Dynamics and Order Theory

Three lecture series

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Ingredients

- Space: (X, d) is a compact metric space;
- Time: $\mathbb{T}$ denotes the time domain, which is either $\mathbb{Z}$ or $\mathbb{R}$, and $\mathbb{T}^+ := \{t \in \mathbb{T} \mid t \geq 0\}$;
- Dynamical system $\phi\colon \mathbb{T}^+ \times \mathrm{X} \to \mathrm{X}$;

Definition

A *dynamical system* on a compact metric space (X, d) is a continuous mapping $\phi : \mathbb{T}^+ \times \mathrm{X} \to \mathrm{X}$ that satisfies the following two properties:

(i) $\phi(0, x) = x$ for all $x \in \mathrm{X}$, and

(ii) $\phi(t, \phi(s, x)) = \phi(t + s, x)$ for all $s, t \in \mathbb{T}^+$ and all $x \in \mathrm{X}$.

The backward extension $\phi\colon \mathbb{T} \times \mathrm{X} \to \mathrm{X}$ is defined by

$$\phi(-t, x) := \{y \in \mathrm{X} \mid \phi(t, y) = x\}, \quad t > 0.$$

Context

- Since ϕ is not surjective necessarily the space X may be a compact forward invariant set for a system defined on a larger locally compact metric space.
- The theory also works if we let X be a compact isolating neighborhood.
- In spirit the theory can be adjusted arbitrary metric spaces. There are compactness issues. These lead to defects that makes the theory more involved.
- We assume compactness to best explain the concepts.

Basic concepts: invariance

- A set $\mathrm{S} \subset \mathrm{X}$ is *invariant* if $\phi(t, \mathrm{S}) = \mathrm{S}$ for all $t \in \mathbb{T}^+$.
  The set of invariant sets is denoted by $\mathsf{Invset}(\phi)$.
- A set S is *forward-backward invariant* if $\phi(t, \mathrm{S}) = \mathrm{S}$ for all $t \in \mathbb{T}^-$.
  The set of forward-backward invariant sets is denoted by $\mathsf{Invset}^{\pm}(\phi)$.
- A set S is *strongly invariant* if $\phi(t, \mathrm{S}) = \mathrm{S}$ for all $t \in \mathbb{T}$.
  Set of strongly invariant sets is denoted by $\mathsf{SInvset}(\phi)$.
- A set $\mathrm{S} \subset \mathrm{X}$ is *forward invariant* if $\phi(t, \mathrm{S}) \subset \mathrm{S}$ for all $t \in \mathbb{T}^+$.
  The set of forward invariant sets is denoted by $\mathsf{Invset}^{+}(\phi)$. *Backward invariant* sets $\mathsf{Invset}^{-}(\phi)$ are defined similarly.

Remark

For flows and homeomorphisms the first three notions are equivalent.

Basic concepts: limit sets

For a set $\mathrm{U} \subset \mathrm{X}$ define

$$\alpha(\mathrm{U}) = \bigcap_{t\le 0} \overline{\phi\big((-\infty,t],\mathrm{U}\big)} \quad \text{and} \quad \omega(\mathrm{U}) = \bigcap_{t\ge 0} \overline{\phi\big([t,\infty),\mathrm{U}\big)},$$

which are called the *alpha-limit and omega-limit sets* of U respectively.

For noninvertible dynamical systems there is a lack of symmetry between alpha-limit and omega-limit sets.

Both alpha and omega limit sets are compact and $\alpha(\mathrm{U}) \in \mathsf{Invset}^+(\phi)$ and $\omega(\mathrm{U}) \in \mathsf{Invset}(\phi)$. Other properties will be mentioned as we go along.

Basic concept: distributive lattice

A *lattice* is a set L with the binary operations $\vee, \wedge : \mathsf{L} \times \mathsf{L} \to \mathsf{L}$ satisfying the following axioms:

(i) (idempotent) $a \wedge a = a \vee a = a$ for all $a \in \mathsf{L}$,

(ii) (commutative) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ for all $a, b \in \mathsf{L}$,

(iii) (associative) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$ for all $a, b, c \in \mathsf{L}$,

(iv) (absorption) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ for all $a, b \in \mathsf{L}$.

A *distributive lattice* satisfies the additional axiom

(v) (distributive) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in \mathsf{L}$.

A lattice is *bounded* if there exist *neutral* elements 0 and 1 with property that

(vi) $0 \wedge a = 0$, $0 \vee a = a$, $1 \wedge a = a$, and $1 \vee a = 1$ for all $a \in \mathsf{L}$.

A subset $\mathsf{K} \subset \mathsf{L}$ is called a *sublattice* of L, if $a, b \in \mathsf{K}$ implies that $a \vee b \in \mathsf{K}$ and $a \wedge b \in \mathsf{K}$. For sublattices we impose the additional condition that $0, 1 \in \mathsf{K}$.

Poset structure: $a \leq b$ if $a = a \wedge b$, or if $b = a \vee b$.

An example

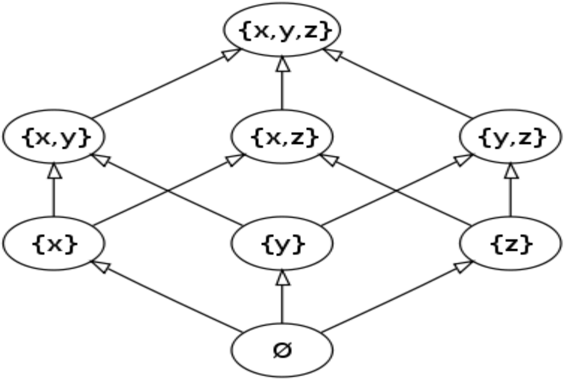


Figure: Lattice of subsets.

An example

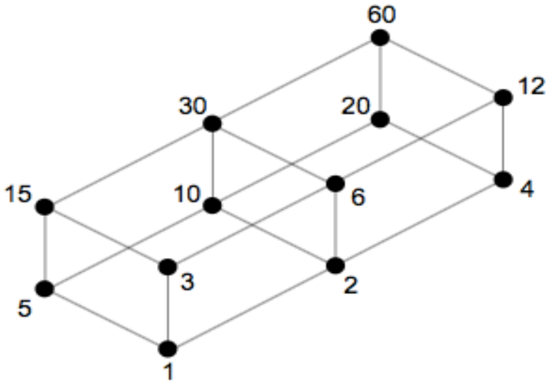


Figure: Divisors of 60, partially ordered by divisibility.

Lattices of invariant sets

With the binary relations

$$\mathrm{S} \wedge \mathrm{S}' = \mathrm{S} \cap \mathrm{S}', \quad \mathrm{S} \vee \mathrm{S}' = \mathrm{S} \cup \mathrm{S}',$$

$\mathsf{Invset}^{\pm}(\phi)$ and $\mathsf{SInvset}(\phi)$ are distributive lattice (big!).

With the binary relations

$$\mathrm{S} \wedge \mathrm{S}' = \mathsf{Inv}(\mathrm{S} \cap \mathrm{S}')^{1} = \omega(\mathrm{S} \cap \mathrm{S}'), \quad \mathrm{S} \vee \mathrm{S}' = \mathrm{S} \cup \mathrm{S}',$$

$\mathsf{Invset}(\phi)$ is a distributive lattice.

Remark

The forward invariant and backward invariant sets $\mathsf{Invset}^{+}(\phi)$ and $\mathsf{Invset}^{-}(\phi)$ are bounded distributive lattices with respect to $\cap$ and $\cup$. The neutral elements are $0 = \varnothing$ and $1 = X$.

[1] $\mathsf{Inv}(U, \phi) = \bigcup\{\mathrm{S} \subset U \mid \mathrm{S} \in \mathsf{Invset}(\phi)\}$.

Attractors and repellers

A (regular) closed set $\mathrm{U} \subset \mathrm{X}$ is an *attracting block* if

$$\phi(t, \mathrm{U}) \subset \operatorname{int} \mathrm{U}, \quad \forall t > 0.$$

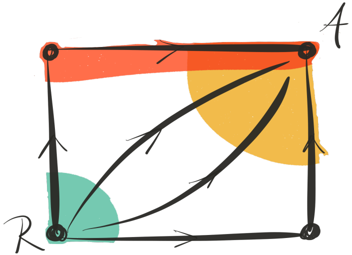
A set $\mathrm{A} \subset \mathrm{X}$ is called an *attractor* if there exists an attracting block U such that $\mathrm{A} = \omega(\mathrm{U})$.

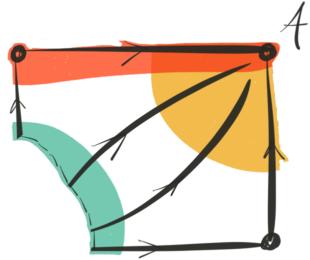
A (regular) closed set $\mathrm{U} \subset \mathrm{X}$ is an *repelling block* if

$$\phi(t, \mathrm{U}) \subset \operatorname{int} \mathrm{U}, \quad \forall t < 0.$$

A set $\mathrm{R} \subset \mathrm{X}$ is called a *repeller* if there exists an repelling block U such that $\mathrm{R} = \alpha(\mathrm{U})$.

An example





Attractors and repellers: duality

Attractors are compact, invariant and the set of attractors $\mathsf{Att}(\phi) \subset \mathsf{Invset}(\phi)$ is a sublattice.

