Symmetry breaking and restoration in the Ginzburg-Landau model of nematic liquid crystals

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(joint work with Marcel Clerc and Michal Kowalczyk)

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The mathematical model

- We consider the energy functional:

$$E(u) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 - \frac{1}{2\epsilon^2}\mu(x)|u|^2 + \frac{1}{4\epsilon^2}|u|^4 - \frac{a}{\epsilon}f(x) \cdot u, \quad (1)$$

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- $f = (f_1, f_2) \in C^\infty(\mathbb{R}^2, \mathbb{R}^2) \cap L^1(\mathbb{R}^2, \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is also radial i.e. $f(x) = f_{\mathrm{rad}}(|x|)\frac{x}{|x|}$, and $f_{\mathrm{rad}} > 0$ on $(0, \infty)$.

The mathematical model

- E admits a global minimizer v solving

$$\epsilon^2 \Delta v + \mu(x) v - |v|^2 v + \epsilon a f(x) = 0, \qquad x \in \mathbb{R}^2. \tag{2}$$

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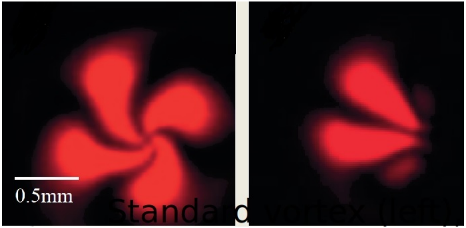
- This mathematical model was proposed by Marcel Clerc to describe the light-matter interaction in a liquid crystal:
- the parameter a represents the intensity of the light,
- the vector field v is related to the orientation of the molecules in the liquid crystal.
- The model seems to be good since it confirms the experiments.



Problems

- Qualitative properties of the global minimizers v as $\epsilon > 0$ is small and $a(\epsilon) \geq 0$ is bounded uniformly in ϵ. More precisely: existence and location of zeros; profile of v in the regions where $\mu > 0$, $|\mu| \ll 1$, and $\mu < 0$.

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- Symmetry breaking and restoration phenomena as a and ϵ vary. The energy (1) and equation (2) are invariant under the transformations $v(x) \mapsto g^{-1}v(gx)$, $\forall g \in O(2)$. In addition, $\forall \epsilon > 0$, $\forall a > 0$ there exists a unique radial solution $u(x) = u_{\mathrm{rad}}(|x|)\frac{x}{|x|}$, (i.e. a solution invariant under the previous transformations).

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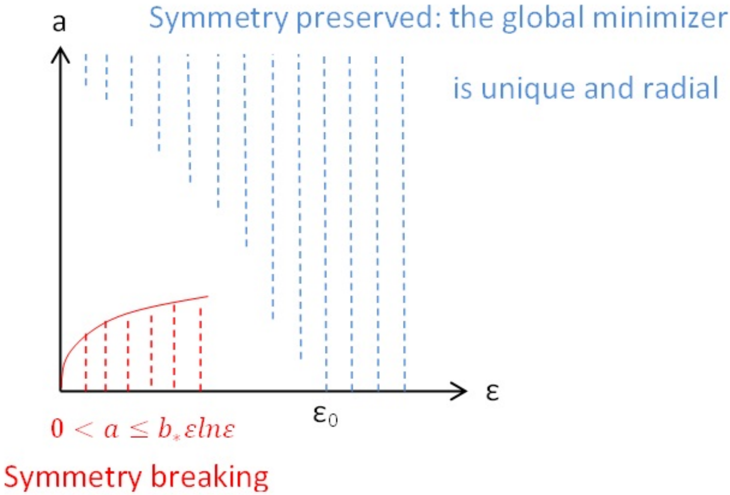
- However, the symmetry is broken as soon as $a > 0$; it is restored for sufficiently large values of a.
- The case $a = 0$ is special since the global minimizer inherits the radial and one dimensional profile of μ: $v(x) = (v_{\mathrm{rad}}(|x|), 0)$ up to change of v by gv with $g \in SO(2)$.

Problems


Symmetry preserved: the global minimizer
is unique and radial
a
ε
ε0
$0 < a \leq b_* \varepsilon ln \varepsilon$
Symmetry breaking

Method to determine $\lim_{\epsilon \to 0} v_{\epsilon, a(\epsilon)}$

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- Next, by the theorem of Ascoli we obtain the convergence of the rescaled minimizers to a minimal solution η of a new equation:

$$\Delta \eta = \nabla W(\eta), \ \ \eta : \mathbb{R}^2 \to \mathbb{R}^2, \ \ W : \mathbb{R}^2 \to \mathbb{R}, \tag{3}$$

that is, $E_{\mathrm{W}}(\eta, \operatorname{supp} \phi) \leq E_{\mathrm{W}}(\eta + \phi, \operatorname{supp} \phi)$, for all $\phi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$, where

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  is the energy associated to (3).
- To determine the limit η we need to know the classification of all minimal solutions of (3) and some properties of v.

