})$ to the additive group of polynomials of degree $\leq n$ in one variable, which take integral values at all integers. Note that P takes non-zero values at classes of non-trivial sheaves.

2) More generally, consider the case where X is a Grassmannian. Let L be the ample generator of $\mathrm{Pic}(X)$, then the boundary of each Schubert variety X_I (regarded as a reduced Weil divisor on X_I) is the divisor of the section p_I of $L|_{X_I}$ by Remark 1.4.6.3. Thus, we have an exact sequence

$$0 \to L^{-1}|_{X_I} \to \mathcal{O}_{X_I} \to \mathcal{O}_{\partial X_I} \to 0,$$

where the map on the left is the multiplication by p_I. It follows that

$$[\mathcal{O}_{X_I}(-\partial X_I)] = [L^{-1}|_{X_I}].$$

Thus, the dual basis of the basis of Schubert classes $\{\mathcal{O}_{X_I} := \mathcal{O}_I\}$ is the basis $\{[L^{-1}] \cdot \mathcal{O}^I\}$.

Notes. Our degeneration of the diagonal of a flag variety was first constructed in [5], by using canonical compactifications of adjoint semisimple groups; see [7] for further developments realizing these compactifications as irreducible components of Hilbert schemes. The direct construction of 3.1 follows [9] with some simplifications. In [loc.cit.], this degeneration was combined with vanishing theorems for unions of Richardson varieties, to obtain a geometric approach to standard monomial theory. Conversely, this theory also yields the degeneration of the diagonal presented here, see [30].

The normality criterion in 3.2 appears first in [8]. It is also proved there that a B-invariant multiplicity-free subvariety Y of a G-variety Z is normal and Cohen-Macaulay (resp. has rational singularities), if Y is normal and Cohen-Macaulay (resp. has rational singularities). This yields an alternative proof for the rationality of singularities of Schubert varieties.

The exposition in 3.3 is based on [3] regarding fundamental results on the Grothendieck ring $K(X)$, where X is any nonsingular variety, and on [19] regarding the relation of this ring to intersection theory on X.

The dual bases of the K-theory of the flag manifold presented in 3.4 were first constructed by Kostant and Kumar [27], in the more general framework of T-equivariant K-theory. In fact, our approach fits into this framework. Indeed, T acts on $X \times X \times \mathbb{P}^1$ via $t(x,y,z) = (tx,ty,z)$. This action commutes with the $\mathbb{C}^*$-action via λ and leaves $\mathcal{X}$ invariant; clearly, the morphism $\pi : \mathcal{X} \to \mathbb{P}^1$ is T-invariant as well. So π is a degeneration of T-varieties. Further, the filtration of $\mathcal{O}_{\pi^{-1}}(0)$ constructed in Proposition 3.1.3 is also T-invariant. So Theorem 3.4.1 extends readily to the T-equivariant Grothendieck group.

The idea of determining the (equivariant) class of a subvariety by an (equivariant) degeneration to a union of simpler subvarieties plays an essential role in the works of Graham [21] on the structure constants of the equivariant cohomology ring of flag varieties, and of Knutson and Miller [26] on Schubert polynomials.

Finally, it would be very interesting to have other examples of varieties with a torus action, where the diagonal admits an equivariant degeneration to a reduced union of products of subvarieties. The Bott-Samelson varieties should provide such examples.

4 Positivity in the Grothendieck group of the flag variety

Let Y be a subvariety of the full flag variety $X = G/B$. By the results of Section 3, we may write in the Grothendieck group $K(X)$:

$$[\mathcal{O}_Y] = \sum_{w \in W} c^w(Y)\ \mathcal{O}_w,$$

where the $\mathcal{O}_w = [\mathcal{O}_{X_w}]$ are the Schubert classes. Further, $c^w(Y) = 0$ unless $\dim(Y) \geq \dim(X_w) = \ell(w)$, and we have in the cohomology group $H^*(X)$:

$$[Y] = \sum_{w \in W,\ \ell(w) = \dim(Y)} c^w(Y)\ [X_w].$$

By Proposition 1.3.6, it follows that $c^w(Y) = \#\{Y \cap gX^w\}$ if $\ell(w) = \dim(Y)$; in particular, $c^w(Y) \geq 0$ in this case.

One may ask for the signs of the integers $c^w(Y)$, where w is arbitrary. In this section, we show that these signs are *alternating*, i.e.,

$$(-1)^{\dim(Y) - \ell(w)} c^w(Y) \geq 0,$$

if Y has rational singularities (but not for arbitrary Y, see Remark 4.1.4.2).

We also show that the Richardson varieties have rational singularities, and we generalize to these varieties the results of Section 2 for cohomology groups of homogeneous line bundles on Schubert varieties. From this, we deduce that the structure constants of the ring $K(X)$ in its basis of Schubert classes have alternating signs as well, and we present several related positivity results.

Finally, we obtain a version in $K(X)$ of the Chevalley formula, that is, we decompose the product $[L_\lambda] \cdot \mathcal{O}_w$ in the basis of Schubert classes, where λ is any dominant weight, and X_w is any Schubert variety.