Repellers are compact, forward-backward invariant and the set of repellers $\mathsf{Rep}(\phi) \subset \mathsf{Invset}^{\pm}(\phi)$ is a sublattice.

Define $\mathrm{U} \vee \mathrm{U}' = \mathrm{U} \cup \mathrm{U}'$, $\mathrm{U} \wedge \mathrm{U}' = \overline{\mathrm{int}\,\mathrm{U} \cap \mathrm{int}\,\mathrm{U}'}$ and $\mathrm{U}^{\#} = \overline{\mathrm{U}^c}$ in the Boolean algebra of regular closed sets $\mathbf{R}(\mathrm{X})$.

$\mathrm{A} = \omega(\mathrm{U}) \mapsto \mathrm{A}^* = \alpha(\mathrm{U}^{\#})$ the dual repeller (well-defined).

Theorem, KMV I, [2]

$$\begin{array}{ccc} \mathsf{ABlock}(\phi) & \xleftrightarrow{\;\#\;} & \mathsf{RBlock}(\phi) \\ \Big\downarrow{\scriptstyle \omega(\cdot)} & & \Big\downarrow{\scriptstyle \alpha(\cdot)} \\ \mathsf{Att}(\phi) & \xleftrightarrow{\;*\;} & \mathsf{Rep}(\phi) \end{array}$$

Attractors and repellers: reflection

- The surjective homomorphism $\mathrm{ABlock}(\phi) \longrightarrow \mathrm{Att}(\phi)$ is an algebraic description of a dynamical system.
- The homomorphism $\mathrm{RBlock}(\phi) \longrightarrow \mathrm{Rep}(\phi)$ is the natural (anti-isomorphic) dual.
- Finite sublattices provide partial information about the system: resolution.
- The lattices $\mathrm{Att}(\phi)$ and $\mathrm{Rep}(\phi)$ are at most countably infinite!

Theorem, Robbin-Salamon, [5], KMV I, [2]

For $A, R \subset X$ the following statements are equivalent.

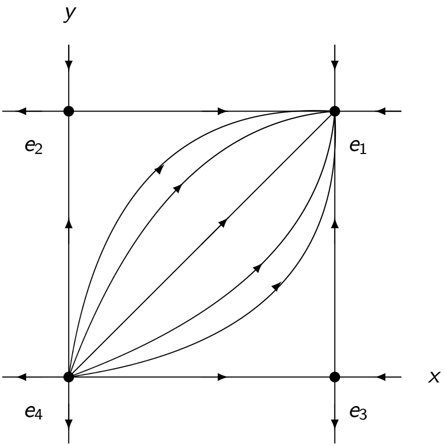
(i) (A, R) is an attractor-repeller pair.

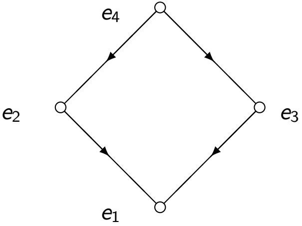
(ii) A and R are disjoint, compact sets with $A \in \mathrm{Invset}(\phi)$ and $R \in \mathrm{Invset}^+(\phi)$ such that for every $x \in X \setminus (A \cup R)$ and every backward orbit γ_x^- through x we have $\alpha_o(\gamma_x^-) \subset R$ and $\omega(x) \subset A$.

An example

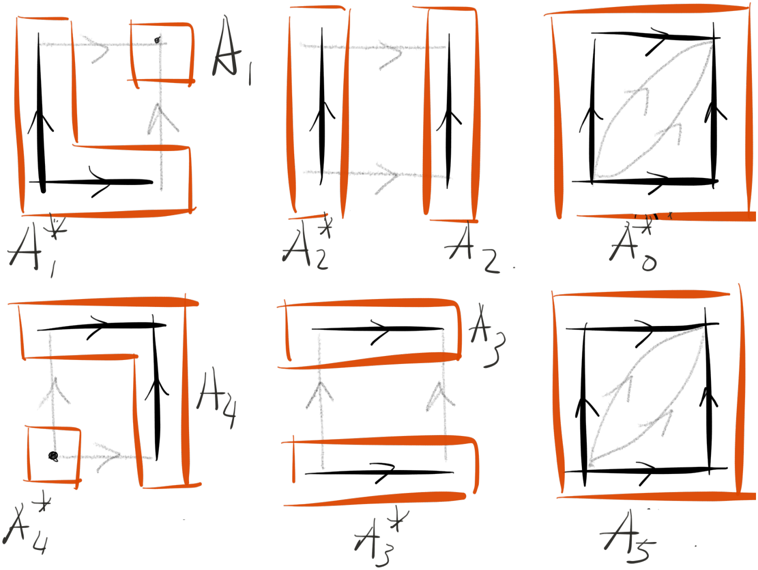
Let $(x, y) \in \mathbb{R}^2$. Consider the gradient flow generated by

$$\begin{aligned} \dot{x} &= x - x^2 \\ \dot{y} &= y - y^2 \end{aligned}$$





An example



An example

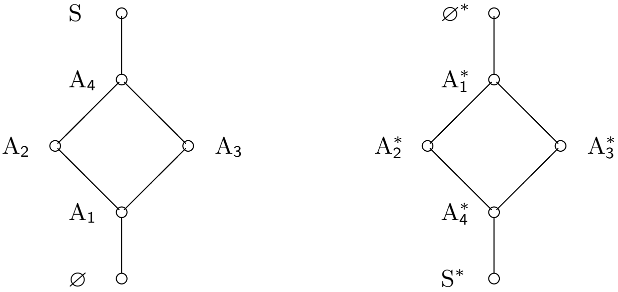


Figure: The lattices Att(ϕ) and Rep(ϕ).

Basic lattice theory

Let L be a finite distributive lattice.

An element $a \in \mathsf{L}$ is *join-irreducible* if it has a unique predecessor $\overleftarrow{a} \prec a$. The join-irreducible elements are denoted by $\mathsf{J}(\mathsf{L})$: poset.

Let P be a finite poset.

A *down-set* in P is a subset I of P characterized by the property: $p \in \mathrm{I}$, $q \leq p \implies q \in \mathrm{I}$. The set of down-sets is denoted by $\mathsf{O}(\mathsf{P})$: distributive lattice.

$$
\begin{aligned}
\mathsf{L} &= \{0, a, b, c, d, 1\},\\
\mathsf{J(L)} &= \{a, b, c, 1\}.
\end{aligned}
$$

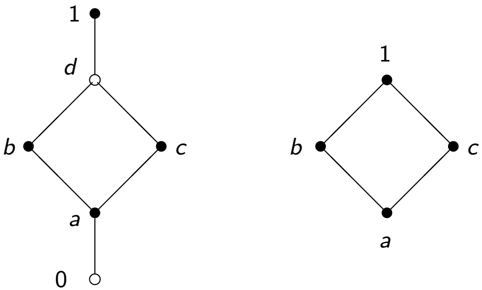


Figure: The lattice L [left] and the poset J(L) [right]. The join-irreducible elements are solid dots.

$$\begin{aligned} \mathsf{P} &= \{1,2,3,4\}, \\ \mathsf{O}(\mathsf{P}) &= \Big\{\varnothing, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}\Big\}. \end{aligned}$$

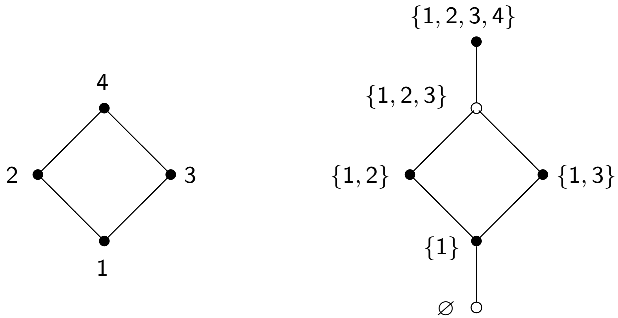


Figure: The poset $\mathsf{P} = \{1,2,3,4\}$ and the lattice of down-sets $\mathsf{O}(\mathsf{P})$. The join-irreducible elements are solid dots.

The Birkhoff representation theorem

Theorem (Birkhoff)

$$\mathsf{L} \cong \mathsf{O}(\mathsf{J}(\mathsf{L})), \qquad \mathsf{P} \cong \mathsf{J}(\mathsf{O}(\mathsf{P})).$$

The ‘recipes’ J and O are contravariant functors.

$$\begin{array}{ccc} \mathsf{K} & & \mathsf{J}(\mathsf{K}) \\ h \downarrow & \overset{\mathsf{J}}{\Longrightarrow} & \uparrow \mathsf{J}(h) \\ \mathsf{L} & & \mathsf{J}(\mathsf{L}) \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{P} & & \mathsf{O}(\mathsf{P}) \\ \phi \downarrow & \overset{\mathsf{O}}{\Longrightarrow} & \uparrow \mathsf{O}(\phi) \\ \mathsf{Q} & & \mathsf{O}(\mathsf{Q}) \end{array}$$

$$\mathsf{J}(h)(a) = \min h^{-1}(\uparrow a), \quad a \in \mathsf{J}(\mathsf{L})$$

$$\mathsf{O}(\phi)(\mathrm{I}) = \phi^{-1}(\mathrm{I}), \quad \mathrm{I} \in \mathsf{O}(\mathsf{Q}).$$

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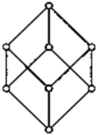
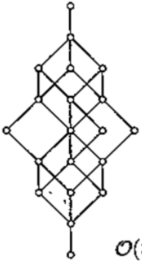
Free distributive lattices on n elements.

