Geometric Aspects of Phase Separation

Susanna Terracini

joint works with A. Salort, G. Verzini and A. Zilio

Dipartimento di Matematica "Giuseppe Peano"
Università di Torino
ERC Advanced Grant n. 339958 - COMPAT

Emerging issues in nonlinear elliptic equations:
Singularities, singular perturbations and non local problems June 18 - 23, 2017
Mathematical Research and Conference Center
Będlewo, Poland

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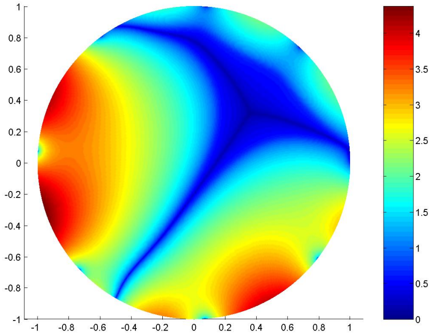
Competition diffusion systems with Lotka-Volterra interactions: symmetric competition rates

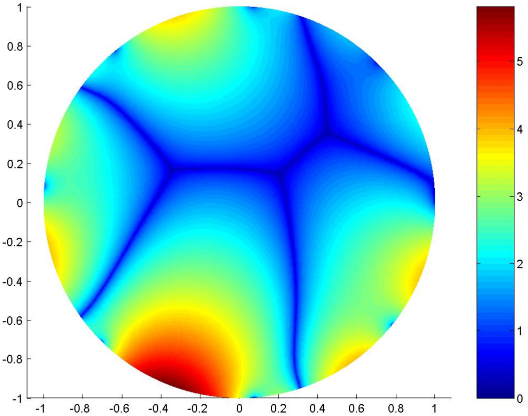
With large and symmetric interspecific competition rates $\beta_{i,j} = \beta_{j,i}$ and three populations:

$$\frac{\partial u_i}{\partial t} - \mathrm{div}(d_i \nabla u_i) = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j \neq i}}^{h} \beta_{i,j} u_j \text{ in } \Omega,$$



Competition diffusion systems with Lotka-Volterra interactions : asymmetric competition rates

With large and symmetric interspecific competition rates $\beta_{i,j} = \beta_{j,i}$ and five populations:



Energy minimizing segregated configurations

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ and let us call segregated state a k–uple $U = (u_1, \ldots, u_k) \in (H^1(\Omega))^k$ where

$$u_i(x) \cdot u_j(x) = 0 \qquad i \neq j, \text{ a.e. } x \in \Omega$$

We define the internal energy of U as

$$J(U) = \sum_{i=1,\cdots,k} \left\{ \int_\Omega \left(\frac{1}{2} d_i^2(x) \, |\nabla u_i(x)|^2 - F_i(x, u_i(x)) \right) dx \right\},$$

Our goal is to minimize J among a class of segregated states subject to some boundary and positivity conditions.

A weak reflection law

Theorem (M. Conti, S. T. Terracini, G. Verzini 2005, L. Caffarelli, F. Lin 2008)

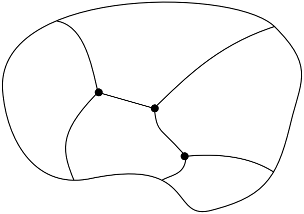
For reasonable F's, the minimization problem has a (unique) segregated solution U, which is Lipschitz. Let Γ_U its nodal set, Then, there exists a set $\Sigma_U \subseteq \Gamma_U$ the regular part, relatively open in Γ_U, such that

- $\mathcal{H}_{dim}(\Gamma_U \setminus \Sigma_U) \leq N-2$, *and if $N = 2$ then actually $\Gamma_U \setminus \Sigma_U$ is a locally finite set;*
- Σ_U *is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0 < \alpha < 1$).*

Furthermore for every $x_0 \in \Sigma_U$

$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0,$$

where the limits as $x \to x_0^{\pm}$ are taken from the opposite sides of the hyper-surface. Furthermore, if $N = 2$ then Σ_U consists in a locally finite collection of curves meeting with equal angles at singular points.



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The general model

We consider the semilinear system:

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j \qquad \text{in } \Omega, \qquad i = 1, \dots, k, \qquad \text{(LV)}$$

where $u_i \geq 0$, $\beta > 0$, $a_{ij} > 0$ (+ boundary conditions).