Method to determine $\lim_{\epsilon\to 0} v_{\epsilon,a(\epsilon)}$

- In particular, in the case of the Ginzburg-Landau equation

$$\Delta\eta = (|\eta|^2 - 1)\eta,\ \eta : \mathbb{R}^2 \to \mathbb{R}^2,\ W(u) = \frac{1}{4}(|u|^2 - 1)^2, \quad (4)$$

we utilize a result of Mironescu (1996): any minimal solution of (4) is either a constant of modulus 1 or (up to orthogonal transformation in the range and translation in the domain) the radial solution $\eta(x) = \eta_{\mathrm{rad}}(|x|)\frac{x}{|x|}$.

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- In many situations the limiting equation is trivial, for instance when W is convex.
- Finally, if the classification of minimal solutions to (3) is not available, our method only establishes the existence of a minimal solution to (3) (but we cannot determine it).

Theorems

Theorem

Let $v_{\epsilon,a}$ be the global minimizer of E, let $a \geq 0$ be bounded (possibly dependent on ϵ), let $\rho > 0$ be the zero of μ_{rad} and let $\mu_1 := \mu'_{\mathrm{rad}}(\rho) < 0$. The following statements hold:

(i) Let $\Omega \subset D(0;\rho)$ be an open set such that $v_{\epsilon,a} \neq 0$ on Ω, for every $\epsilon \ll 1$. Then $|v_{\epsilon,a}| \to \sqrt{\mu}$ in $C^0_{\mathrm{loc}}(\Omega)$.

(ii) For every $\xi = \rho e^{i\theta}$, we consider the local coordinates $s = (s_1, s_2)$ in the basis $(e^{i\theta}, ie^{i\theta})$, and the rescaled minimizers:

$$w_{\epsilon,a}(s) = 2^{-1/2}(-\mu_1\epsilon)^{-1/3} v_{\epsilon,a}\Big(\xi + \epsilon^{2/3}\frac{s}{(-\mu_1)^{1/3}}\Big).$$

As $\epsilon \to 0$, the function $w_{\epsilon,a}$ converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ up to subsequence, to a bounded in the half-planes $[s_0, \infty) \times \mathbb{R}$ solution of

$$\Delta y(s) - s_1 y(s) - 2|y(s)|^2 y(s) - \alpha = 0, \qquad \forall s = (s_1, s_2) \in \mathbb{R}^2, \tag{5}$$

with $\alpha = \lim_{\epsilon\to 0} \frac{a(\epsilon) f(\xi)}{\sqrt{2}\mu_1}$.

Theorems

(iii) For every $r_0 > \rho$, we have $\lim_{\epsilon \to 0} \frac{v_{\epsilon,a}((r_0+t\epsilon)e^{i\theta})}{\epsilon} = -\frac{a_0}{\mu_{\mathrm{rad}}(r_0)} f(r_0 e^{i\theta})$ uniformly when t remains bounded and $\theta \in \mathbb{R}$, with $a_0 := \lim_{\epsilon \to 0} a(\epsilon)$.

Theorem

Assume that $a(\epsilon) > 0$, a is bounded and $\lim_{\epsilon \to 0} \epsilon^{1-\frac{3\gamma}{2}} \ln a = 0$ for some $\gamma \in [0, 2/3)$.

(i) For $\epsilon \ll 1$, the global minimizer $v_{\epsilon,a}$ has at least one zero $\bar{x}_\epsilon$ such that

$$|\bar{x}_\epsilon| \leq \rho + o(\epsilon^\gamma).$$

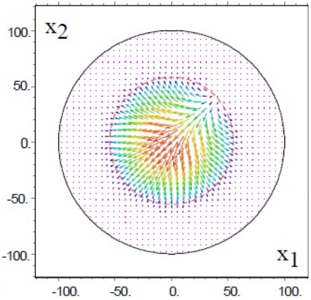
In addition, any sequence of zeros of $v_{\epsilon,a}$, either satisfies (6) or it diverges to ∞.