$\lvert X \rvert$	$\lvert \wp(X) \rvert$	$\lvert \mathcal{O}(\wp(X)) \rvert$
1	2	3
2	4	6
3	8	20
4	16	168
5	32	7581
6	64	7828354
7	128	2414682040998
8	256	56130437228687557907788



$\wp(\{1,2,3\})$



$\mathcal{O}(\wp(\{1,2,3\}))$

Morse decompositions

$$\begin{array}{ccc} \mathsf{N} & \overset{\subset}{\rightarrowtail} & \mathsf{ABlock}(\phi) \\ {\scriptstyle\omega}\big\downarrow\!\!\downarrow & & \big\downarrow\!\!\downarrow{\scriptstyle\omega} \\ \mathsf{A} & \overset{\subset}{\rightarrowtail} & \mathsf{Att}(\phi) \end{array}$$

A choice of finite sublattices of $\mathsf{N} \subset \mathsf{ABlock}(\phi)$ and $\mathsf{A} \subset \mathsf{Att}(\phi)$ with $\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A}$ may be regarded as a finite rendering of the global dynamics of a system.

The Birkhoff representation theorem is applicable to the homomorphism

$$\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A}.$$

Morse decompositions

In a more general setting
Let O(P) be a finite distributive lattice and consider a mapping

$$\mathsf{O}(\mathsf{P}) \twoheadrightarrow \mathsf{A} \subset \mathsf{Att}(\phi).$$

The Birkhoff representation theorem yields:

$$\mathsf{P} \cong \mathsf{J}(\mathsf{O}(\mathsf{P})) \hookleftarrow \mathsf{J}(\mathsf{A}).$$

P gives representation of J(O(P)). Find a representation for J(A).

$$
\begin{aligned}
P &= \{1,2,3,4\},\\
O(P) &= \Big\{\varnothing, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\}\Big\},\\
J(O(P)) &= \Big\{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3,4\}\Big\}.
\end{aligned}
$$

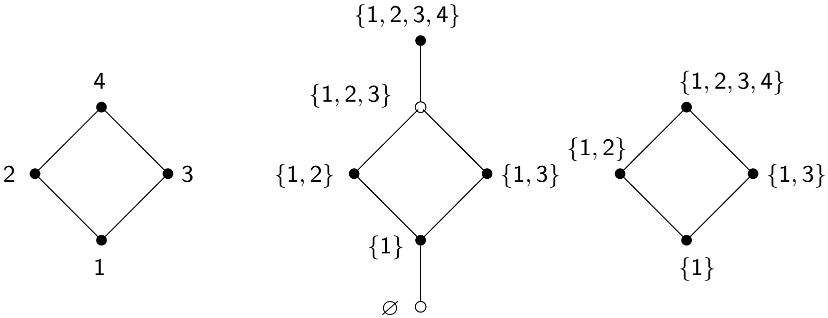


Figure: The isomorphisms $P \cong J(O(P))$.

- The join-irreducible elements are of the form $\downarrow p$.
- Duality: $\mathsf{O}(\mathsf{P}) \ni I \mapsto I^c \in \mathsf{U}(\mathsf{P})$ — the *up-sets* in P.
- Determine the maximal element in $\downarrow p$ via duality.

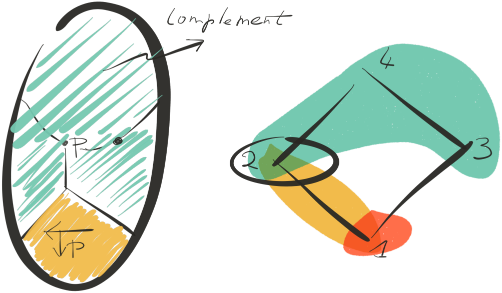


Figure: $p = I \cap \left(\overleftarrow{I}\right)^c$, where $I = \downarrow p$.

Morse decompositions: Conley form, [4]

$$\mathsf{O}(\mathsf{P}) \times \mathsf{O}(\mathsf{P}) \longrightarrow \mathsf{I}(\mathsf{P}) \quad \text{convex sets in} \quad \mathsf{P}$$

$$(a, b) \mapsto a - b := a \cap b^c.$$

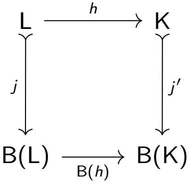
(p1) (absorption) $(a \vee b) - a = b - a$ and $a - (a \wedge b) = a - b$;

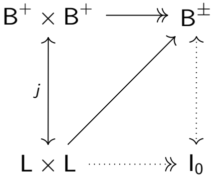
(p2) (distributivity) $(a \wedge c) - (b \vee d) = (a - b) \wedge (c - d)$;

(p3) (normalization) $1 - 0 = 1$ and $0 - 1 = 0$;

(p4) (monotonicity) $a - b = 0$ implies $b \geq a$.

Booleanization





$\mathsf{B}(\mathsf{L}) = \mathsf{Set}(\mathsf{J}(\mathsf{L}))$, $\mathsf{B}^+ = \mathsf{O}(\mathsf{J}(\mathsf{L}))$ and $(\alpha, \beta) \mapsto \alpha - \beta$.

Morse decompositions: Conley form

$$\mathsf{Att}(\phi) \times \mathsf{Att}(\phi) \longrightarrow \mathsf{Morse}(\phi) \subset \mathsf{Invset}(\phi),$$
$$(\mathrm{A}, \mathrm{A}') \mapsto \mathrm{A} - \mathrm{A}' := \mathrm{A} \cap \mathrm{A}'^{*},$$

where $\mathsf{Morse}(\phi) := \{\mathrm{A} \cap \mathrm{R} \mid \mathrm{A} \in \mathsf{Att}(\phi),\ \mathrm{R} \in \mathsf{Rep}(\phi)\}$ is a *semi-lattice* with binary operation $\wedge$ and with zero.
Axioms (p1)-(p4) are satisfied. Unique up to isomorphism. Generalizes set-difference in Boolean algebras. In general:

$$\begin{array}{ccccc} \mathsf{K} \times \mathsf{K} & \twoheadrightarrow & \mathsf{J}_0 & \rightarrowtail & \mathsf{J} \\ \downarrow{\scriptstyle h \times h} & & \downarrow{\scriptstyle \jmath} & \searrow & \\ \mathsf{L} \times \mathsf{L} & \twoheadrightarrow & \mathsf{I}_0 & \rightarrowtail & \mathsf{I} \end{array}$$

where $\jmath(a - b) := h(a) - h(b)$.

$$\mathsf{ABlock}(\phi) \times \mathsf{ABlock}(\phi) \longrightarrow \mathsf{MTile}(\phi) \subset \mathsf{INbhd}(\phi),$$
$$(\mathrm{U}, \mathrm{U}') \mapsto \mathrm{U} - \mathrm{U}' := \mathrm{U} \wedge \mathrm{U}'^{\#},$$

where $\mathsf{MTile}(\phi) := \{\mathrm{U} \wedge \mathrm{V} \mid \mathrm{U} \in \mathsf{ABlock}(\phi),\ \mathrm{V} \in \mathsf{RBlock}(\phi)\}$ is a *semi-lattice* with binary operation $\wedge$ and with zero.

Morse decompositions: Conley form

Apply to

$$
\begin{array}{ccc}
\mathsf{O}(\mathsf{J}(\mathsf{A})) \times \mathsf{O}(\mathsf{J}(\mathsf{A})) & \twoheadrightarrow & \mathsf{I}(\mathsf{J}(\mathsf{A})) \\
h \times h \updownarrow & & \updownarrow \mu \\
\mathsf{A} \times \mathsf{A} & \twoheadrightarrow & \mathsf{I}_0(\mathsf{A})
\end{array}
$$

with $\mu(a - b) = h(a) - h(b)$, $h(a) = \bigvee\{A \in a\}$ and $\mu(\{\mathrm{A}\}) = h(\downarrow \mathrm{A}) - h(\downarrow \mathrm{A} \setminus \{\mathrm{A}\}) = \mathrm{A} - \overleftarrow{\mathrm{A}}$.

$$
\begin{array}{ccc}
\mathsf{J}(\mathsf{O}(\mathsf{P})) & \hookleftarrow & \mathsf{J}(\mathsf{A}) \\
\updownarrow & & \updownarrow \\
\mathsf{P} & \overset{\pi}{\hookleftarrow} & \mathsf{M}(\mathsf{A})
\end{array}
$$

where $\mathsf{M}(\mathsf{A}) := \{\mu(\{\mathrm{A}\}) \mid \mathrm{A} \in \mathsf{J}(\mathsf{A})\}$.
There are no empty sets $\mu(\{A\})$!