4.1 The class of a subvariety

In this subsection, we sketch a proof of the alternation of signs for the coefficients $c^w(Y)$. By Theorem 3.4.1, we have

$$c^w(Y) = \chi([\mathcal{O}_Y] \cdot [\mathcal{O}_{X^w}(-\partial X^w)]) = \chi([\mathcal{O}_Y] \cdot [\mathcal{O}_{X^w}]) - \chi([\mathcal{O}_Y] \cdot [\mathcal{O}_{\partial X^w}]).$$

Our first aim is to obtain a more tractable formula for $c^w(Y)$. For this, we need the following version of a lemma of Fulton and Pragacz (see [FP98] p. 108).

4.1.1 Lemma. *Let Y, Z be equidimensional Cohen-Macaulay subschemes of a nonsingular variety X. If Y meets Z properly in X, then the scheme-theoretic intersection $Y \cap Z$ is equidimensional and Cohen-Macaulay, of dimension* $\dim(Y) + \dim(Z) - \dim(X)$. *Further,*

$$Tor_i^X(\mathcal{O}_Y, \mathcal{O}_Z) = 0 = Tor_i^X(\omega_Y, \omega_Z)$$

for any $j \geq 1$, and $\omega_{Y\cap Z} = \omega_Y \otimes \omega_Z \otimes \omega_X^{-1}$.

Thus, we have in $K(X)$:

$$[\mathcal{O}_{Y\cap Z}] = [\mathcal{O}_Y] \cdot [\mathcal{O}_Z] \quad \textit{and} \quad [\omega_{Y\cap Z}] = [\omega_Y] \cdot [\omega_Z] \cdot [\omega_X^{-1}].$$

We also need another variant of Kleiman's transversality theorem (Lemma 1.3.1):

4.1.2 Lemma. *Let Y be a Cohen-Macaulay subscheme of the flag variety X. Then there exists a non-empty open subset Ω of G such that, Y meets properly gX^w for any $g \in \Omega$ and $w \in W$; further, $Y \cap gX^w$ is equidimensional and Cohen-Macaulay.*

If, in addition, Y is a variety with rational singularities, then $Y \cap gX^w$ is a disjoint union of varieties with rational singularities, for any g in a (possibly smaller) non-empty open subset of G.

We refer to [8] p. 142–144 for the proof of these results. Together with the fact that the boundary of any Schubert variety is Cohen-Macaulay (Corollary 2.2.7), they imply that

$$\begin{aligned} c^w(Y) &= \chi(\mathcal{O}_{Y\cap gX^w}) - \chi(\mathcal{O}_{Y\cap g\partial X^w}) \\ &= \chi(\mathcal{O}_{Y\cap gX^w}(-Y\cap g\partial X^w)) = \sum_{j=0}^{\dim(Y\cap gX^w)} (-1)^j \, h^j(\mathcal{O}_{Y\cap gX^w}(-Y\cap g\partial X^w)). \end{aligned}$$

Further, $\dim(Y \cap gX^w) = \dim(Y) + \dim(X^w) - \dim(X) = \dim(Y) - \ell(w)$. Thus, the assertion on the sign of $c^w(Y)$ will result from the following vanishing theorem, which holds for any partial flag variety X.

4.1.3 Theorem. *Let $Y \subseteq X$ be a subvariety with rational singularities and let $Z \subseteq X$ be a Schubert variety. Then we have for all g in a non-empty open subset of G:*

$$H^j(Y\cap gZ, \mathcal{O}_{Y\cap gZ}(-Y\cap g\partial Z)) = 0 \quad \textit{whenever} \quad j < \dim(Y) + \dim(Z) - \dim(X).$$

Proof. First we present the argument in the simplest case, where $X = \mathbb{P}^n$ and Y is nonsingular. Then $Z = \mathbb{P}^j$ and $\mathcal{O}_Z(-\partial Z) = \mathcal{O}_{\mathbb{P}^j}(-1)$, see Example 3.4.3.1. Thus, $Y \cap gZ =: V$ is a general linear section of Y. By Bertini's theorem, V is nonsingular (and irreducible if its dimension is positive). Further, $\mathcal{O}_{Y\cap gZ}(-Y\cap g\partial Z) = \mathcal{O}_V(-1)$. Thus, we are reduced to showing the vanishing of $H^j(V, \mathcal{O}(-1))$ for $j < \dim(V)$, where V is a nonsingular subvariety of $\mathbb{P}^n$. But this follows from the Kodaira vanishing theorem.

Next we consider the case where X is a Grassmannian, and Y is allowed to have rational singularities. Let L be the ample generator of $\mathrm{Pic}(X)$ and recall that $\mathcal{O}_Z(-\partial Z) = L^{-1}|_Z$. It follows that $\mathcal{O}_{Y\cap gZ}(-Y \cap g\partial Z) = L^{-1}|_{Y\cap gZ}$. Further, by Lemma 4.1.2, $Y \cap gZ$ is a disjoint union of varieties with rational singularities, of dimension $\dim(Y)+\dim(Z)-\dim(X)$. Thus, it suffices to show that $H^j(V, L^{-1}) = 0$ whenever V is a variety with rational singularities, L is an ample line bundle on V, and $j < \dim(V)$. Let $\pi : \tilde{V} \to V$ be a desingularization and put $\tilde{L} := \pi^* L$. Since $R^i\pi_*\mathcal{O}_{\tilde{V}} = 0$ for any $i \geq 1$, we obtain isomorphisms $H^j(V, L^{-1}) \cong H^j(\tilde{V}, \tilde{L}^{-1})$ for all j. Thus, the Grauert-Riemenschneider theorem (see [16] Corollary 5.6) yields the desired vanishing.