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(LV) is the stationary version of the competition-diffusion system with Lotka-Volterra interactions:

$$\partial_t u - \Delta u_i = f_i(u_i) - \beta u_i \sum_{j \neq i} a_{ij} u_j.$$

(LV) is never variational. It can be either symmetric ($a_{ij} = a_{ji}$) or asymmetric ($a_{ij} \neq a_{ji}$).

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where $u_i \geq 0$, $\beta > 0$, $a_{ij} > 0$ (+ boundary conditions).

Starting from [Dancer-Du, *JDE* (1994), *Nonl. Anal.* (1995)] two main issues has been addressed:

- existence
- asymptotics as $\beta \to +\infty$ (segregation, i.e. $u_i u_j \to 0$)

The reactions f_i and the boundary conditions are crucial for existence, less important for segregation.

Here we mainly deal with the segregation limit: free boundaries, nodal structure, in particular in dimension $N = 2$ (also in connection with entire solutions on $\mathbb{R}^2$ or $\mathbb{R}^2_+$).

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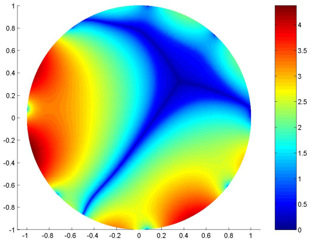

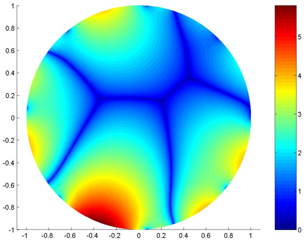

Figure: on the left $k = 3$, on the right $k = 5$; in both $a_{ij} = a_{ji}$ and β large.

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The symmetric case for $k \geq 3$ populations

We assume $a_{ij} = a_{ji}$ ($= 1$ w.l.o.g.). The system becomes

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j \neq i} u_j \qquad \text{in } \Omega, \qquad i = 1, \ldots, k,$$

Theorem (Conti, Terracini, Verzini '05)

Let U_β be a family of H^1–bounded solutions. For every $\alpha < 1$ there exists $L_\alpha > 0$ such that

$$\sup_{x,y \in \Omega} \frac{|u_{i,\beta}(x) - u_{i,\beta}(y)|}{|x - y|^\alpha} < L_\alpha$$

for all $i = 1, \ldots, k$ and for all $\beta > 0$.

This allows to pass to the limit as $\beta \to +\infty$.

Recently, (optimal) uniform Lipschitz bounds have been obtained [Soave-Zilio, *ARMA* 2015]

Segregation limit in the symmetric case

Theorem (Conti, Terracini, Verzini '05)

Let $U_\beta = (u_{1,\beta}, ..., u_{k,\beta})$ be a solution of the system at fixed β, and $\beta \to \infty$. There exists U such that, for all $i = 1, \ldots, k$:

1. *up to subsequences, $u_{i,\beta} \to u_i$ strongly in H^1 and in $\mathcal{C}^\alpha$, for any $\alpha \in (0,1)$*
2. *if $i \neq j$ then $u_i \cdot u_j = 0$ a.e. in Ω*
3. $-\Delta u_i \leq f(x, u_i)$
4. $$-\Delta \left(u_i - \sum_{j \neq i} u_j \right) \geq f(x, u_i) - \sum_{j \neq i} f(x, u_j)$$
5. *the segregated limiting profiles are Lipschitz.*

This agrees with the case $k = 2$, which reads

$$-\Delta(u_1 - u_2) \geq 0, \qquad -\Delta(u_2 - u_1) \geq 0.$$

The class $\mathcal{S}$

Define

$$\widehat{u}_i = u_i - \sum_{j \neq i} u_j$$

and similarly

$$\widehat{f}(x, \widehat{u}_i) = \begin{cases} f_i(x, u_i) & \text{if } x \in \mathrm{supp}(u_i) \\ -f_j(x, u_j) & \text{if } x \in \mathrm{supp}(u_j),\ j \neq i. \end{cases}$$

Then the segregation limits belong to the class

$$\mathcal{S} = \left\{ (u_1, \cdots, u_k) : \begin{array}{c} u_i \geq 0,\ u_i \cdot u_j = 0 \text{ if } i \neq j \\ -\Delta u_i \leq f(x, u_i) \\ -\Delta \widehat{u}_i \geq \widehat{f}(x, \widehat{u}_i),\ \forall i \end{array} \right\}$$

(+ boundary conditions)

Basic properties in $\mathcal{S}$

The multiplicity of a point $x \in \Omega$ is

$$m(x) = \sharp \{ i : meas(\{u_i > 0\} \cap B(x,r)) > 0 \; \forall \, r > 0 \} \; .$$

Proposition

Let $x_0 \in \Omega$:

(a) If $m(x_0) = 0$, then there is $r > 0$ such that $u_i \equiv 0$ on $B(x,r)$, for every i.