(ii) For every $\rho_0 \in (0, \rho)$, there exists $b_* > 0$ such that when $\limsup_{\epsilon \to 0} \frac{a}{\epsilon |\ln \epsilon|} < b_*$ then any limit point $l \in \mathbb{R}^2$ of the set of zeros of the global minimizer satisfies

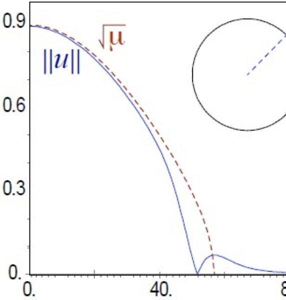
$$\rho_0 \leq |l| \leq \rho.$$

Theorems

In addition if $a = o(\epsilon|\ln\epsilon|)$ then $|l| = \rho$, and $\lim_{\epsilon\to 0}|v(x)| = \sqrt{\mu^+(x)}$ uniformly on $\mathbb{R}^2$.



In black: the region where the product $v_1 v_2$ is close to 0



$a(\varepsilon) = o(\varepsilon\, ln\varepsilon)$ shadow vortex

Theorems

(iii) On the other hand, for every $\rho_0 \in (0, \rho)$, there exists $b^* > 0$ such that when $\limsup_{\epsilon\to 0} \frac{a}{\epsilon|\ln\epsilon|^2} > b^*$, the set of zeros of the global minimizer has a limit point l such that

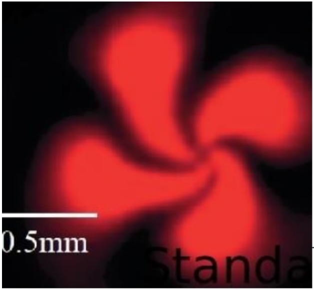
$$|l| \leq \rho_0.$$

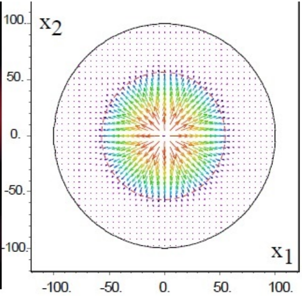
If $v_{\epsilon,a}(\bar{x}_\epsilon) = 0$ and $\bar{x}_\epsilon \to l$ then up to a subsequence

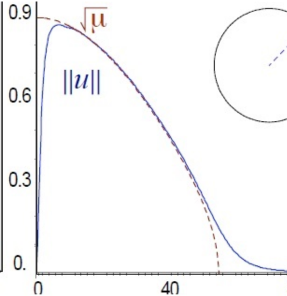
$$\lim_{\epsilon\to 0} v_{\epsilon,a}(\bar{x}_\epsilon + \epsilon s) \to \sqrt{\mu(l)}(g \circ \eta)(\sqrt{\mu(l)}s),$$

in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$, for some $g \in O(2)$. In addition if $\limsup_{\epsilon\to 0} \frac{a}{\epsilon|\ln\epsilon|^2} = \infty$ then $l = 0$.

Theorems







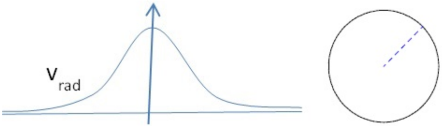
In black: the region where the product $v_1 v_2$ is close to 0 $\quad limsup \frac{a(\varepsilon)}{\varepsilon(ln\varepsilon)^2} = \infty$ standard vortex

Theorems

Theorem

(i) *Given* $\epsilon > 0$, *there exists* $A > 0$ *such that for every* $a > A$, *the global minimizer* $v_{\epsilon,a}$ *is unique and radial i.e.* $v(x) = v_{\mathrm{rad}}(|x|)\frac{x}{|x|}$.

(ii) *When* $a = 0$ *the global minimizer can be written as* $v(x) = (v_{\mathrm{rad}}(|x|), 0)$ *with* $v_{\mathrm{rad}} \in C^{\infty}(\mathbb{R})$ *positive. It is unique up to change of* v *by* gv *with* $g \in SO(2)$.



How we locate the vortices

- It is convenient to renormalize the energy: $\mathcal{E}(u) := E(u) + \int_{|x|<\rho} \frac{\mu^2}{4\epsilon^2} =$ $\int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \int_{|x|<\rho} \frac{(|u|^2-\mu)^2}{4\epsilon^2} + \int_{|x|>\rho} \frac{|u|^2(|u|^2-2\mu)}{4\epsilon^2} - \frac{a}{\epsilon}\int_{\mathbb{R}^2} f \cdot u$
- We compute an upper bound of $\mathcal{E}(v)$ by choosing an appropriate test function:

$$\mathcal{E}(v_{\epsilon,a}) \leq \frac{\pi|\mu_1|\rho}{6}|\ln\epsilon| + \mathcal{O}(1).$$

- Then we proceed by contradiction. If v has a zero $\bar{x}_\epsilon$ such that $|\bar{x}_\epsilon| \leq \rho_0 < \rho$, it follows that

$$\liminf_{\epsilon\to 0} \mathcal{E}(v_{\epsilon,a}) \geq \left(\lambda(\rho_0) + \frac{\pi|\mu_1|\rho}{6}\right)|\ln\epsilon| - K\frac{a}{\epsilon} + \mathcal{O}(1).$$

- We reach a contradiction if $\limsup_{\epsilon\to 0} \frac{a(\epsilon)}{\epsilon|\ln\epsilon|} < \frac{\lambda(\rho_0)}{K}$.