The proof for arbitrary flag varieties goes along similar lines, but is much more technical. Like in the proof of Theorem 2.3.1, one applies the Kawamata-Viehweg theorem to a desingularization of $Y \cap gZ$, see [8] p. 153–156 for details. □

4.1.4 Remarks. 1) As a consequence of Theorem 4.1.3, we have

$$c^w(Y) = (-1)^{\dim(Y)-\ell(w)}\ h^{\dim(Y)-\ell(w)}(\mathcal{O}_{Y\cap gX^w}(-Y \cap g\partial X^w)).$$

By using Serre duality on $Y \cap gX^w$, it follows that

$$c^w(Y) = (-1)^{\dim(Y)-\ell(w)}\ h^0(Y \cap gX^w, L_\rho \otimes \omega_Y).$$

2) The property of alternation of signs for the coefficients of $[\mathcal{O}_Y]$ on Schubert varieties fails for certain (highly singular) subvarieties Y of a flag variety X. Indeed, there exist surfaces $Y \subset X = \mathbb{P}^4$ such that the coefficient of $[\mathcal{O}_Y]$ on $[\mathcal{O}_x]$ (where x is any point of $\mathbb{P}^4$) is arbitrarily negative.

Specifically, let $d \geq 3$ be an integer and let C be the image of the morphism $\mathbb{P}^1 \to \mathbb{P}^3$, $(x, y) \mapsto (x^d, x^{d-1}y, xy^{d-1}, y^d)$ (a closed immersion). Then C is a nonsingular rational curve of degree d in $\mathbb{P}^3$. Regarding C as a curve in $\mathbb{P}^4 \supset \mathbb{P}^3$, choose $x \in \mathbb{P}^4 \setminus \mathbb{P}^3$ and denote by $Y \subset \mathbb{P}^4$ the projective cone over C with vertex x, that is, the union of all projective lines containing x and meeting C. Then Y is a surface, so that we have by Example 3.4.3.1:

$$[\mathcal{O}_Y] = c_2(Y)\ [\mathcal{O}_{\mathbb{P}^2}] + c_1(Y)\ [\mathcal{O}_{\mathbb{P}^1}] + c_0(Y)\ [\mathcal{O}_x].$$

We claim that $c_0(Y) \leq 4 - d$.

To see this, first notice that $c_0(Y) = \chi(\mathcal{O}_Y(-1))$, as $\chi(\mathcal{O}_{\mathbb{P}^j}(-1)) = 0$ for all $j \geq 1$. Thus,

$$c_0(Y) = \chi(\mathcal{O}_Y) - \chi(\mathcal{O}_{Y\cap\mathbb{P}^3}) = \chi(\mathcal{O}_Y) - \chi(\mathcal{O}_C) = \chi(\mathcal{O}_Y) - 1.$$

To compute $\chi(\mathcal{O}_Y)$, consider the desingularization $\pi : Z \to Y$, where Z is the total space of the projective line bundle $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-1))$ on C (that is, the blow-up of x in Y). Then we have an exact sequence

$$0 \to \mathcal{O}_Y \to \pi_*\mathcal{O}_Z \to \mathcal{C} \to 0,$$

where the sheaf $\mathcal{C}$ is supported at x. Further, $R^i\pi_*\mathcal{O}_Z = 0$ for all $i \geq 1$. (Indeed, since the affine cone $Y_0 := Y \setminus C$ is an affine neighborhood of x in Y, it suffices to show that $H^i(Z_0, \mathcal{O}_{Z_0}) = 0$ for $i \geq 1$, where $Z_0 := \pi^{-1}(Y_0)$. Now Z_0 is the total space of the line bundle $\mathcal{O}_C(-1) \cong \mathcal{O}_{\mathbb{P}^1}(-d)$ on $C \cong \mathbb{P}^1$, whence

$$H^i(Z_0, \mathcal{O}_{Z_0}) \cong \bigoplus_{n=0}^{\infty} H^i(Z_0, \mathcal{O}_{\mathbb{P}^1}(nd))$$

for any $i \geq 0$.) Thus, we obtain $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_Z) - \chi(\mathcal{C}) = 1 - h^0(\mathcal{C})$, so that $c_0(Y) = -h^0(\mathcal{C})$. Further, $\mathcal{C}$ identifies with the quotient $\pi_*\mathcal{O}_{Z_0}/\mathcal{O}_{Y_0}$. Since $Y_0 \subset \mathbb{C}^4$ is the affine cone over $C \subset \mathbb{P}^3$, this quotient is a graded vector space, with component of degree 1 being $H^0(\mathcal{O}_{\mathbb{P}^1}(d))/H^0(\mathcal{O}_{\mathbb{P}^3}(1))$, of dimension $d-4$. Thus, $h^0(\mathcal{C}) \geq d-4$.