(b) If $m(x_0) = 1$, then there are i and $r > 0$ such that $u_i > 0$ and

$$-\Delta u_i = f_i(x, u_i) \quad \text{on } B(x,r).$$

(c) If $m(x_0) = 2$, then are i, j and $r > 0$ such that $u_k \equiv 0$ for $k \neq i, j$ and

$$-\Delta(u_i - u_j) = g_{ij}(x, u_i - u_j) \quad \text{on } B(x,r),$$

where $g_{i,j}(x,s) = f_i(x, s^+) - f_j(x, s^-)$.

Multiple junctions of nodal lines

We wish to analyze the structure of the zero set of a k-tuple $U \in \mathcal{S}$, i.e. the set

$$\mathcal{Z} = \{x : u_i(x) = 0 \text{ for every } i = 1, \dots, k\}.$$

Such set naturally splits into the union of the regular part $\mathcal{Z}_2 = \{x \in \mathcal{Z} : m(x) = 2\}$, which is itself the union of the interfaces

$$\Gamma_{ij} = \partial\omega_i \cap \partial\omega_j \cap \mathcal{Z}_2,$$

and of the singular part

$$\mathcal{W} = \mathcal{Z} \setminus \mathcal{Z}_2.$$

With this respect, the sets ω_i and ω_j are said to be adjacent whenever $\Gamma_{ij} \neq \emptyset$.

Structure of the nodal set

Theorem (Conti-Terracini-Verzini '05, Caffarelli-Karakayan-Lin '08, Tavares-Terracini '12)

Let U be in the class $\mathcal{S}$, and let $\mathcal{Z} = \{x \in \Omega \ : \ U(x) = 0\}$. Then, there exists a set $\mathcal{Z}_2 \subseteq \mathcal{Z} =$ the regular part, relatively open in $\mathcal{Z}$, such that

- *$\mathcal{Z}_2$ is a collection of hyper-surfaces of class $C^{1,\alpha}$ (for every $0 < \alpha < 1$). Furthermore for every $x_0 \in \mathcal{Z}_2$*

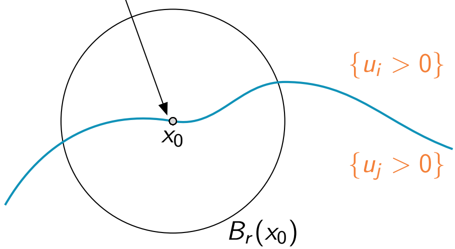
$$\lim_{x \to x_0^+} |\nabla U(x)| = \lim_{x \to x_0^-} |\nabla U(x)| \neq 0,$$

  where the limits as $x \to x_0^{\pm}$ are taken from the opposite sides of the hyper-surface;
- *$\mathcal{H}_{dim}(\mathcal{Z} \setminus \mathcal{Z}_2) \leq N - 2$, and $\lim_{x \to x_0} |\nabla U(x)| = 0$.*

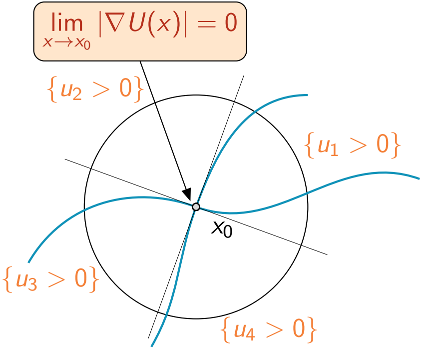
Furthermore, if $N = 2$ then $\mathcal{Z}$ consists in a locally finite collection of curves meeting with equal angles at a locally finite number of singular points.