Morse decompositions: Conley form

$$\begin{array}{ccccc} \mathsf{ABlock}(\phi)\times\mathsf{ABlock}(\phi) & \twoheadrightarrow & \mathsf{MTile}(\phi) & \rightarrowtail & \mathsf{INbhd}(\phi) \\ \Big\downarrow{\scriptstyle \omega\times\omega} & & \Big\downarrow{\scriptstyle \mathsf{Inv}} & & \Big\downarrow{\scriptstyle \mathsf{Inv}} \\ \mathsf{Att}(\phi)\times\mathsf{Att}(\phi) & \twoheadrightarrow & \mathsf{Morse}(\phi) & \rightarrowtail & \mathsf{Isol}(\phi). \end{array}$$

$$\begin{array}{ccccc} \mathsf{N}\times\mathsf{N} & \twoheadrightarrow & \mathsf{MTile}(\mathsf{N}) & \rightarrowtail & \mathsf{INbhd}(\phi) \\ \Big\downarrow{\scriptstyle \omega\times\omega} & & \Big\downarrow{\scriptstyle \mathsf{Inv}} & & \Big\downarrow{\scriptstyle \mathsf{Inv}} \\ \mathsf{A}\times\mathsf{A} & \twoheadrightarrow & \mathsf{Morse}(\mathsf{A}) & \rightarrowtail & \mathsf{Isol}(\phi). \end{array}$$

Morse decompositions: Conley form

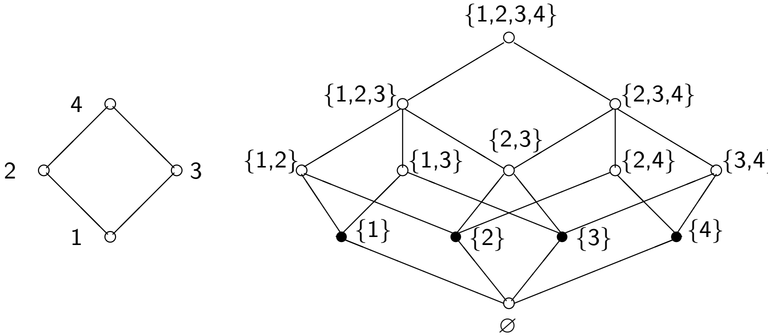


Figure: The poset P [left] and the lattice of convex sets in I(P) [right]. The join-irreducible elements are denoted by solids.

Morse decompositions

KMV III, [4]

The injection

$$\pi\colon \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{P},$$

is called a *Morse decompositon*. The poset $\mathsf{M}(\mathsf{A})$ is called a *Morse representation*.
Apply to $\omega\colon \mathsf{N} \twoheadrightarrow \mathsf{A}$ yields

$$\pi\colon \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N}),$$

where $\mathsf{T}(\mathsf{N})$ is called *Morse tiling* and π is a *tesselated Morse decomposition*.

The mapping π has a left-inverse: $\mathsf{Inv} \circ \pi = \mathrm{id}$ on $\mathsf{M}(\mathsf{A})$.

An example

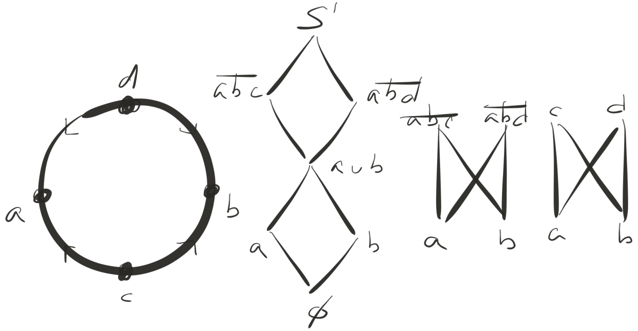


Figure: The double diamond lattice.

Morse decompositions

Since X is compact the order-theoretic definition of Morse decomposition is equivalent to the traditional dynamical systems definition of Morse decomposition:

Definition (Conley)

A *Morse decomposition* is an order embedding $\pi\colon \mathsf{M} \hookrightarrow \mathsf{P}$, where M and P are finite posets and where M consists of nonempty, compact, pairwise disjoint invariant sets $\mathrm{M} \subset \mathrm{X}$ of ϕ such that for every complete orbit γ_x through a point $x \in \mathrm{X} \setminus \cup_{\mathsf{M}} \mathrm{M}$ there exist $p, p' \in \mathsf{P}$ with $p < p'$ such that

$$\omega(x) \subset \pi^{-1}(p) \quad \text{and} \quad \alpha_{\mathrm{o}}(\gamma_x^-) \subset \pi^{-1}(p').$$

$\alpha_{\mathrm{o}}(\gamma_x^-) = \bigcap_{t \leq 0} \overline{\gamma_x((-\infty, t])}$ is the orbital alpha-limit set.

Reconstruction

Given $\mathsf{O}(\mathsf{P}) \twoheadrightarrow \mathsf{A}$ and its dual $\pi\colon \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{P}$ it holds:

$$\nu = \mathsf{O}(\pi)\colon \mathsf{O}(\mathsf{P}) \twoheadrightarrow \mathsf{O}(\mathsf{M}),$$

and

$$\mathsf{O}(\mathsf{M}) \ni a \mapsto \nu(a) = \mathrm{A} = \bigcup_{\mathrm{M}\in \mathrm{a}} W^u(\mathrm{M}),$$

where $W^u(\mathrm{M}) = \{\mathrm{x} \in \mathrm{X} \mid \exists \gamma_{\mathrm{x}}^- \ni \alpha_{\mathrm{o}}(\gamma_{\mathrm{x}}^-) \subset \mathrm{M}\}$.

Proof: Let $\mathrm{M} \in \mathrm{a}$ be a maximal element, then $a \setminus \{\mathrm{M}\} \in \mathsf{O}(\mathsf{M})$ and

$$\mathrm{A} = \nu(\downarrow \mathrm{M}) \cup \bigcup_{\mathrm{M}'\in \mathrm{a}'} \nu(\downarrow \mathrm{M}'), \quad \mathrm{a}' = \mathrm{a} \setminus \{\mathrm{M}\}.$$

Since $a - a' = \{\mathrm{M}\}$ the Conley form implies that $\mathrm{A} - \mathrm{A}' = \mathrm{M}$, where $\mathrm{A}' = \bigcup_{\mathrm{M}'\in \mathrm{a}'} \nu(\downarrow \mathrm{M}')$ and $\mathrm{A} = W^u(\mathrm{M}) \cup \mathrm{A}'$.

Repeat the same procedure to obtain the expression $\nu(a) = \bigcup_{\mathrm{M}\in \mathrm{a}} W^u(\mathrm{M})$

Reconstruction

Conversely, given a Morse decomposition $\pi \colon \mathsf{M} \hookrightarrow \mathsf{P}$, then

$$\mathsf{O}(\mathsf{P}) \twoheadrightarrow \mathsf{O}(\mathsf{M}) \leftrightarrow \mathsf{A} \subset \mathsf{Att}(\phi),$$

is a lattice surjection. The latter isomorphism is given by

$$a \mapsto \bigcup_{\mathrm{M}\in\mathrm{a}} W^u(\mathrm{M}).$$

Moreover, $\mathsf{M}(\mathsf{A}) = \mathsf{M}$.

An example

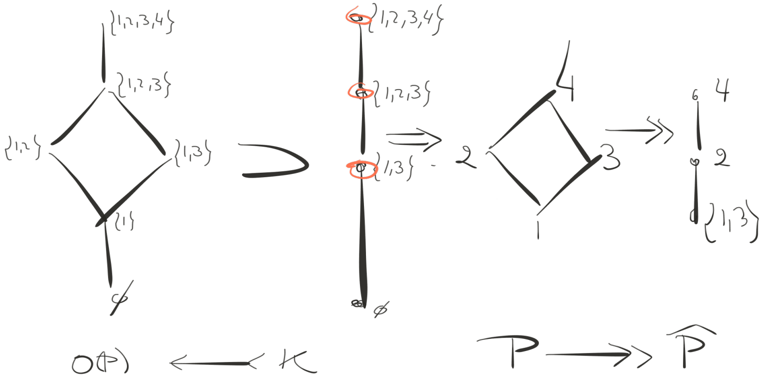


Figure: Coarsening via sublattices.

Example

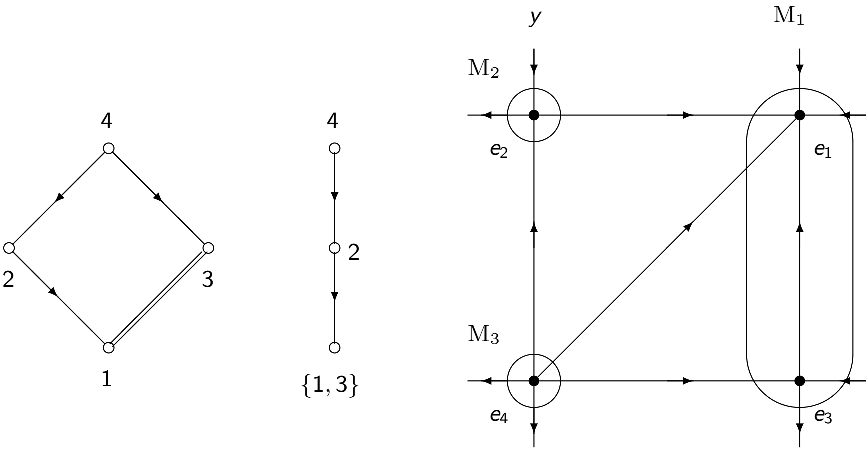


Figure: A Morse representation via sublattices.