On the other hand, for any surface $Y \subset \mathbb{P}^n$, the coefficient $c_2(Y)$ is the degree of Y, a positive integer. Further, one checks that

$$c_1(Y) = \chi(\mathcal{O}_Y(-1)) - \chi(\mathcal{O}_Y(-2)) = \chi(\mathcal{O}_{Y\cap\mathbb{P}^{n-1}}(-1)) = -h^1(\mathcal{O}_{Y\cap\mathbb{P}^{n-1}}(-1))$$

for any hyperplane $\mathbb{P}^{n-1}$ which does not contain Y. Thus, $c_1(Y) \leq 0$.

Likewise, one may check that the property of alternation of signs holds for any curve in any flag variety. In other words, the preceding counterexample has the smallest dimension.

4.2 More on Richardson varieties

We begin with a vanishing theorem for these varieties that generalizes Theorem 2.3.1. Let v, w in W such that $v \leq w$ and let X_w^v be the corresponding Richardson variety. Then X_w^v has two kinds of boundaries, namely

$$(\partial X_w)^v := (\partial X_w) \cap X^v \quad \text{and} \quad (\partial X^v)_w := (\partial X^v) \cap X_w,$$

where $\partial X^v = X^v \setminus C^v = \bigcup_{u>v} X^u$ denotes the boundary of the opposite Schubert variety X^v. Define the total boundary by

$$\partial X_w^v := (\partial X_w)^v \cup (\partial X^v)_w,$$

this is a closed subset of pure codimension 1 in X_w^v. We may now state

4.2.1 Theorem. *(i) The Richardson variety X_w^v has rational singularities, and its dualizing sheaf equals $\mathcal{O}_{X_w^v}(-\partial X_w^v)$. Further, we have in $K(X)$:*

$$[\mathcal{O}_{X_w^v}] = \mathcal{O}_w \cdot \mathcal{O}^v = \mathcal{O}_w \cdot \mathcal{O}_{w_0 w}.$$

(ii) $H^j(X_w^v, L_\lambda) = 0$ for any $j \geq 1$ and any dominant weight λ.
(iii) $H^j(X_w^v, L_\lambda(-(\partial X^v)_w)) = 0$ for any $j \geq 1$ and any dominant weight λ.
(iv) $H^j(X_w^v, L_\lambda(-\partial X_w^v)) = 0$ for any $j \geq 1$ and any regular dominant weight λ.

Proof. (i) follows from the rationality of singularities of Schubert varieties and the structure of their dualizing sheaves, together with Lemmas 4.1.1 and 4.1.2.

(ii) We adapt the proof of Theorem 2.3.1 to this setting. Choose a reduced decomposition $\underline{w}$ of w and let $Z_{\underline{w}}$ be the associated Bott-Samelson variety, with morphism

$$\pi_{\underline{w}} : Z_{\underline{w}} \to X_w.$$

Likewise, a reduced decomposition $\underline{v}$ of v yields an opposite Bott-Samelson variety $Z^{\underline{v}}$ (defined via the opposite Borel subgroup B^- and the corresponding minimal parabolic subgroups) together with a morphism

$$\pi^{\underline{v}} : Z^{\underline{v}} \to X^v.$$

Now consider the fibered product

$$Z = Z_{\underline{w}}^{\underline{v}} := Z_{\underline{w}} \times_X Z^{\underline{v}},$$

with projection $\pi = \pi_{\underline{w}}^{\underline{v}} : Z_{\underline{w}}^{\underline{v}} \to X_w \cap X^v = X_w^v$. Using Kleiman's transversality theorem, one checks that $Z_{\underline{w}}^{\underline{v}}$ is a nonsingular variety and that π is a desingularization of X_w^v. Let ∂Z be the union of the boundaries

$$(\partial Z_{\underline{w}})^{\underline{v}} := (\partial Z_{\underline{w}}) \times_X Z^{\underline{v}}, \quad (\partial Z^{\underline{v}})_{\underline{w}} := Z_{\underline{w}}) \times_X (\partial Z^{\underline{v}}).$$

This is a union of nonsingular divisors intersecting transversally, and one checks that $\omega_Z \cong \mathcal{O}_Z(-\partial Z)$.

Since X_w^v has rational singularities, it suffices to show that $H^j(Z, \pi^*\mathcal{L}_\lambda) = 0$ for $j \geq 1$. By Lemma 2.3.2 and the fact that Z is a subvariety of $Z_{\underline{w}} \times Z^{\underline{v}}$, the boundary ∂Z is the support of an effective ample divisor E on Z. Applying the Kawamata-Viehweg theorem with $D := N\partial Z - E$, where N is a large integer, and $\mathcal{L} := (\pi^*\mathcal{L}_\lambda)(\partial Z)$, we obtain the desired vanishing as in the proof of Theorem 2.3.1.

