Attainability of Hardy inequality through sharp Sobolev-Lorentz embeddings

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in collaboration with

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Emerging issues in nonlinear elliptic equations: singularities, singular perturbations and non local problems

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I have never done anything "useful". No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. (Hardy, *A Mathematician's Apology*, 1940)

Hardy inequality

Kufner, Maligranda and Persson, *The Prehistory of the Hardy Inequality* (2006)

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Let $\Omega \subseteq \mathbb{R}^n$, $0 \in \Omega$, $1 < p < n$: then for any $u \in \mathcal{D}^{1,p}(\Omega)$

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Many generalization in several directions (Adimurthi,Brezis, Chaudhuri, Cianchi, del Pino, Davila, Dolbeault, Dupaigne, Ferone, Filippas, Ghoussoub, Kufner, Marcus, Mitidieri, Mizel, Moradifam, Opic, Pinchover, Ramaswamy, Tarantello, Tertikas, Tintarev, Trombetti, Vazquez and many others)

We focus on improvements of Hardy inequality in Ω bounded s.t. $0 \in \Omega$. The first result is due to Brezis and Vazquez (1997):

$$\frac{(n-2)^2}{4}\int_\Omega \frac{u^2}{|x|^2}\,dx + H_2\left(\frac{\omega_n}{\Omega}\right)^{\frac{2}{n}}\int_\Omega u^2 \le \int_\Omega |\nabla u|^2$$

The presence of remainder terms avoids the attainability of the best constant on bounded domains.

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- some classes of remainders are excluded (Ghoussoub-Moradifam);
- the best constant is not attained : the associated E-L equation is solved by

$$\psi = |x|^{-\frac{n-p}{p}}$$

  which are not in $W^{1,p}(\mathbb{R}^n)$ (Brezis-Vazquez, (1997));

Lorentz spaces

If $1 \leq q < \infty$ we define

$$L^{p,q}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \|u\|_{p,q} := \left(\int_0^\infty \left(u^*(t) t^{1/p} \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}$$

whereas for $q = \infty$

$$\|u\|_{p,\infty} := \sup_{t>0} t^{\frac{1}{p}} u^*(t)$$

the so-called Marcinkiewicz or weak-L^p space.

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Lorentz scale:

$$\boxed{L^{p,1} \subset L^{p,q_1} \subset L^{p,q_2} \subset L^{p,\infty}, \quad \text{if } 1 \leq q_1 < q_2 \leq \infty}$$

Optimal Sobolev Embeddings

The optimal Sobolev embedding

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$$

can be refined in the context of r.i. spaces:

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(Peetre (1966)) and generalized to Sobolev Lorentz sp.

$$W^1L^{p,q}(\mathbb{R}^n) \hookrightarrow L^{p^*,q}(\mathbb{R}^n), \quad 1 \leq q \leq \infty$$

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(Edmunds, Kerman and Pick, (2000)). The same embedding holds for the larger homogeneous space:

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The corresponding sharp inequality is known only for $1 \leq q \leq p$ (Alvino (1977))

$$\|u\|_{p^*,q} \leq \frac{p}{n-p}\,\omega_n^{-\frac{1}{n}}\,\|\nabla u\|_{p,q}, \quad u \in \mathcal{C}_c^\infty$$

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There is an explicit relation between the boundedness of certain Hardy operator and Sobolev type inequalities involving r.i. spaces (Kerman and Pick (2006)).

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- ghost solution of the EL-equation: $\psi = |x|^{-\frac{n-p}{p}}$

Hardy inequality VS Sobolev Lorentz embedding

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$\Leftrightarrow$ Optimal emb. in rearr. invariant sp.

$$\mathcal{D}^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*,p}(\mathbb{R}^n)$$

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- the sharp constant does not depend on q

Main results

Cassani, Ruf, T. (2017)

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx \leq \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |\nabla u|^p dx \quad \overset{known}{\Longleftrightarrow} \quad \|u\|_{p^*,p} \leq \frac{p}{n-p}\, \omega_n^{-\frac{1}{n}} \|\nabla u\|_p$$

$$new \Uparrow \qquad\qquad \Downarrow new$$

$$\|u\|_{p^*,\infty} \leq \frac{p}{n-p}\, \omega_n^{-\frac{1}{n}} \|\nabla u\|_{p,\infty} \quad \overset{new}{\Longleftarrow} \quad \|u\|_{p^*,q} \leq \frac{p}{n-p}\, \omega_n^{-\frac{1}{n}} \|\nabla u\|_{p,q},$$

$$q = \infty \qquad\qquad p \leq q < \infty$$

with sharp constants.