Coarsening

Choose a sublattice $\mathsf{O}(Q) \rightarrowtail \mathsf{O}(P) \twoheadrightarrow \mathsf{A}$. This yields the commutative diagram:

$$
\begin{array}{ccc}
\mathsf{O}(Q) & \rightarrowtail & \mathsf{O}(P) \\
\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
\mathsf{A}_0 & \rightarrowtail & \mathsf{A}
\end{array}
$$

where A_0 is the range of $\mathsf{O}(Q) \to \mathsf{A}$.
Morse duals:

$$
\begin{array}{ccc}
Q & \twoheadleftarrow & P \\
\uparrow & & \uparrow \\
\mathsf{M}(\mathsf{A}_0) & \twoheadleftarrow & \mathsf{M}(\mathsf{A})
\end{array}
$$

Morse tilings

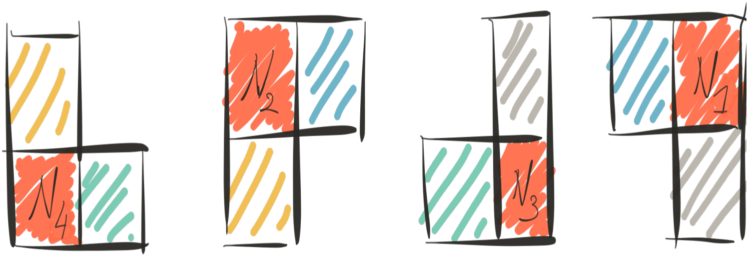


Figure: A tesselated Morse decomposition with isomorphic Morse tiling.

Tesselated Morse decompositions

- Tesselated Morse decompositions $\pi\colon \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N})$ play a crucial role in the theory of connection matrices.
- Given a Morse decomposition $\mathsf{M} \hookrightarrow \mathsf{P}$. Does such a Morse decomposition factor through a tesselated Morse decomposition?

$$\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(\mathsf{N}) \hookrightarrow \mathsf{P}.$$

- This question is equivalent to

$$\mathsf{O}(\mathsf{P}) \twoheadrightarrow \mathsf{N} \twoheadrightarrow \mathsf{A} \subset \mathsf{Att}(\phi).$$

- Is there a sublattice $\mathsf{N} \subset \mathsf{ABlock}(\phi)$ such that $\omega(\mathsf{N}) = \mathsf{A}$.

Tesselated Morse decompositions: lifting

We seek

$$\mathsf{O}(\mathsf{P}) \twoheadrightarrow \mathsf{N} \longleftrightarrow \mathsf{A} \subset \mathsf{Att}(\phi).$$

Finding an isomorphic N is called a lift of A into $\mathsf{ABlock}(\phi)$.

Theorem, Franzosa, KMV I, [2]

Let i denote the inclusion map. Then, for every finite sublattice $\mathsf{A} \subset \mathsf{Att}(\phi)$, there exists a lattice monomorphism k such that the following diagram

$$\begin{array}{ccc} & & \mathsf{ABlock}(\phi) \\ & \overset{k}{\nearrow} & \Big\downarrow \omega \\ \mathsf{A} & \xrightarrow{\;i\;} & \mathsf{Att}(\phi) \end{array}$$

commutes.

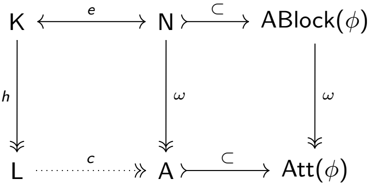
Table of Contents

Combinatorial models

In order to have a computational framework we develop a combinatorial model for $\omega\colon \mathsf{ABlock}(\phi) \to \mathsf{Att}(\phi)$.

(i) a lattice epimorphism $h\colon \mathsf{K} \twoheadrightarrow \mathsf{L}$ of finite distributive lattices, called the *interior homomorphism* and

(ii) a lattice monomorphism $e\colon \mathsf{K} \rightarrowtail \mathsf{ABlock}(\phi)$, called the *evaluation homomorphism*, which links the combinatorial model to the dynamical system through the commutative diagram



where $\mathsf{N} = e(\mathsf{K})$ and $\mathsf{A} = \omega(\mathsf{N})$.

Combinatorial models

- The *dashed* arrow $c\colon \mathsf{L} \twoheadrightarrow \mathsf{A}$, which may or may not exist if we just choose $\mathsf{K} \twoheadrightarrow \mathsf{L}$, is called the *connecting homomorphism*.
- The latter is essential for the epimorphism h to reflect the dynamics of ω. Combinatorial models for which a dashed arrow exists are called *commutative* combinatorial models.
- The objective is to perform computations on combinatorial models for which K and L are given and translate the result to the underlying system through the evaluation homomorphism.
- To clearly illustrate the concept of a combinatorial model, we describe the well-studied approach of *outer approximations*.

Outer approximations

For sake of simplicity consider discrete time time dynamical systems $f\colon \mathrm{X} \to \mathrm{X}$.

- Build a finite discretization of the phase space X by a *grid*, which is defined as a finite subalgebra of $\mathbf{R}(\mathrm{X})$, the regular closed subsets of X, such as a triangulation or a cubical grid when X is a region in $\mathbb{R}^n$.
- By $\mathcal{X}$ denote an indexing set for the grid. In particular, given $\xi \in \mathcal{X}$ the corresponding grid element is denoted by $|\xi| \in \mathbf{R}(\mathrm{X})$. The *evaluation mapping* $|\cdot| : \mathsf{Set}(\mathcal{X}) \to \mathbf{R}(\mathrm{X})$ is defined by

$$|\mathcal{U}| := \bigcup_{\xi \in \mathcal{U}} |\xi|\,.$$

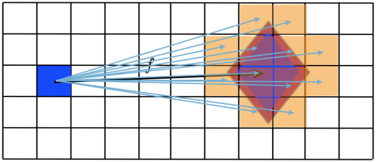
The mapping $f\colon \mathrm{X} \to \mathrm{X}$ is approximated by a relation $\mathcal{F}$ on the set of grid elements $\mathcal{X}$: notation $(\mathcal{X}, \mathcal{F})$. Define $\mathcal{F}(\xi) := \{\eta \in \mathcal{X} \mid (\xi, \eta) \in \mathcal{F}\}$.

Outer approximations

Definition (cf. Mrozek, Mischaikow, Szymczak)

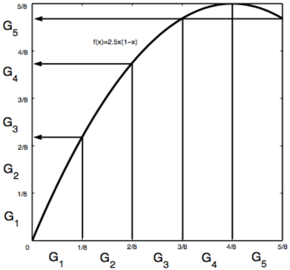
Let $f\colon X \to X$ be a continuous mapping and let $\mathcal{X}$ be the indexing set for a grid on X. A relation $(\mathcal{X}, \mathcal{F})$ is an *outer approximation* of a dynamical system (X, f) if

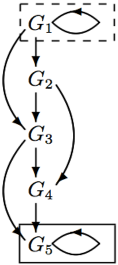
$$f(|\xi|) \subset \operatorname{int} |\mathcal{F}(\xi)| \text{ for all } \xi \in \mathcal{X}.$$



In order to understand outer approximations we regard relations $(\mathcal{X}, \mathcal{F})$ as dynamical systems on the finite point set $\mathcal{X}$.

Outer approximations





We start with a primer on combinatorial dynamics.

Dynamics of binary relations

Let $\mathcal{F} \subset \mathcal{X} \times \mathcal{X}$ be a binary relation on $\mathcal{X}$. Then for every $\mathcal{U} \subset \mathcal{X}$ the relation acts on $\mathcal{U}$ as follows:

$$\mathcal{F}(\mathcal{U}) := \{\eta \in \mathcal{X} \mid \exists \xi \in \mathcal{U} \text{ s.t. } (\xi, \eta) \in \mathcal{F}\}.$$

This defines a mapping on $\mathsf{Set}(\mathcal{X})$ whose element wise representation is a multivalued mapping on $\mathcal{X}$.

As multivalued mapping, $\mathcal{F}^{-1}$ denotes the inverse image.

The concept of binary relation can be equivalently described by the notion of *directed graph* as follows: the set $\mathcal{X}$ represent the vertices and the edges are given by the pairs $(\xi, \eta) \in \mathcal{F}$, where ξ is the source and η the target. We abuse notation and use the symbol $\mathcal{F}$ to represent a binary relation on $\mathcal{X}$ and its equivalent digraph.

Attractors

- A set $\mathcal{U} \subset \mathcal{X}$ is *forward invariant* if $\mathcal{F}(\mathcal{U}) \subset \mathcal{U}$.
- A set $\mathcal{A} \subset \mathcal{X}$ is an *attractor* for $\mathcal{F}$ if $\mathcal{F}(\mathcal{A}) = \mathcal{A}$.
- Forward invariant sets and attractors are denoted by $\mathrm{Invset}^+(\mathcal{F})$ and $\mathrm{Att}(\mathcal{F})$ respectively.
- Define *$\boldsymbol{\omega}$-limit set* of a set $\mathcal{U} \subset \mathcal{X}$ as

$$\boldsymbol{\omega}(\mathcal{U}, \mathcal{F}) = \bigcap_{k \geq 0} \Gamma_k^+(\mathcal{U}),$$

where $\Gamma_k^+(\mathcal{U}) = \bigcup_{n \geq k} \mathcal{F}^n(\mathcal{U})$ for $k > 0$ is *k-forward image* of $\mathcal{U}$.