(iii) is checked similarly : let now E be the pull-back on Z of an effective ample divisor on $Z_{\underline{w}}$ with support $\partial Z_{\underline{w}}$. Let N be a large integer, and put $\mathcal{L} := (\pi^*\mathcal{L}_\lambda)((\partial Z_{\underline{w}})^{\underline{v}})$. Then the assumptions of the Kawamata-Viehweg theorem are still verified, since the projection $Z \to Z_{\underline{w}}$ is generically injective. Thus, we obtain

$$H^j(Z, (\pi^*\mathcal{L}_\lambda)(-(\partial Z^{\underline{v}})_{\underline{w}})) = 0 \quad \text{for} \quad j \geq 1.$$

This implies in turn that

$$R^j\pi_*\mathcal{O}_Z(-(\partial Z^{\underline{v}})_{\underline{w}}) = 0 \quad \text{for} \quad j \geq 1.$$

Together with the isomorphism

$$\pi_*\mathcal{O}_Z(-(\partial Z^{\underline{v}})_{\underline{w}}) = \mathcal{O}_{X_w^v}(-(\partial X^v)_w),$$

and a Leray spectral sequence argument, this completes the proof.

Likewise, (iv) follows from the vanishing of $H^j(Z, (\pi^* L_\lambda) \otimes \omega_Z)$ for $j \geq 1$. In turn, this is a consequence of the Grauert-Riemenschneider theorem, since L_λ is ample on X_w^v. $\square$

4.2.2 Remarks. 1) One may also show that the restriction $H^0(\lambda) \to H^0(X_w^v, L_\lambda)$ is surjective for any dominant weight λ. As in Corollary 2.3.3, it follows that the affine cone over X_w^v has rational singularities in the projective embedding given by any ample line bundle on X. In particular, X_w^v is projectively normal in any such embedding.
2) Theorem 4.2.1 (iv) does not extend to all the dominant weights λ. Indeed, for $\lambda = 0$ we obtain

$$H^j(X_w^v, \mathcal{O}(-\partial X_w^v)) = H^j(X_w^v, \omega_{X_w^v}).$$

By Serre duality, this equals $H^{\ell(w)-\ell(v)-j}(X_w^v, \mathcal{O}_{X_w^v})$; i.e., $\mathbb{C}$ if $j = \ell(w) - \ell(v)$, and 0 otherwise, by Theorem 4.2.1 (iii).

Next we adapt the construction of Section 3 to obtain a degeneration of the diagonal of any Richardson variety X_w^v. Let $\lambda : \mathbb{C}^* \to T$ be as in Subsection 3.1 and let $\mathcal{X}_w^v$ be the closure in $X \times X \times \mathbb{P}^1$ of the subset

$$\{(x, \lambda(t)x, t) \mid x \in X_w^v,\ t \in \mathbb{C}^*\} \subseteq X \times X \times \mathbb{C}^*.$$

We still denote by $\pi : \mathcal{X}_w^v \to \mathbb{P}^1$ the projection, then $\pi^{-1}(\mathbb{C}^*)$ identifies again to the product $\operatorname{diag}(X_w^v) \times \mathbb{C}^*$ above $\mathbb{C}^*$. Further, we have the following analogues of Theorem 3.1.2 and Proposition 3.1.3.

4.2.3 Proposition. *(i) With the preceding notation, we have equalities of subschemes of $X \times X$:*

$$\pi^{-1}(0) = \bigcup_{x \in W,\ v \leq x \leq w} X_x^v \times X_w^x \quad \textit{and} \quad \pi^{-1}(\infty) = \bigcup_{x \in W,\ v \leq x \leq w} X_w^x \times X_x^v.$$

(ii) The sheaf $\mathcal{O}_{\pi^{-1}(0)}$ admits a filtration with associated graded

$$\bigoplus_{x \in W,\ v \leq x \leq w} \mathcal{O}_{X_x^v} \otimes \mathcal{O}_{X_w^x}(-(\partial X^x)_w).$$

Therefore, we have in $K(X \times X)$:

$$[\mathcal{O}_{\operatorname{diag} X_w^v}] = \sum_{x \in W,\ v \leq x \leq w} [\mathcal{O}_{X_x^v}] \times [\mathcal{O}_{X_w^x}(-(\partial X^x)_w)].$$

Proof. Put

$$Y_w^v := \bigcup_{x \in W} X_x^v \times X_w^x.$$

By the argument of the proof of Theorem 3.1.2, we obtain the inclusion $Y_w^v \subseteq \pi^{-1}(0)$. Further, the proof of Proposition 3.1.3 shows that the structure sheaf $\mathcal{O}_{Y_w^v}$ admits a filtration with associated graded given by (ii).

On the other hand, Lemma 3.3.1 implies the equality $[\mathcal{O}_{\pi^{-1}(0)}] = [\mathcal{O}_{\mathrm{diag}\, X_w^v}]$ in $K(X \times X)$. Further, we have

$$[\mathcal{O}_{\mathrm{diag}\, X_w^v}] = [\mathcal{O}_{\mathrm{diag}\, X}] \cdot [\mathcal{O}_{X^v \times X_w}]$$

by Lemma 4.1.1, since $\mathrm{diag}\, X$ and $X^v \times X_w$ meet properly in $X \times X$ along $\mathrm{diag}\, X_w^v$. Together with Theorem 3.4.1 and Lemma 4.1.1 again, this yields

$$[\mathcal{O}_{\mathrm{diag}\, X_w^v}] = \sum_{x \in W} [\mathcal{O}_{X_x^v}] \times [\mathcal{O}_{X_w^x}(-(\partial X^x)_w)] = [\mathcal{O}_{Y_w^v}].$$