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with sharp constants.

Hardy ineq. is equivalent to a sharp weak-weak Sobolev type embedding:

$$\mathcal{D}^1_W L^{p,\infty} \hookrightarrow L^{p^*,\infty}$$

Attainability:

Cassani, Ruf, T. (2017)

The sharp constant in the embedding

$$\mathcal{D}^1_W L^{p,q}(\mathbb{R}^n) \hookrightarrow L^{p^*,q}(\mathbb{R}^n)$$

is attained if and only if $q = \infty$.
A maximizer is

$$\psi(x) = |x|^{-\frac{n-p}{p}}$$

The ghost maximizer of Hardy inequality,

$$\psi(x)$$

is a true maximizer of its equivalent embedding!

Lorentz-Sobolev spaces

(Cianchi, Costea, Edmunds, Kerman, Pick...)

$$W^1 L^{p,q}(\Omega) = \{u : \|\nabla u\|_{p,q} + \|u\|_{p,q} < \infty\}$$

Poincaré type inequality available for bounded domains.

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If $\boxed{\Omega = \mathbb{R}^n}$: Homogeneous Sobolev-Lorentz spaces

$$\boxed{\mathcal{D}^1_H L^{p,q} = cl\,\{u \in \mathcal{C}^\infty_c(\mathbb{R}^n) : \|\nabla u\|_{p,q} < \infty\}}$$

for $1 \leq p < +\infty, 1 \leq q \leq +\infty$.

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for $1 \le p < +\infty, 1 \le q \le +\infty$.
Since $\mathcal{D}^1_H L^{p,q} \hookrightarrow L^{p^*,q}$, one may also consider the alternative definition

$$\boxed{\mathcal{D}^1_W L^{p,q} = \left\{u \in L^{p^*,q}(\mathbb{R}^n) : \|\nabla u\|_{p,q} < \infty\right\}}$$

It turns out that

Non limiting case (Costea 2006)

$$\mathcal{D}^1_W L^{p,q} = \mathcal{D}^1_H L^{p,q} := \mathcal{D}^1 L^{p,q} \qquad \text{if } 1 \le p, q < \infty$$

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whereas

Limiting case (Costea 2017)

$$\mathcal{D}^1_H L^{p,\infty} \subset \mathcal{D}^1_W L^{p,\infty}$$

A function belonging to $\mathcal{D}^1_W L^{p,\infty} \setminus \mathcal{D}^1_H L^{p,\infty}$ is given by

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The framework

We will prove the equivalence between Hardy ineq. and Weak-Sobolev Lorentz emb. in $\mathcal{D}^1_W L^{p,\infty}$.

Alvino's proof is based on

- ⋆ Polya-Szego type inequality (Talenti);
- ⋆ Hardy-Littlewood inequality;

both strongly relying on the assumption $1 \leq q \leq p$!

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Let $\boxed{p < q < \infty}$.