Nodal set: regular points

$$\lim_{\substack{x\to x_0\\ x\in\{u_i>0\}}} \nabla u_i(x) = -\lim_{\substack{x\to x_0\\ x\in\{u_j>0\}}} \nabla u_j(x)$$

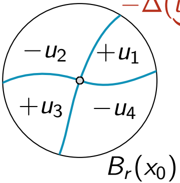


Nodal set: singular points ($N = 2$)



Asymptotic expansion near multiple points

An heuristic argument without reactions:



$$-\Delta(\underbrace{u_1 - u_2 + u_3 - u_4}_{w}) = f_1 - f_2 + f_3 - f_4$$

Then $w(r,\vartheta) = \sum_{k\in\mathbb{Z}} \left[a_k \cos(k\vartheta) + b_k \sin(k\vartheta)\right] r^k$

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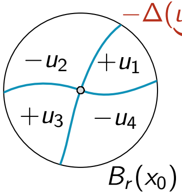

$$-\Delta(\underbrace{u_1 - u_2 + u_3 - u_4}_{w}) = f_1 - f_2 + f_3 - f_4$$

Then $w(r,\vartheta) = \sum_{k\in\mathbb{Z}} [a_k \cos(k\vartheta) + b_k \sin(k\vartheta)]\, r^k$ and

- $a_k^2 + b_k^2 = 0$ for $k < 0$ as w is not singular in x_0,
- $a_k^2 + b_k^2 = 0$ for $k = 0, 1$ as $m(x_0) = 4$,

$$w(r,\vartheta) = r^2 \cos(2\vartheta + \vartheta_0) + o(r^2) \text{ as } r \to 0.$$

In general, $w \sim r^{m(x_0)/2}$, also in the odd case.

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The limiting profiles

Back to the original problem

$$-\Delta u_i = f_i(x, u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j \qquad \text{in } \Omega, \qquad i = 1, \dots, k,$$

assume now $a_{ij} \neq a_{ji}$

- **Passing to the limit as $\beta \to \infty$ we find a new class $\mathcal{S}$:**

Define, for every $i = 1, \dots, k$,

$$\widehat{u}_i := u_i - \sum_{j\neq i} \frac{a_{ij}}{a_{ji}} u_j,$$

and $\widehat{f}_i$ accordingly. The differential inequalities take the usual form

$$-\Delta \widehat{u}_i \geq \widehat{f}_i(x, \widehat{u}_i) \qquad \text{in } \Omega.$$

Asymptotics and nodal set

- What doesn't changes:
  - equi–hölderianity w.r.t. β
  - proportional gradients at points x_0 with $m(x_0) = 2$:

$$\lim_{\substack{x \to x_0 \\ x \in \{u_i > 0\}}} a_{ji} \nabla u_i(x) = - \lim_{\substack{x \to x_0 \\ x \in \{u_j > 0\}}} a_{ij} \nabla u_j(x)$$

  - vanishing of the gradient at points x_0 with $m(x_0) \geq 3$
- What changes:
  - local expansion at multiple points (in dimension $N = 2$).

Asymptotics and nodal set

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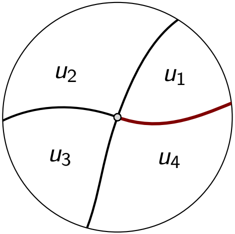
  - vanishing of the gradient at points x_0 with $m(x_0) \geq 3$
- What changes:
  - local expansion at multiple points (in dimension $N = 2$).

Near an isolated point x_0 with (e.g.) $m(x_0) = 4$ we have that

$$w = u_1 - \frac{a_{12}}{a_{21}} u_2 + \frac{a_{12} a_{23}}{a_{21} a_{32}} u_3 - \frac{a_{12} a_{23} a_{34}}{a_{21} a_{32} a_{43}} u_4$$

satisfies

$$-\Delta w = 0 \qquad \text{in } B_{r_0}(x_0) \setminus \underbrace{\left(\overline{\{u_1 > 0\}} \cap \overline{\{u_4 > 0\}} \right)}_{\tilde{\Gamma}} .$$



The main Theorem

Let $(u_1, \ldots, u_k)$ be a segregated limiting profile in the asymmetric case.

Theorem (S. T. , G. Verzini, A. Zilio, 2017)

Let $\mathcal{Z}$ be a compact connected component of $\{x : m(x) \geq 3\}$. Then $\mathcal{Z} = \{x_0\}$.