Thus, the structure sheaves of Y_w^v and of $\pi^{-1}(0)$ have the same class in $K(X \times X)$. But we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_{Y_w^v} \to \mathcal{O}_{\pi^{-1}(0)} \to 0,$$

where $\mathcal{F}$ is a sheaf on $X \times X$. So $[\mathcal{F}] = 0$, and it follows that $\mathcal{F} = 0$ (e.g., by Example 3.4.3.1). In other words, $Y_w^v = \pi^{-1}(0)$. This proves (ii) and the first assertion of (i); the second assertion follows by symmetry. □

4.3 Structure constants and bases of the Grothendieck group

Let c_{vw}^x be the structure constants of the Grothendieck ring $K(X)$ in its basis $\{\mathcal{O}_w\}$ of Schubert classes, that is, we have in $K(X)$:

$$\mathcal{O}_v \cdot \mathcal{O}_w = \sum_{x \in W} c_{vw}^x \, \mathcal{O}_x.$$

Then Theorem 4.2.1 (i) yields the equality $c_{vw}^x = c^x(X_w^{w_0 v})$. Together with Theorem 4.1.3, this implies a solution to Buch's conjecture:

4.3.1 Theorem. *The structure constants c_{vw}^x satisfy*

$$(-1)^{\ell(v)+\ell(w)+\ell(x)+\ell(w_0)} \, c_{vw}^x \geq 0.$$

Another consequence of Theorem 4.2.1 is the following relation between the bases $\{\mathcal{O}_w\}$ and $\{\mathcal{I}_w\}$ of the group $K(X)$.

4.3.2 Proposition. *We have in* $K(X)$

$$\mathcal{O}_w = \sum_{v \in W,\, v \le w} \mathcal{I}_v \quad \textit{and} \quad \mathcal{I}_w = \sum_{v \in W,\, v \le w} (-1)^{\ell(w)-\ell(v)}\, \mathcal{O}_v.$$

Proof. By Theorem 3.4.1, we have

$$\mathcal{O}_w = \sum_{v \in W} \chi(\mathcal{O}_w \cdot \mathcal{O}^v)\, \mathcal{I}_v.$$

Further,

$$\chi(\mathcal{O}_w \cdot \mathcal{O}^v) = \chi(\mathcal{O}_{X_w^v}) = \sum_j (-1)^j\, h^j(\mathcal{O}_{X_w^v})$$

equals 1 if $v = w$, and 0 otherwise, by Theorem 4.2.1.

Likewise, we obtain

$$\mathcal{I}_w = \sum_{v \in W} \chi(\mathcal{I}_w \cdot \mathcal{I}^v)\, \mathcal{O}_v \quad \text{and} \quad \chi(\mathcal{I}_w \cdot \mathcal{I}^v) = \chi(\mathcal{O}_{X_w^v}(-\partial X_w^v)) = \chi(\omega_{X_w^v})$$

by using the equalities $\mathcal{I}_w = [\mathcal{O}_{X_w}] - [\mathcal{O}_{\partial X_w}]$, $\mathcal{I}^v = [\mathcal{O}_{X^v}] - [\mathcal{O}_{\partial X^v}]$, together with Lemma 4.1.2 and Cohen-Macaulayness of Schubert varieties and of their boundaries. Further, we have by Serre duality and Theorem 4.2.1:

$$\chi(\omega_{X_w^v}) = (-1)^{\dim(X_w^v)}\, \chi(\mathcal{O}_{X_w^v}) = (-1)^{\ell(w)-\ell(v)}.$$

□

4.3.3 Remark. The preceding proposition implies that the Möbius function of the Bruhat order on W maps any $(v, w) \in W \times W$ to $(-1)^{\ell(w)-\ell(v)}$ if $v \le w$, and to 0 otherwise. We refer to [14] for a direct proof of this combinatorial fact.

Using the duality involution $\alpha \mapsto \alpha^\vee$ of $K(X)$, we now introduce another natural basis of this group for which the structure constants become positive.

4.3.4 Proposition. *(i) We have in* $K(X)$

$$[L_\rho|_{X_w}(-\partial X_w)] = (-1)^{\ell(w_0)-\ell(w)}\, \mathcal{O}_w^\vee.$$

In particular, the classes

$$\mathcal{I}_w(\rho) := [L_\rho|_{X_w}(-\partial X_w)] = [L_\rho] \cdot \mathcal{I}_w$$

form a basis of the Grothendieck group $K(X)$.

(ii) For any Cohen-Macaulay subscheme Y of X with relative dualizing sheaf $\omega_{Y/X} = \omega_Y \otimes \omega_X^{-1}$, we have

$$[\omega_{Y/X}] = \sum_{w \in W} (-1)^{\dim(Y)-\ell(w)}\, c^w(Y)\, \mathcal{I}_w(\rho).$$

Thus, if Y is a variety with rational singularities, then the coordinates of $\omega_{Y/X}$ in the basis $\{\mathcal{I}_w(\rho)\}$ are the absolute values of the $c^w(Y)$.
(iii) The structure constants of $K(X)$ in the basis $\{\mathcal{I}_w(\rho)\}$ are the absolute values of the structure constants c^x_{vw}.