- For attractors $\mathcal{A}, \mathcal{A}' \in \mathrm{Att}(\mathcal{F})$ define

$$\mathcal{A} \vee \mathcal{A}' = \mathcal{A} \cup \mathcal{A}', \quad \mathcal{A} \wedge \mathcal{A}' = \boldsymbol{\omega}(\mathcal{A} \cap \mathcal{A}').$$

- With $\wedge$ and $\vee$ as defined above $\mathrm{Att}(\mathcal{F})$ is a finite distributive lattice and

$$\boldsymbol{\omega} \colon \mathrm{Invset}^+(\mathcal{F}) \twoheadrightarrow \mathrm{Att}(\mathcal{F})$$

is a lattice epimorphism.

Strongly connected and recurrent components

- If there exists a sequence $\{\xi_0, \cdots, \xi_k\}$, with $\xi_0 = \xi$ and $\xi_k = \xi'$, such that $\xi_{i+1} \in \mathcal{F}(\xi_i)$, then ξ' is *reachable* from ξ, denoted by $\xi' \leftarrow \xi$.
- The reachability relation is the transitive closure $\mathcal{F}^+$ of the relation $\mathcal{F}$.
- If $\xi \leftarrow \xi'$ and $\xi' \leftarrow \xi$, then ξ is *connected* to ξ', denoted by $\xi \leftrightarrow \xi'$. If $\xi \not\leftarrow \xi'$ and $\xi' \not\leftarrow \xi$, then ξ, ξ' are *parallel* elements, denoted by $\xi \parallel \xi'$.
- The relation $\leftrightarrow$ is symmetric and transitive and hence defines the *partial* equivalence relation of *connectivity* on $\mathcal{X}$. It is an equivalence relation on the set of cyclic vertices, and the partial equivalence classes of $\leftrightarrow$ are called the *cyclic strongly connected components* $[\xi]_{\leftrightarrow}$ of $\mathcal{F}$.
- We denote the set of recurrent components by $\mathrm{RC}(\mathcal{F})$ — *Morse graph*.

Strongly connected and recurrent components

- For $\mathcal{S}, \mathcal{S}' \in \mathsf{RC}(\mathcal{F})$, we say $\mathcal{S}' \leq \mathcal{S}$ if and only if there exist $\xi \in \mathcal{S}$ and $\xi' \in \mathcal{S}'$, such that $\xi' \twoheadleftarrow \xi$.
- We refer to the poset $\big(\mathsf{RC}(\mathcal{F}), \leq\big)$ as the poset of *recurrent components* of $\mathcal{F}$, or Morse graph.
- Consider the reflexive closure $\circlearrowleft = (\twoheadleftarrow\!\!\!\twoheadrightarrow)^{=}$, i.e. given $\xi, \xi' \in \mathcal{X}$ define $\xi \circlearrowleft \xi'$ if $\xi \twoheadleftarrow\!\!\!\twoheadrightarrow \xi'$ or $\xi' = \xi$.
- The equivalence relation $\circlearrowleft$ on $\mathcal{X}$ is called *strong connectivity*. In graph theory the equivalence classes $[\xi]_{\circlearrowleft}$ are called the *strongly connected components* which we denote by $\mathsf{SC}(\mathcal{F})$.
- The partial order on $\mathsf{RC}(\mathcal{F})$ can be extended to a partial order on $\mathsf{SC}(\mathcal{F})$ as follows. Given $\mathcal{S}, \mathcal{S}' \in \mathsf{SC}(\mathcal{F})$ define $\mathcal{S}' \leq \mathcal{S}$ if there exist $\xi \in \mathcal{S}$ and $\xi' \in \mathcal{S}'$ such that $\xi' \twoheadleftarrow \xi$ or $\mathcal{S} = \mathcal{S}'$. We refer to the poset $\big(\mathsf{SC}(\mathcal{F}), \leq\big)$ as the poset of *strongly connected components* of $\mathcal{F}$. Set inclusion defines the order embedding

$$\imath\colon \mathsf{RC}(\mathcal{F}) \hookrightarrow \mathsf{SC}(\mathcal{F}).$$

An example

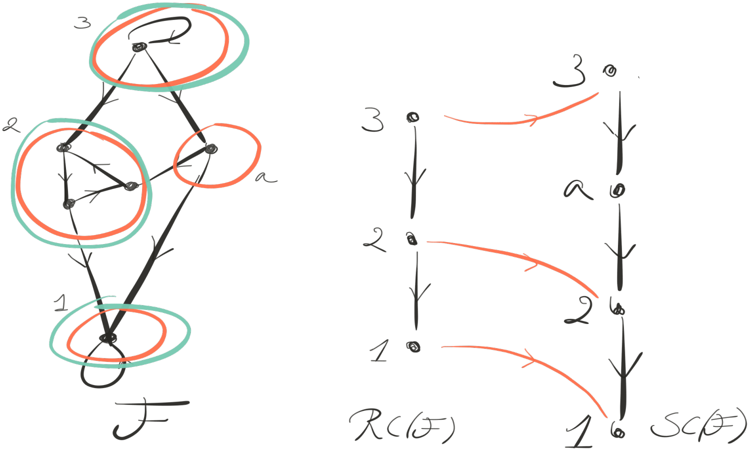


Figure: The recurrent and strongly connected components of $\mathcal{F}$.

Conley form

- J(Att($\mathcal{F}$)) is isomorphic to RC($\mathcal{F}$) via the Conley form; elements are of the form $\mathcal{A} \cap \mathcal{R}$ ($\mathcal{F}^{-1}(\mathcal{R}) = \mathcal{R}$).
- J(Invset$^+$($\mathcal{F}$)) is isomorphic to SC($\mathcal{F}$) via the Conley form; elements are of the form $\mathcal{U} \cap \mathcal{V}$ ($\mathcal{F}^{-1}(\mathcal{R}) \subset \mathcal{V}$).

Generalizations of Birkhoff's theorem, I

For every finite distributive lattice L may be represented as the lattice of down-sets of a finite poset $(\mathsf{P}, \leq)$, i.e. $\mathsf{L} \cong \mathsf{O}(\mathsf{P})$.

Theorem (1st Generalized Birkhoff theorem)

Let $h\colon \mathsf{K} \twoheadrightarrow \mathsf{L}$ be a lattice epimorphism between finite distributive lattices. Then, there exists, up to transitivity, condensation, and isomorphisms, a unique binary relation $\mathcal{F}$ on a finite point set $\mathcal{X}$, such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathsf{Invset}^+(\mathcal{F}) & \longleftrightarrow & \mathsf{K} \\
\Big\downarrow \omega & & \Big\downarrow h \\
\mathsf{Att}(\mathcal{F}) & \longleftrightarrow & \mathsf{L}
\end{array}
$$

Generalizations of Birkhoff's theorem, II

For every finite poset P is isomorphic to the set of join-irreducible elements of a finite distributive lattice L, ordered with respect to set inclusion, i.e. $\mathsf{P} \cong \mathsf{J}(\mathsf{L})$.

Theorem (2nd Generalized Birkhoff theorem)

Let $\mathcal{F}$ be a finite binary relation on a finite point set $\mathcal{X}$. Then, there exists, up to isomorphisms, a unique lattice epimorphism $h\colon \mathsf{K} \twoheadrightarrow \mathsf{L}$ between finite distributive lattices such that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{SC}(\mathcal{F}) & \longleftrightarrow & \mathsf{J}(\mathsf{K}) \\ \subset \big\uparrow & & \big\uparrow \mathsf{J}(h) \\ \mathsf{RC}(\mathcal{F}) & \longleftrightarrow & \mathsf{J}(\mathsf{L}) \end{array}$$

Generalizations of Birkhoff's theorem

- The uniqueness part in the above theorems can be made more precise as follows: Two binary relations $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{X}', \mathcal{F}')$ are equivalent if and only if the following diagram commutes

$$\begin{array}{ccc} \mathsf{SC}(\mathcal{F}) & \longleftrightarrow & \mathsf{SC}(\mathcal{F}') \\ \subset \big\uparrow & & \big\uparrow \subset \\ \mathsf{RC}(\mathcal{F}) & \longleftrightarrow & \mathsf{RC}(\mathcal{F}') \end{array}$$

- In particular, $\mathsf{Invset}^+(\mathcal{F}) \cong \mathsf{O}(\mathsf{SC}(\mathcal{F}))$ and $\mathsf{Att}(\mathcal{F}) \cong \mathsf{O}(\mathsf{RC}(\mathcal{F}))$;
- $\mathsf{SC}(\mathcal{F}) \cong \mathsf{J}(\mathsf{Invset}^+(\mathcal{F}))$ and $\mathsf{RC}(\mathcal{F}) \cong \mathsf{J}(\mathsf{Att}(\mathcal{F}))$.