Theorem (S. T. , G. Verzini, A. Zilio, 2017)

Let $x_0 \in \Omega$ with $m(x_0) = h \geq 3$. Then there exists $\alpha \in \mathbb{R}$ and ϑ_0 such that

$$w(r, \vartheta) = Cr^{h/2} \exp(\alpha\theta) \cos\left(\frac{h}{2}\theta - \alpha \log r + \vartheta_0\right) + o(r^{h/2})$$

as $r \to 0$, where (r, θ) denotes a system of polar coordinates about x_0 and $\tilde{U}$ is a suitably weighted sum of the components u_i.

The value of α

The value of α is explicit in terms of the coefficients a_{ij}, with i and j belonging to the set of indexes associated to the $h \leq k$ densities which do not identically vanish near x_0. For instance, when u_1, u_2 and u_3 meet at x_0, with $m(x_0) = 3$, then (up to a change of sign)

$$\alpha = \frac{1}{2\pi} \log \left(\frac{a_{12}}{a_{21}} \cdot \frac{a_{23}}{a_{32}} \cdot \frac{a_{31}}{a_{13}} \right).$$

In particular, in case $a_{ij} = a_{ji}$ for every $j \neq i$, then $\alpha = 0$, and the spirals reduce to straight lines; in this way we recover the equal-angles-property for multiple points. On the other hand it is easy to construct examples in which $\alpha \neq 0$, see Fig. 25.

Spirals

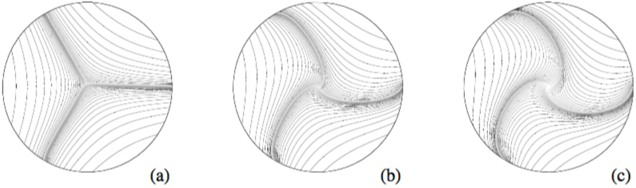


Figure: numerical simulations of functions belonging to the class $\mathcal{S}$ for different values of α. In this particular case, we have considered a system of 3 components (labeled in counterclockwise order as u_1, u_2 and u_3) in the unit ball, with boundary conditions given by suitable restrictions of $|\cos(3/2\,\vartheta)|$. In picture (a), $a_{ij} = 1$ for all i, j, which yields $\alpha = 0$. In picture (b), $a_{ij} = 4$ if $j - i = 1 \mod 3$ and $a_{ij} = 1$ otherwise, which yields $\alpha = 3\log 4/2\pi$ (> 0, which implies that the free-boundary is described asymptotically by rotations of the clockwise logarithmic spiral $\vartheta = \log 4/\pi\ \log r$). In picture (c), $a_{ij} = 10$ if $j - i = 1 \mod 3$ and $a_{ij} = 1$ otherwise, which yields $\alpha = 3\log 10/2\pi$.

Variational principles?

In the asymmetric case, the nodal partition determined by the components supports can not be optimal with respect to any Lagrangian energy (in particular, the Almgren monotonicity formula can not hold). Indeed, it is known that boundaries of optimal partitions in two dimension have the equal angle property . Hence they can not exhibit logarithmic spirals. This fact is in striking contrast with the picture for symmetric inter-specific competition rates: indeed, in such a case we know that solutions to the reaction diffusion system are unique, together with their limit profiles in the class $\mathcal{S}$. Hence, though system does not possess a variational nature, it fulfills a minimization principle in the segregation limit, while this is impossible in the asymmetric setting.

Finite vanishing order

Nevertheless, even in the asymmetric case, functions in the class $\mathcal{S}$ still share with the solutions of variational problems, including harmonic functions, the following fundamental features:

- singular points are isolated and have a finite vanishing order;
- the possible orders are quantized;
- the regular part is smooth.

It is natural to wonder whether similar analogies still hold in dimensions higher than two. However, in higher dimensions, new strategies and unconventional techniques have to be designed to treat the asymmetric case.

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Starting the proof

Lemma

We can reduce the problem to the case in which, for each $i = 1, \ldots, k$, $\partial \operatorname{supp}(u_i) \cap \Omega$ is made of two connected components and we call Γ_{ij} such connected arcs. Furthermore, we can assume that any Γ_{ij} reaches the boundary at most once, there are at least three non trivial densities in Ω and $\mathcal{W}$ is not empty.

Lemma

Under the previous reductions, the set $\mathcal{W}$ is connected and Ω is simply connected.