Proof. We have

$$[\mathcal{O}_{X_w}]^\vee = (-1)^{\operatorname{codim}(X_w)}\, [\omega_{X_w}] \cdot [\omega_X^{-1}] = (-1)^{\operatorname{codim}(X_w)}\, [L_{-\rho}|_{X_w}(-\partial X_w)] \cdot [L_{2\rho}] = (-1)^{\ell(w_0)-\ell(w)}\, \mathcal{I}_w(\rho).$$

This proves (i). Assertions (ii), (iii) follow by applying the duality involution to Theorems 4.1.3 and 4.3.1. □

By similar arguments, we obtain the following relations between the bases $\{\mathcal{I}_w(\rho)\}$ and $\{\mathcal{O}_w\}$.

4.3.5 Proposition. *We have*

$$\mathcal{I}_w(\rho) = \sum_{v \in W} h^v_w\, \mathcal{O}_v, \quad \textit{where} \quad h^v_w := h^0(X^v_w, L_\rho(-\partial X^v_w)).$$

In particular, $h^v_w \neq 0$ only if $v \leq w$. Further,

$$\mathcal{O}_w = \sum_{v \in W,\, v \leq w} (-1)^{\ell(w)-\ell(v)} h^v_w\, \mathcal{I}_v(\rho).$$

Next we consider the decomposition of the products $[L_\lambda] \cdot \mathcal{O}_w$ in the basis $\{\mathcal{O}_v\}$, where λ is a dominant weight. These products also determine the multiplication in $K(X)$. Indeed, by [38], this ring is generated by the classes of line bundles (thus, going to the associated graded $\operatorname{Gr} K(X) \cong H^*(X)$, it follows that the cohomology ring is generated over the rationals by the Chern classes of line bundles). Since any weight is the difference of two dominant weights, it follows that the ring $K(X)$ is generated by the classes $[L_\lambda]$, where λ is dominant. This motivates the following

4.3.6 Theorem. *For any dominant weight λ and any $w \in W$, we have in $K(X)$:*

$$[L_\lambda] \cdot \mathcal{O}_w = [L_\lambda|_{X_w}] = \sum_{v \in W,\, v \leq w} h^0(X^v_w, L_\lambda(-(\partial X^v)_w))\, \mathcal{O}_v.$$

In particular, the coefficients of $[L_\lambda] \cdot \mathcal{O}_w$ in the basis of Schubert classes are non-negative.

Proof. By Theorem 3.4.1, we have

$$[L_\lambda] \cdot \mathcal{O}_w = \sum_{v \in W} \chi([L_\lambda] \cdot \mathcal{O}_w \cdot \mathcal{I}^v)\ \mathcal{O}_v.$$

Further, as in the proof of Proposition 4.3.2, we obtain

$$\chi([L_\lambda] \cdot \mathcal{O}_w \cdot \mathcal{I}^v) = \chi(X_w^v, L_\lambda(-(\partial X^v)_w)).$$

The latter equals $h^0(X_w^v, L_\lambda(-(\partial X^v)_w))$ by Theorem 4.2.1. □

Next let σ be a non-zero section of L_λ on X. Then the structure sheaf of the zero subscheme $Z(\sigma) \subset X$ fits into an exact sequence

$$0 \to L_{-\lambda} \to \mathcal{O}_X \to \mathcal{O}_{Z(\sigma)} \to 0.$$

Thus, the class $[\mathcal{O}_{Z(\sigma)}] = 1 - [L_{-\lambda}]$ depends only on λ; we denote this class by $\mathcal{O}_\lambda$. Note that the image of $\mathcal{O}_\lambda$ in the associated graded $\operatorname{Gr} K(X) \cong H^*(X)$ is the class of the divisor of σ, i.e., $c_1(L_\lambda)$. We now decompose the products $\mathcal{O}_\lambda \cdot \mathcal{O}_w$ in the basis of Schubert classes.

4.3.7 Proposition. *For any dominant weight λ and any $w \in W$, we have in $K(X)$:*

$$\mathcal{O}_\lambda \cdot \mathcal{O}_w = \sum_{v \in W,\ v < w} (-1)^{\ell(w)-\ell(v)-1}\ h^0(X_w^v, L_\lambda(-(\partial X_w)^v))\ \mathcal{O}_v.$$

Proof. We begin by decomposing the product $[L_\lambda] \cdot \mathcal{I}_w$ in the basis $\{\mathcal{I}_v\}$. As in the proof of Theorem 4.3.6, we obtain

$$\begin{aligned}[L_\lambda] \cdot \mathcal{I}_w &= \sum_{v \in W} \chi([L_\lambda] \cdot \mathcal{I}_w \cdot \mathcal{O}^v)\ \mathcal{I}_v \\ &= \sum_{v \in W,\ v \le w} \chi(X_w^v, L_\lambda(-(\partial X_w)^v))\ \mathcal{I}_v = \sum_{v \in W,\ v \le w} h^0(X_w^v, L_\lambda(-(\partial X_w)^v))\ \mathcal{I}_v.\end{aligned}$$