- reduction to the radial case, mimicking Alvino-Lions-Trombetti (1989): for any $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ there exists $u_{rad} \in \mathcal{D}^{1,\sharp}L^{p,q}(\mathbb{R}^n)$:

$$\boxed{\|u_{rad}\|_{p^*,q} \geq \|u\|_{p^*,q} \quad \text{and} \quad \|\nabla u_{rad}\|_{p,q} \leq \|\nabla u\|_{p,q}}$$

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$$\boxed{\|u_{rad}\|_{p^*,q} \geq \|u\|_{p^*,q} \quad \text{and} \quad \|\nabla u_{rad}\|_{p,q} \leq \|\nabla u\|_{p,q}}$$

- scaling: for any $u \in \mathcal{D}^{1,\sharp}L^{p,q}$ set

$$\boxed{v(x) = \left[u(|x|^{\frac{p}{q}})\right]^{\frac{q}{p}} = \left[u^*(|x|^{\frac{np}{q}}\omega_n)\right]^{\frac{q}{p}}}$$

Then

$$\|v\|_{p^*,p} = \left(\frac{q}{p}\right)^{\frac{1}{p}} \omega_n^{\frac{p-q}{pp^*}} \|u\|_{p^*,q}^{\frac{q}{p}}$$

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and

$$|\nabla v(x)| = |u(|x|^{\frac{p}{q}})|^{\frac{q-p}{p}} |\nabla u(|x|^{\frac{p}{q}})| |x|^{\frac{p-q}{q}}$$

$p < q$ implies $|x|^{\frac{p-q}{q}}$ is decreasing.

$$\|\nabla v\|_p \leq \left\{ \int_{\mathbb{R}^n} \left[|u|^{\frac{q-p}{p}} (|x|^{\frac{p}{q}}) |\nabla u|^{\sharp} (|x|^{\frac{p}{q}}) |x|^{\frac{p-q}{q}} \right]^p dx \right\}^{1/p} \cdots$$

Then

$$\|v\|_{p^*,p} = \left(\frac{q}{p}\right)^{\frac{1}{p}} \omega_n^{\frac{p-q}{pp^*}} \|u\|_{p^*,q}^{\frac{q}{p}}$$

and

$$|\nabla v(x)| = |u(|x|^{\frac{p}{q}})|^{\frac{q-p}{p}} |\nabla u(|x|^{\frac{p}{q}})| |x|^{\frac{p-q}{q}}$$

$p < q$ implies $|x|^{\frac{p-q}{q}}$ is decreasing.

$$\|\nabla v\|_p \leq \left\{ \int_{\mathbb{R}^n} \left[|u|^{\frac{q-p}{p}} (|x|^{\frac{p}{q}}) |\nabla u|^{\sharp} (|x|^{\frac{p}{q}}) |x|^{\frac{p-q}{q}} \right]^p dx \right\}^{1/p} \cdots$$

$$\cdots \leq \left(\frac{q}{p}\right)^{\frac{1}{p}} \omega_n^{-\frac{(n-p)(q-p)}{np^2}} \|u\|_{p^*,q}^{\frac{q-p}{p}} \cdot \|\nabla u\|_{p,q}$$

Combining these two inequalities yields

$$\|u\|_{p^*,q} \leq \frac{p}{n-p} \omega_n^{-\frac{1}{n}} \|\nabla u\|_{p,q}$$

- Sharpness: through an explicit maximizing sequence (see Alvino)

$$v_\varepsilon(x) := \begin{cases} |x|^{-\frac{n-p}{p}+\varepsilon}, & \text{if } |x| < 1 \\ 1 - \left(\frac{n-p}{p} - \varepsilon\right)(|x| - 1), & \text{if } 1 \leq |x| < 1 + \frac{1}{\frac{n-p}{p} - \varepsilon} \end{cases}$$

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$$\frac{\|\nabla v_\varepsilon\|_{p,q}}{\|v_\varepsilon\|_{p^*,q}} \sim \omega_n^{\frac{1}{n}} \left(\frac{n-p}{p}\right) \quad \text{as } \varepsilon \to 0$$

The limiting case $q = \infty$

Given $u \in \mathcal{D}^{1,\sharp}_{W} L^{p,\infty}(\mathbb{R}^n)$, define $\boxed{v(r) = r^{n/p^*} u^\sharp(r)}$.