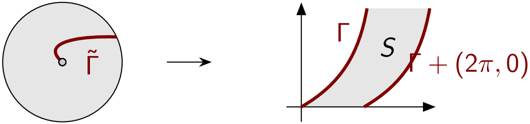
By Riemann Mapping Theorem we can the reduce furthermore to the case when Ω is the unit ball and $0 \in \mathcal{W}$.

Going to the universal covering

The conformal map

$$(r\cos\vartheta, r\sin\vartheta) \mapsto (x,y) = (\vartheta, -\log(r/r_0))$$

allows to map $B_{r_0}(x_0) \setminus \tilde{\Gamma}$ to the strip $S \subset \mathbb{R}^2_+$.



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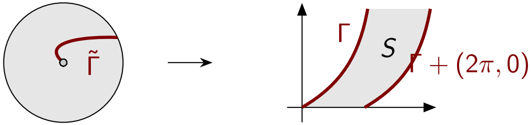

Then a suitably weighted sum of the components u_i corresponds to v, which can be extended from S to $\mathbb{R}^2_+$, with

$$\begin{cases} v\Delta v = 0 & \text{in } \mathbb{R}^2_+ \\ v = 0 & \text{on } \Gamma \\ v(x+2\pi, y) = \lambda v(x, y). \end{cases}$$

where $\lambda = \dfrac{\prod_i a_{i,i+1}}{\prod_i a_{i+1,i}}$.

A representation formula in $\mathbb{R}^2_+$

To start with, we wish to solve the simplest case, when $\mathcal{W} = \{0\}$. In this case,

$$\begin{cases} \Delta v = 0 \quad \text{in } \mathbb{R}^2_+ \\ v = 0 \quad \text{on } \Gamma \\ v(x+2\pi, y) = \lambda v(x,y) \end{cases} \iff \begin{cases} \Delta z + 2\alpha z_x + \alpha^2 z = 0. \quad \text{in } \mathbb{R}^2_+ \\ z = 0 \quad \text{on } \Gamma \\ z(x+2\pi, y) = z(x,y) \\ z(x,y) := e^{-\alpha x} v(x,y), \quad \alpha = \dfrac{\ln \lambda}{2\pi}. \end{cases}$$

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Separating the variables we obtain

$$v(x,y) = \sum_{k\in\mathbb{Z}} \left[a_k \cos\left(kx + \alpha y\right) + b_k \sin\left(kx + \alpha y\right)\right] \exp(\alpha x - ky).$$

From now on, for concreteness, we suppose

$$\lambda = \frac{\prod_i a_{i,i+1}}{\prod_i a_{i+1,i}} > 1, \qquad \text{i.e. } \alpha > 0.$$

Asymptotic spirals

$$v(x,y) = \sum_{k\in\mathbb{Z}} \left[a_k \cos\left(kx+\alpha y\right) + b_k \sin\left(kx+\alpha y\right)\right] \exp(\alpha x - ky)$$

$$= \sum_{k\geq\bar{k}} \cdots + \underbrace{\sum_{k<\bar{k}} \cdots}_{evil}$$

GOAL: show that $a_k^2 + b_k^2 = 0$ for $k < \bar{k} = m(x_0)/2$, $a_{\bar{k}}^2 + b_{\bar{k}}^2 \neq 0$.

Asymptotic spirals

$$v(x,y) = \sum_{k\in\mathbb{Z}} [a_k \cos(kx+\alpha y) + b_k \sin(kx+\alpha y)] \exp(\alpha x - ky)$$

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GOAL: show that $a_k^2 + b_k^2 = 0$ for $k < \bar{k} = m(x_0)/2$, $a_{\bar{k}}^2 + b_{\bar{k}}^2 \neq 0$.

Then

- $v(x,y) \sim \left[a_{\bar{k}} \cos(\bar{k}x+\alpha y) + b_{\bar{k}} \sin(\bar{k}x+\alpha y)\right] e^{(\alpha-\bar{k})y}$ as $y \to +\infty$
- S asymptotically lies in the strip $C_1 \leq \bar{k}x + \alpha y \leq C_2$

and finally

$$w(r,\vartheta) = Cr^{\bar{k}} \exp(\alpha\vartheta) \cos(\bar{k}\vartheta - \alpha \log r + \vartheta_0) + o(r^{\bar{k}}) \qquad \text{as } r \to 0$$

where w is a suitably weighted sum of the components u_i: asymptotic logaritmic spirals!