How we locate the vortices

- On the other hand, if v does not vanish in $D(0;\rho_0)$ with $\rho_0 < \rho$, we have $\mathcal{L}^2\left(\left\{x \in D(0;\rho_0) : \frac{x}{|x|}\cdot\frac{v}{|v|} \leq \frac{1}{2}\right\}\right) \leq \frac{K_2\epsilon|\ln\epsilon|}{a}$.
- Since the degree of v on the circles $|x| = r \in (0,\rho_0)$ is 0 we obtain an increase of energy

$$\int_{D(0;\rho_0)} \left|\nabla\frac{v}{|v|}\right|^2 \geq \frac{(\pi\rho_0)^2 a}{6^2 K_2\epsilon|\ln\epsilon|} - \frac{4\pi}{3},$$

which contradicts the upper bound $\int_{D(0;\rho_0)}\left|\nabla\frac{v}{|v|}\right|^2 \leq K_1|\ln\epsilon|$ when $\limsup_{\epsilon\to 0}\frac{a(\epsilon)}{\epsilon^2|\ln\epsilon|} < \frac{6^2K_1K_2}{(\pi\rho_0)^2}$.

The vector Painlevé equation $y : \mathbb{R}^2 \to \mathbb{R}^2$

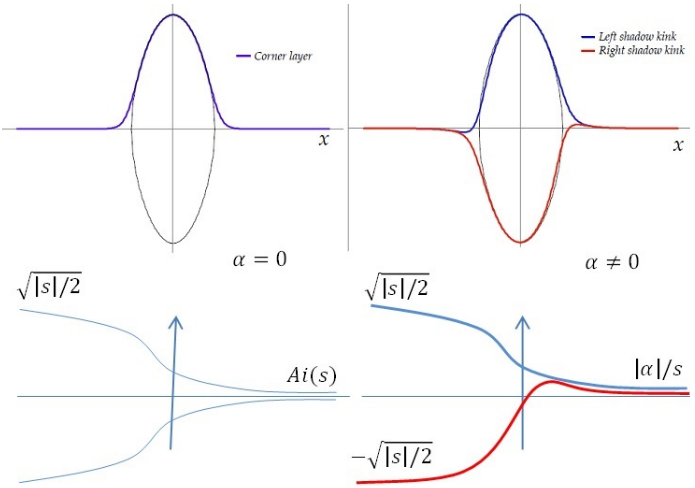
$$\Delta y(s) - s_1 y(s) - 2|y(s)|^2 y(s) - \alpha = 0, \qquad \forall s = (s_1, s_2) \in \mathbb{R}^2, \quad (6)$$

with $\alpha = \lim_{\epsilon \to 0} \frac{a(\epsilon) f(\xi)}{\sqrt{2}\mu_1} \in \mathbb{R}^2$,

- generalizes the second Painlevé O.D.E. $y : \mathbb{R} \to \mathbb{R}$

$$y'' - sy - 2y^3 - \alpha = 0, \qquad s \in \mathbb{R}. \quad (7)$$

- In a previous paper we showed that this last equation plays an analogous role in the one dimensional, scalar version of the problem.


Corner layer
x
Left shadow kink
Right shadow kink
x
α = 0
α ≠ 0
√|s|/2
Ai(s)
√|s|/2
|α|/s
−√|s|/2

Open question

- Numerical simulations suggest that in $1D$ and $2D$, the rescaled profile of the shadow vortex comes from the generalized second Painlevé equation, and not from the Ginzburg-Landau equation (like the standard vortex).
- To deduce this, it is sufficient to show that the zero of the globsl minimizer satisfies $|\bar{x}_\epsilon| = \rho + \mathcal{O}(\epsilon^{2/3})$.

Related problems

- The energy E belongs to the class of Ginzburg-Landau type functionals that appear for example in the theory of superconductivity or in the theory of Bose-Einstein condensates.
- For these problems there also exist threshold values of the parameters that determine the structure of the global minimizers. According to these values, the global minimizer may or may not have vortices.

Reference: same problem in $1D$

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Reference: physical aspect of the problem

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Reference: Painlevé equation

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