Then

$$\infty > \|u\|_{p^*,\infty} = \omega_n^{1/p^*} \|v\|_\infty \Rightarrow \|v\|_\infty = \lim_{\gamma\to\infty} \|v\|_\gamma$$

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and

$$\|u\|_{p^*,\infty} = \omega_n^{1/p^*} \cdot \lim_{\gamma \to +\infty} \|v\|_\gamma \leq \omega_n^{1/p^* - 1/n} \cdot \lim_{\gamma \to +\infty} \omega_n^{-1/\tilde{p}^* + 1/\tilde{p}} \frac{\tilde{p}}{n-\tilde{p}} \|\nabla u\|_{\tilde{p},\gamma}$$

$$= \cdots = \omega_n^{1/p - 1/n} \frac{p}{n-p} \|\nabla u\|_{p,\infty}$$

We have then proved

$$
\begin{aligned}
\|u\|_{p^*,p} &\leq \tfrac{p}{n-p}\,\omega_n^{-\frac{1}{n}}\|\nabla u\|_p \;\; \forall\, u \in \mathcal{D}^{1,p} \\
&\Downarrow \\
\|u\|_{p^*,q} &\leq \tfrac{p}{n-p}\,\omega_n^{-\frac{1}{n}}\|\nabla u\|_{p,q} \;\; \forall\, u \in \mathcal{D}^{1}L^{p,q} \quad p \leq q < \infty \\
&\Downarrow \\
\|u\|_{p^*,\infty} &\leq \tfrac{p}{n-p}\,\omega_n^{-\frac{1}{n}}\|\nabla u\|_{p,\infty} \;\; \forall\, u \in \mathcal{D}^{1}_{W}L^{p,\infty}
\end{aligned}
$$

that is,

$$\boxed{\textit{Hardy ineq.} \quad \Rightarrow \quad \mathcal{D}^{1}_{W}L^{p,\infty} \hookrightarrow L^{p^*,\infty}}$$

with sharp constant.

Let us now prove the reverse implication:

we assume

$$\|v\|_{p^*,\infty} \leq \frac{p}{n-p}\omega_n^{-\frac{1}{n}}\|\nabla v\|_{p,\infty} \qquad \forall\ v \in \mathcal{D}^1_W L^{p,\infty}(\mathbb{R}^n)$$

Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^n)$:

we want to prove Hardy inequality

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^n}\, dx \leq \left(\tfrac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |\nabla u|^p dx$$

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By symmetrization + density + dilation invariance of the norms, we may assume

$$\boxed{u \in \mathcal{D}^{1,p,\sharp}(B_1)}.$$

The proof relies upon an auxiliary function

$$v(r) = \int_r^1 \rho^{-\frac{n}{p}} \int_\rho^1 |u'(t)|^p t^{n-1} dt \, d\rho$$

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so that

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$\cdots$ at the end $\boxed{v \in \mathcal{D}_W^1 L^{p,\infty}}$.

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writing u as integral of its derivatives u' and applying Fubini's Thm

$$\cdots \leq \left(\frac{p}{n-p}\right)^{p-1} \omega_n^{1-\frac{1}{p^*}} \|v\|_{p^*,\infty} \overset{weak\ emb.}{\leq} \omega_n^{1-\frac{1}{p}} \left(\frac{p}{n-p}\right)^p \|\nabla v\|_{p,\infty}$$

$$= \left(\frac{p}{n-p}\right)^p \|\nabla u\|_p^p$$

$q = +\infty$

Sharp constant attained:
'model' extremal:

$$\psi = |x|^{-\frac{n-p}{p}} \in \mathcal{D}^1_W L^{p,\infty}$$

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Sharp constant never attained:
Let $u \in \mathcal{D}^1 L^{p,q}$ any (radial) maximizer. Then,

$$v(x) = \left[u(|x|^{\frac{p}{q}}) \right]^{\frac{q}{p}}$$

is a maximizer for Hardy ineq.: a contradiction.

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- Is it possible to obtain, via equivalent weak embedding, a proof of nonexistence of remainders?

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..to be continued..

- Is it possible to obtain, via equivalent weak embedding, a proof of nonexistence of remainders?
- Limiting cases $p = n$?
- Higher order case (Rellich inequality)?


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ATTENTION
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