How to kill the evil part?

Further condition on

$$v(x,y) = \sum_{k\in\mathbb{Z}} [a_k \cos(kx+\alpha y) + b_k \sin(kx+\alpha y)] \exp(\alpha x - ky).$$

By conformality

$$\int_S |\nabla v|^2 < +\infty.$$

In terms of z, which is 2π-periodic in y, this reads

$$\int_S e^{2\alpha x} |\nabla z(x,y)|^2 \, dxdy < +\infty.$$

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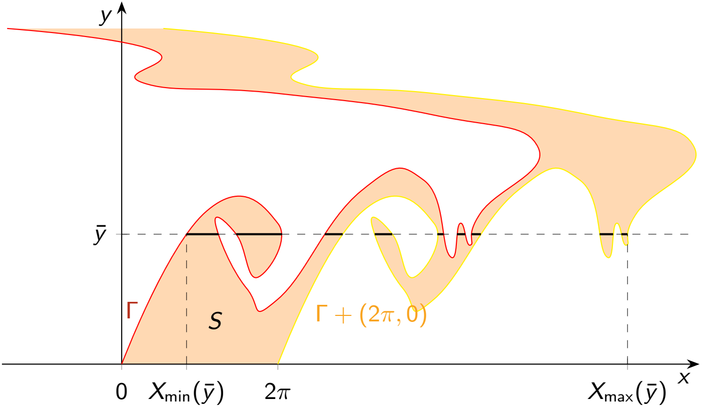
Since we do not know the actual position of S, we can not exclude the integrability on S of quantities of order $e^{2(\alpha x + ky)}$, $k > 0$, even for arbitrarily large k.

Description of the strip S

We denote by

$$S_y := \{(x, y) \in S\}$$

the horizontal sections of S, having endpoints $X_{\min}(y)$ and $X_{\max}(y)$. While $|S_y| \leq 2\pi$, diam S_y may be arbitrarily large.



The battle against evil - part I (the easy stuff)

$$v(x,y) = \sum_{k\in\mathbb{Z}} \left[a_k \cos(kx+\alpha y) + b_k \sin(kx+\alpha y)\right] \exp(\alpha x - ky),$$

$$\int_S |\nabla v(x,y)|^2\, dxdy < +\infty \qquad \left(\text{or } \int_S e^{2\alpha x} |\nabla z(x,y)|^2\, dxdy < +\infty\right).$$

Two cases can be easily ruled out:

- $S \subset \{(x,y) : x \geq my + q\}$ for some m, q
- $\inf\left\{k \in \mathbb{Z} : a_k^2 + b_k^2 \neq 0\right\} > -\infty.$

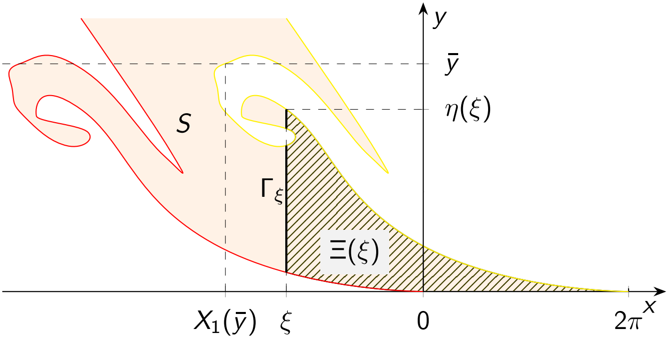
The battle against evil - part II (the angular A-C-F)

More difficult: rule out

- $S \subset \{(x,y) : x \leq my + q\}$ for some m, q

Lemma

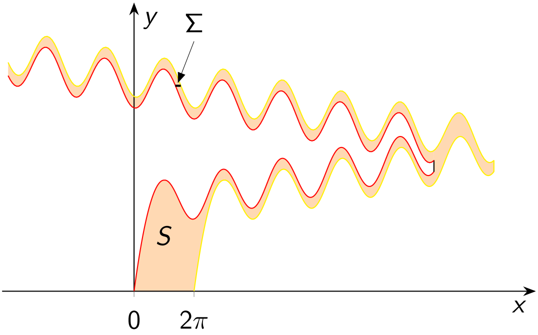
$$\int_{S\setminus\Xi(\bar{y})} |\nabla v|^2\, dxdy \leq e^{-X_1^2(\bar{y})/\bar{y}} \int_S |\nabla v|^2\, dxdy.$$