Applying the duality involution and using the equality

$$\mathcal{I}_w^\vee = (-1)^{\ell(w_0)-\ell(w)}\ [L_\rho] \cdot \mathcal{O}_w,$$

we obtain

$$[L_{-\lambda}] \cdot \mathcal{O}_w = \sum_{v \in W, v \le w} (-1)^{\ell(w)-\ell(v)}\ h^0(X_w^v, L_\lambda(-(\partial X_w)^v))\ \mathcal{O}_v.$$

Further, $[L_{-\lambda}] = 1 - \mathcal{O}_\lambda$. Substituting in the previous equality completes the proof. □

4.3.8 Remarks. 1) In the case of a fundamental weight χ_d, the divisor of the section $p_{w_0\chi_d}$ equals $[X_{w_0 s_d}]$, and hence $\mathcal{O}_{w_0\chi_d}$ is the class of the Schubert divisor $X_{w_0 s_d}$. Thus, Proposition 4.3.7 expresses the structure constants arising from the product of the classes of Schubert divisors by arbitrary Schubert classes. These structure constants have alternating signs, as predicted by Theorem 4.3.1.
2) Proposition 4.3.7 gives back the Chevalley formula in $H^*(X)$ obtained in Proposition 1.4.3. Indeed, going to the associated graded $\operatorname{Gr} K(X) \cong H^*(X)$ yields

$$c_1(L_\lambda) \cup [X_w] = \sum_v h^0(X_w^v, L_\lambda(-(\partial X_w)^v))\,[X_v],$$

the sum over the $v \in W$ such that $v \le w$ and $\ell(v) = \ell(w) - 1$. For such a v, we know that the Richardson variety X_w^v is isomorphic to $\mathbb{P}^1$, identifying the restriction of L_λ to $\mathcal{O}_{\mathbb{P}^1}(\lambda_i - \lambda_j)$, where $v = ws_{ij}$ and $i < j$. Further, $(\partial X_w)^v$ is just the point vF, so that $L_\lambda|_{X_w^v}(-(\partial X_w)^v)$ identifies to $\mathcal{O}_{\mathbb{P}^1}(\lambda_i - \lambda_j - 1)$. Thus, $h^0(X_w^v, L_\lambda(-(\partial X_w)^v)) = \lambda_i - \lambda_j$.
3) The results of this subsection adapt to any partial flag variety $X = G/P$. In particular, if X is the Grassmannian $\operatorname{Grass}(d, n)$ and L is the ample generator of $\operatorname{Pic}(X)$, then we have $L_{X_I}(-\partial X_I) \cong \mathcal{O}_{X_I}$, so that Theorem 4.3.6 yields the very simple formula

$$[L|_{X_I}] = [L] \cdot \mathcal{O}_I = \sum_{J,\, J \le I} \mathcal{O}_J.$$

In particular, $[L] = \sum_I \mathcal{O}_I$ (sum over all the multi-indices I).

By Möbius inversion, it follows that $[L^{-1}] \cdot \mathcal{O}_I = \sum_{J,\, J \le I} (-1)^{|I|-|J|}\, \mathcal{O}_J$. This yields

$$\mathcal{O}_{\omega_d} \cdot \mathcal{O}_I = \sum_{J,\, J < I} (-1)^{|I|-|J|-1}\, \mathcal{O}_J,$$

where $\mathcal{O}_{\omega_d}$ is the class of the Schubert divisor.

Notes. A general reference for this section is [6], from which much of the exposition is taken.

Stronger versions of Theorem 4.2.1 were obtained in [9] by the techniques of Frobenius splitting, and Proposition 4.2.3 was also proved there. These results also follow from standard monomial theory by work of Kreiman and Lakshmibai for Grassmannians [29], and of Lakshmibai and Littelmann in general [30].

Propositions 4.3.2, 4.3.4 (i) and 4.3.5 are due to Kostant and Kumar [27] in the framework of T-equivariant K-theory; again, the present approach is also valid in this framework.

Theorem 4.3.6 also extends readily to T-equivariant K-theory. In this form, it is due to Fulton and Lascoux in the case of the general linear group [17]. Then the general case was settled by Pittie and Ram [42], Mathieu [39], Littelmann and Seshadri [33], via very different methods. The latter authors obtained a more precise version by using standard monomial theory. Specifically, they constructed a B-stable filtration of the sheaf $L_\lambda|_{X_w}(-\partial X_w)$, with associated graded sheaf being the direct sum of structure sheaves of Schubert varieties (with twists by characters). This was generalized to Richardson varieties by Lakshmibai and Littelmann [30], again by using standard monomial theory.

This theory constructs bases for spaces of sections of line bundles over flag varieties, consisting of T-eigenvectors which satisfy very strong compatibility properties to Schubert and opposite Schubert varieties. It was completed by Littelmann [31], [32] after contributions of Lakshmibai, Musili, and Seshadri. Littelmann's approach is based on methods from combinatorics (the path model in representation theory) and algebra (quantum groups at roots of unity). It would be highly desirable to obtain a completely geometric derivation of standard monomial theory; some steps in this direction are taken in [9].

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