Back to outer approximations, KMV II, [3]

For outer approximations we have the following combinatorial model

$$\begin{array}{ccccc} \mathsf{Invset}^+(\mathcal{F}) & \overset{|\cdot|}{\longleftrightarrow} & \mathsf{N} & \rightarrowtail & \mathsf{ABlock}(f) \\ \Big\downarrow\!\!\!\downarrow \boldsymbol{\omega} & & \Big\downarrow\!\!\!\downarrow \omega & & \Big\downarrow\!\!\!\downarrow \omega \\ \mathsf{Att}(\mathcal{F}) & \overset{\omega(|\cdot|)}{\twoheadrightarrow} & \mathsf{A} & \rightarrowtail & \mathsf{Att}(f) \end{array} \tag{1}$$

Dualize

$$\begin{array}{ccc} \mathsf{SC}(\mathcal{F}) & \overset{\delta}{\longleftrightarrow} & \mathsf{T}(\mathsf{N}) \\ i \Big\uparrow \subset & & \Big\uparrow \pi \\ \mathsf{RC}(\mathcal{F}) & \hookleftarrow & \mathsf{M}(\mathsf{A}) \end{array}$$

Consequences

- Combinatorial models as described above contain computable information about the underlying system. The latter diagram gives a tesselated Morse decomposition

$$\pi\colon \mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{RC}(\mathcal{F}) \hookrightarrow \mathsf{SC}(\mathcal{F}) \longleftrightarrow \mathsf{T}(\mathsf{N}),$$

- In general the tesselation $\mathsf{T}(\mathsf{N}) \cong \mathsf{SC}(\mathcal{F})$ is a very large set, while $\mathsf{RC}(\mathcal{F})$ is relatively small in size. Two central questions are:
  - (i) Can every sublattice $\mathsf{A} \subset \mathsf{Att}(f)$ in Diagram (1) be realized via an outer approximation?
  - (ii) Do coarser tesselations exist, for example tesselated Morse decompositions of the form $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{RC}(\mathcal{F}) \longleftrightarrow \mathsf{T}(\mathsf{N})$?

Consequences

To answer these questions we consider the following commutative diagram

$$
\begin{array}{ccccccc}
\mathsf{U} & \xrightarrow{\subset} & \mathsf{Invset}^+(\mathcal{F}) & \xleftrightarrow{|\cdot|} & \mathsf{N} & \longrightarrow & \mathsf{ABlock}(f) \\
\downarrow \omega & & \downarrow \omega & & \downarrow \omega & & \downarrow \omega \\
\mathsf{L} & \xrightarrow{\subset} & \mathsf{Att}(\mathcal{F}) & \xrightarrow{\omega(|\cdot|)} & \mathsf{A} & \longrightarrow & \mathsf{Att}(f)
\end{array}
$$

- Every U for which $\omega(|\mathsf{L}|) = \mathsf{A}$ yields a tesselated Morse decomposition $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(|\mathsf{U}|)$, which is coarser than $\mathsf{T}(\mathsf{N})$.
- Without knowing A this may not be computable and therefore does not answer the second question. The first question may be answered as a variation on the lifting theorem.

Consequences

Convergence of a 'sequence' $\mathcal{F}_n$ of outer approximations on grids $\mathcal{X}_n$ corresponds to both the diameters of the grid elements and the errors in images of grid elements under $\mathcal{F}_n$ to tend to zero as $n \to \infty$.

Theorem (Theorems 1.2 and 4.20 in [3])

Let $(\mathcal{X}_n, \mathcal{F}_n)$ be a convergent cofiltration of outer approximations. Then for every finite sublattice $\mathsf{A} \subset \mathrm{Att}(f)$ there exists $n_{\mathsf{A}} \in \mathbb{N}$ such that for all $n \geq n_{\mathsf{A}}$ there exists a lift $\ell_n : \mathsf{A} \rightarrowtail \mathrm{Invset}^+(\mathcal{F}_n)$ of the inclusion map $i \colon \mathsf{A} \rightarrowtail \mathrm{Att}(f)$ through $\omega(|\cdot|) \colon \mathrm{Invset}^+(\mathcal{F}_n) \to \mathrm{Att}(f)$, i.e. the following diagram commutes

$$\begin{array}{ccc} & & \mathrm{Invset}^+(\mathcal{F}_n) \\ & \overset{\ell_n}{\nearrow} & \downarrow \omega(|\cdot|) \\ \mathsf{A} & \xrightarrow{\ \ i\ \ } & \mathrm{Att}(f) \end{array}$$

Consequences

- Using the above theorem we set $\mathsf{U}_n = \ell_n(\mathsf{A})$ and $\mathsf{N}_n := |\ell_n(\mathsf{A})| \cong \mathsf{A}$, which induces an isomorphic tesselated Morse decomposition $\pi_n \colon \mathsf{M}(\mathsf{A}) \leftrightarrow \mathsf{T}(\mathsf{N}_n)$ for all $n \geq n_{\mathsf{A}}$. This also provides an alternative proof for the existence of index lattices for Morse decompositions as given before.
- The asymptotic lifting theorem answers both questions in (i) and (ii) with respect to existence.
- However, a fundamental issue still remains unanswered by this asymptotic result. If a computation is performed at a certain fixed resolution, how can we algorithmically determine whether a lift exists, and if so construct a lift and a 'coarse' tesselated Morse decomposition.
- To develop such an algorithm, we make use of a generalization Birkhoff's representation theorem. We characterize the existence of a lift in terms of the existence of a certain mapping between the strongly connected components and the recurrent components of the directed graph defined from $\mathcal{F}$.

Order retractions

An order-preserving surjection $\mathsf{SC}(\mathcal{F}) \twoheadrightarrow \mathsf{RC}(\mathcal{F})$ is called an *order retraction*.

The existence of such a mapping implies the existence of a lift $\mathsf{Att}(\mathcal{F}) \rightarrowtail \mathsf{Invset}^+(\mathcal{F})$, which yields the diagram:

$$
\begin{array}{ccccccc}
\mathsf{U} & \xrightarrow{\ \subset\ } & \mathsf{Invset}^+(\mathcal{F}) & \xleftrightarrow{\ |\cdot|\ } & \mathsf{N} & \rightarrowtail & \mathsf{ABlock}(f) \\
\big\updownarrow \boldsymbol{\omega} & & \boldsymbol{\omega} \big\downarrow & & \big\downarrow \omega & & \big\downarrow \omega \\
\mathsf{Att}(\mathcal{F}) & \xleftrightarrow{\ \mathrm{id}\ } & \mathsf{Att}(\mathcal{F}) & \xrightarrow{\ \omega(|\cdot|)\ } & \mathsf{A} & \rightarrowtail & \mathsf{Att}(f)
\end{array}
$$

The sublattice U yields a tesselated Morse decomposition

$$\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(|\mathsf{U}|) \cong \mathsf{RC}(\mathcal{F}).$$

Order retractions

- We are developing an algorithm to decide whether order retractions exist and construct them.
- If an order retraction does not exist the algorithm provides the inclusions

$$\mathsf{RC}(\mathcal{F}) \hookrightarrow \mathsf{RC}(\mathcal{F})^{\dagger} \hookrightarrow \mathsf{SC}(\mathcal{F}),$$

and an order retraction $\mathsf{SC}(\mathcal{F}) \twoheadrightarrow \mathsf{RC}^{\dagger}(\mathcal{F})$.

- This yields the diagram

$$\begin{array}{ccccccc}
\mathsf{U} & \xrightarrow{\ \subset\ } & \mathsf{Invset}^{+}(\mathcal{F}) & \xleftrightarrow{\ |\cdot|\ } & \mathsf{N} & \rightarrowtail & \mathsf{ABlock}(f) \\
\big\updownarrow \boldsymbol{\omega} & & \boldsymbol{\omega} \big\downarrow & & \big\downarrow \omega & & \big\downarrow \omega \\
\mathsf{L} & \twoheadrightarrow & \mathsf{Att}(\mathcal{F}) & \xrightarrow{\ \omega(|\cdot|)\ } & \mathsf{A} & \rightarrowtail & \mathsf{Att}(f)
\end{array}$$

As before the sublattice U yields a tesselated Morse decomposition

$$\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{T}(|\mathsf{U}|) \cong \mathsf{RC}^{\dagger}(\mathcal{F}).$$

An example: the Leslie model (Arai, Kalies, Kokubu, Mischaikow, Oka, Pilarczyk)

- Consider the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x_1 + \theta_2 x_2) e^{-\phi(x_1+x_2)} \\ p x_1 \end{bmatrix}, \tag{2}$$

  where we choose parameters $\theta_1 = 20.0$, $\theta_2 = 20.0$, $\phi = 0.1$, and $p = 0.7$. The phase space region is taken to be $\mathrm{X} = [0, 74] \times [0, 52]$, which is a forward invariant region.

- A (rigorous) outer approximation $\mathcal{F}$ is computed for f on X, and the poset structure of the recurrent components $\mathrm{RC}(\mathcal{F})$ is shown on the next slide. The results of [2, 4] imply that there is a Morse decomposition $\pi\colon \mathsf{M} \hookrightarrow \mathrm{RC}(\mathcal{F})$ where each of the invariant sets in M lies in exactly one of the recurrent components.

- In this example, $\mathrm{SC}(\mathcal{F})$ has 16,343,562 elements; the 6 recurrent components contain 433,654 boxes. We run the algorithm the order retraction, cf. [1], and verify that an order retraction $\mathrm{SC}(\mathcal{F}) \twoheadrightarrow \mathrm{RC}(\mathcal{F})$ exists so that $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathrm{RC}(\mathcal{F}) \leftrightarrow \mathsf{T}(|\mathsf{U}|)$ is a tesselated Morse decomposition, whose Morse tiles are shown on the next slides (left). However, we do not know whether this is an isomorphic decomposition where $\mathsf{M}(\mathsf{A}) \leftrightarrow \mathrm{RC}(\mathcal{F})$.