The battle against evil - part II (the angular A-C-F)

The battle against evil - part III (the bottlenecks argument)

Last case to rule out:



Lemma

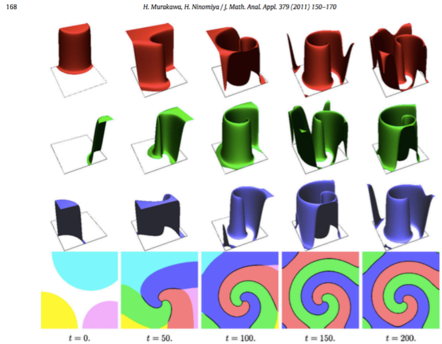
$$\int_{above\ \Sigma} |\nabla v|^2\, dxdy \leq \frac{C}{|\ln(|\Sigma|)|} \int_{around\ \Sigma} |\nabla v|^2\, dxdy.$$

Table of contents

The parabolic problem

With asymmetric interspecific competition rates $\beta_{i,j} \neq \beta_{j,i}$ large and three populations:

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - u_i \sum_{\substack{j=1 \\ j \neq i}}^{h} \beta_{i,j} u_j \text{ in } \Omega,$$



H. Murakawa and H. Ninomiya, *Fast reaction limit of a three-component reaction-diffusion system*. J. Math. Anal. Appl. 379 (2011), no. 1, 150-170,

The spiralling wave ansatz in two-dimension

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_i) - \beta u_i \sum_{\substack{j=1 \\ j\neq i}}^{h} a_{i,j} u_j \text{ in } \mathbb{C},$$

Ansatz:

$$u_i(t,x) = v_i\left(e^{i\omega t}x\right)\ , \qquad x \in \mathbb{C}$$

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Ansatz:

$$u_i(t,x) = v_i\left(e^{i\omega t}x\right)\ , \qquad x \in \mathbb{C}$$

Then $(v_1, \ldots, v_i)$ solve

$$\omega x^{\perp} \cdot \nabla v_i - \Delta v_i = f_i(v_i) - \beta v_i \sum_{\substack{j=1 \\ j\neq i}}^{h} a_{i,j} v_j \text{ in } \mathbb{C}.$$

Spiralling limiting profiles:

$$\omega x^{\perp} \cdot \nabla v_i - \Delta v_i = f_i(v_i) - \beta v_i \sum_{\substack{j=1 \\ j \neq i}}^{h} a_{i,j} v_j \text{ in } \mathbb{C}, \qquad (*)$$

Next we pass to the limit as $\beta \to +\infty$.

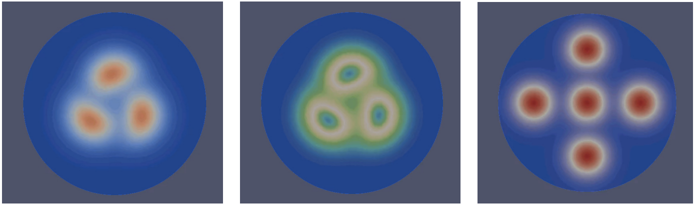
Theorem (Salort,Terracini,Verzini, Zilio 2016)

For every ω, for a codimension two set of boundary traces, there exists a unique solution in the class $\mathcal{S}$ associated with $(*)$ in the unit disk. Furthermore, there exists $\alpha \in \mathbb{R}$ and θ_0 such that

$$\tilde{V}(r,\theta) = C r^{h/2} \exp(\alpha\vartheta) \left| \cos\left(\frac{h}{2}\vartheta - \alpha \log r + \vartheta_0\right) \right| + o(r^{h/2})$$

as $r \to 0$, where (r,θ) denotes a system of polar coordinates about 0 and $\tilde{U}$ is a suitably weighted sum of the components v_i.

Some numerical simulations (by courtesy of Alessandro Zilio)



Final remarks

- Wave spirals appear in reaction-diffusion systems related with ventricular fibrillation.
- Work on the dynamics of spiral waves by Björn Sandstede, Arnd Scheel, Claudia Wulff (no singular limit).