An example: the Leslie model

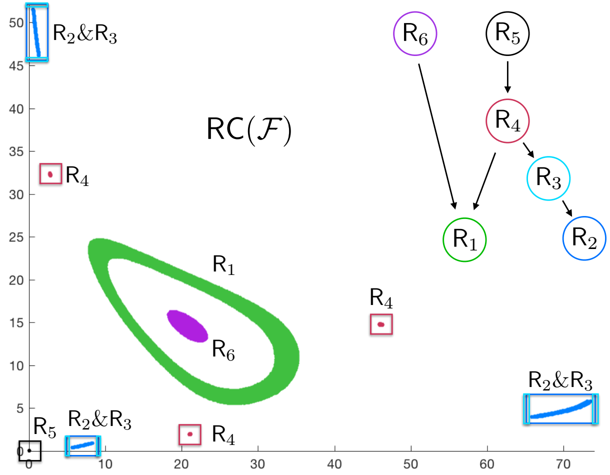


Figure: Poset structure of the recurrent components of an outer approximation of the Leslie map Each labeled region is a recurrent component.

An example: the Leslie model

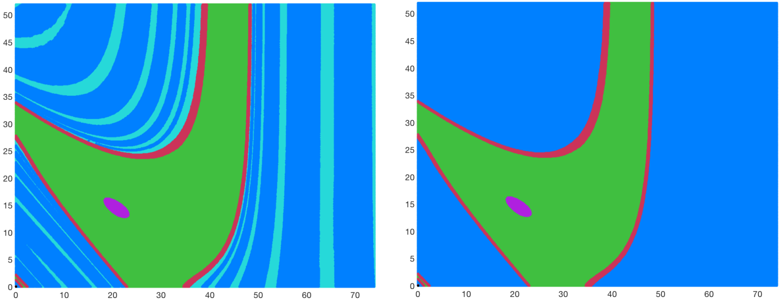


Figure: Morse tiles of tesselated Morse decompositions $\mathsf{M}(\mathsf{A}) \hookrightarrow \mathsf{RC}(\mathcal{F}) \leftrightarrow \mathsf{T}(|\mathsf{U}|)$ (left), and $\mathsf{M}(\mathsf{A}_{\rm II}) \leftrightarrow \mathsf{RC}_{\rm II}(\mathcal{F}) \leftrightarrow \mathsf{T}(|\mathsf{U}_{\rm II}|)$ where $\mathsf{RC}(\mathcal{F})$ is coarsened to $\mathsf{RC}_{\rm II}(\mathcal{F})$ by the retraction of R_3 onto R_2 (right).

The right picture is associated with an inclusion

$$\mathsf{RC}_{\rm II}(\mathcal{F}) \hookrightarrow \mathsf{RC}(\mathcal{F}) \hookrightarrow \mathsf{SC}(\mathcal{F}),$$

and the existence of an order retraction $\mathsf{SC}(\mathcal{F}) \twoheadrightarrow \mathsf{RC}_{\rm II}(\mathcal{F})$.

In general

Recall a combinatorial model:

$$
\begin{array}{ccccc}
\mathsf{K} & \xleftrightarrow{\;e\;} & \mathsf{N} & \xrightarrow{\;\subset\;} & \mathsf{ABlock}(\phi) \\
\Big\downarrow h & & \Big\downarrow \omega & & \Big\downarrow \omega \\
\mathsf{L} & \overset{c}{\twoheadrightarrow} & \mathsf{A} & \xrightarrow{\;\subset\;} & \mathsf{Att}(\phi)
\end{array}
$$

By the generalized Birkhoff theorems we can rephrase:

$$
\begin{array}{ccc}
\mathsf{Invset}^+(\mathcal{F}) & \xleftrightarrow{\;e\;} & \mathsf{N} \\
\Big\downarrow h & & \Big\downarrow \omega \\
\mathsf{Att}(\mathcal{F}) & \overset{c}{\twoheadrightarrow} & \mathsf{A}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{SC}(\mathcal{F}) & \xleftrightarrow{\;\delta\;} & \mathsf{T}(\mathsf{N}) \\
\Big\uparrow i \;\subset & & \Big\uparrow \pi \\
\mathsf{RC}(\mathcal{F}) & \leftarrow & \mathsf{M}(\mathsf{A})
\end{array}
$$

for some binary relation $(\mathcal{X}, \mathcal{F})$.

In general

- Example of combinatorial models (not commutative)
  - polygonal models by Boczko, Kalies, Mischaikow;
  - braid dynamics (Spendlove);
  - lectures Konstantin, regulatory networks.
- In the construction of $\mathcal{F}$ recurrence is built-in based on the dynamical information of the system — self-connections, etc.
- The example of outer approximations is special since a yield commutative combinatorial models. This is not the case in general.
- Binary relations yield weak outer approximations

$$\phi(\tau, |\xi|) \subset \operatorname{int} |\Gamma^+(\xi)|, \quad \tau > 0.$$

- How to proceed with a given combinatorial model?

Analysis

(i): Existence of an order retraction $\sigma\colon \mathsf{SC}(\mathcal{F}) \twoheadrightarrow \mathsf{RC}(\mathcal{F})$.

Equivalent to $\mathsf{O}(\sigma)\colon \mathsf{O}(\mathsf{RC}(\mathcal{F})) \rightarrowtail \mathsf{O}(\mathsf{SC}(\mathcal{F}))$, i.e. a lift $\mathsf{Att}(\mathcal{F}) \rightarrowtail \mathsf{Invset}^+(\mathcal{F})$. Denote the image of the latter by U. Then,

$$
\begin{array}{ccccc}
\mathsf{U} & \xleftrightarrow{\;e\;} & |\mathsf{U}| & \overset{\subset}{\rightarrowtail} & \mathsf{N} \\
\updownarrow{\scriptstyle \ell} & & \downarrow\!\!\downarrow{\scriptstyle \omega} & & \downarrow\!\!\downarrow{\scriptstyle \omega} \\
\mathsf{Att}(\mathcal{F}) & \overset{c}{\twoheadrightarrow} & \mathsf{A}_0 & \overset{\subset}{\rightarrowtail} & \mathsf{A}
\end{array}
$$

where $c = \omega \circ e \circ \ell$.

Dual

$$
\begin{array}{ccccc}
\widehat{\mathsf{SC}(\mathcal{F})} & \xleftrightarrow{\;\delta\;} & \mathsf{T}(|\mathsf{U}|) & \twoheadleftarrow & \mathsf{T}(\mathsf{N}) \\
\updownarrow & & \uparrow{\scriptstyle \pi} & & \uparrow \\
\mathsf{RC}(\mathcal{F}) & \hookleftarrow & \mathsf{M}(\mathsf{A}_0) & \twoheadleftarrow & \mathsf{M}(\mathsf{A})
\end{array}
$$

Analysis

(ii): Existence of an order retraction $\sigma\colon \mathsf{SC}(\mathcal{F}) \twoheadrightarrow \mathsf{RC}^\dagger(\mathcal{F})$, with $\mathsf{RC}(\mathcal{F}) \hookrightarrow \mathsf{RC}^\dagger(\mathcal{F}) \hookrightarrow \mathsf{SC}(\mathcal{F})$, or $\mathsf{RC}^\dagger(\mathcal{F}) \hookrightarrow \mathsf{RC}(\mathcal{F}) \hookrightarrow \mathsf{SC}(\mathcal{F})$.

$\mathsf{O}(\mathsf{RC}^\dagger(\mathcal{F})) = \mathsf{L} \rightarrowtail \mathsf{Invset}^+(\mathcal{F})$. Denote the image of the latter by U. Then,

$$
\begin{array}{ccccc}
\mathsf{U} & \overset{e}{\longleftrightarrow} & |\mathsf{U}| & \overset{\subset}{\rightarrowtail} & \mathsf{N} \\
{\scriptstyle \ell}\updownarrow & & \downarrow\!\!\downarrow{\scriptstyle \omega} & & \downarrow\!\!\downarrow{\scriptstyle \omega} \\
\mathsf{L} & \overset{c}{\twoheadrightarrow} & \mathsf{A}_0 & \overset{\subset}{\rightarrowtail} & \mathsf{A}
\end{array}
$$

Dual

$$
\begin{array}{ccccc}
\widehat{\mathsf{SC}(\mathcal{F})} & \overset{\delta}{\longleftrightarrow} & \mathsf{T}(|\mathsf{U}|) & \twoheadleftarrow & \mathsf{T}(\mathsf{N}) \\
\updownarrow & & \uparrow{\scriptstyle \pi} & & \uparrow \\
\mathsf{RC}^\dagger(\mathcal{F}) & \hookleftarrow & \mathsf{M}(\mathsf{A}_0) & \twoheadleftarrow & \mathsf{M}(\mathsf{A})
\end{array}
$$

Concluding remarks

- Finite sublattices play an important role. The set of (finite) sublattices of a bounded distributive lattice forms a complete lattice. Meet and join are defined on sublattices.
- The binary operations on sublattices can be dualized to binary operations on Morse decompositions and Morse representations: coarsening an refining.
- The above lattice structures allow limits: chain-recurrence.
- The above algebraic structures also play a role in the treatment of parameter dependence of systems.
- Role of Stone and Priestley spaces for infinite lattices.

Thanks for your attention!