On the Measure of the Feigenbaum Julia Set

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Będlewo, Poland

22 March 2018

Feigenbaum polynomial

The Feigenbaum polynomial is the quadratic polynomial

$$q_*(z) = z^2 - 1.4011551890\cdots \qquad \text{or, equivalently,}$$
$$f_*(z) \approx \; 1 - 1.4011551890z^2 \,,$$

which is the limit of the period-doubling cascade in the quadratic family.

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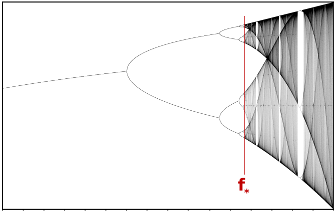

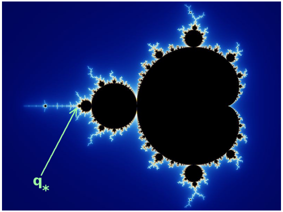

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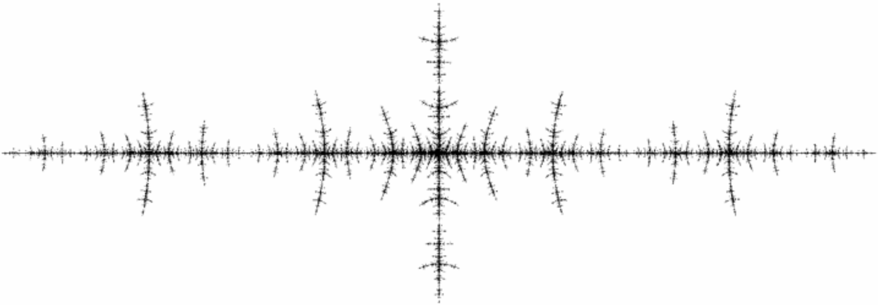
Feigenbaum Julia set $J(f_*)$



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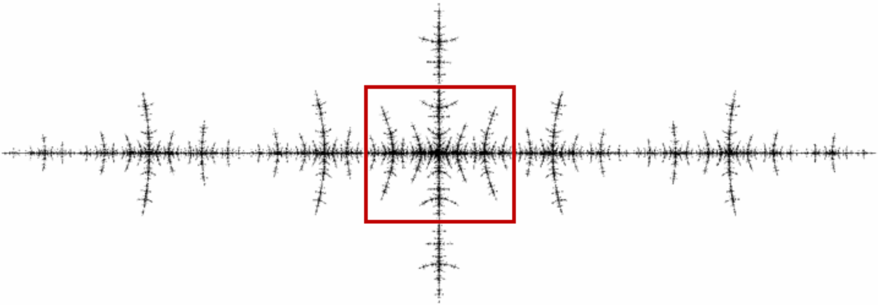
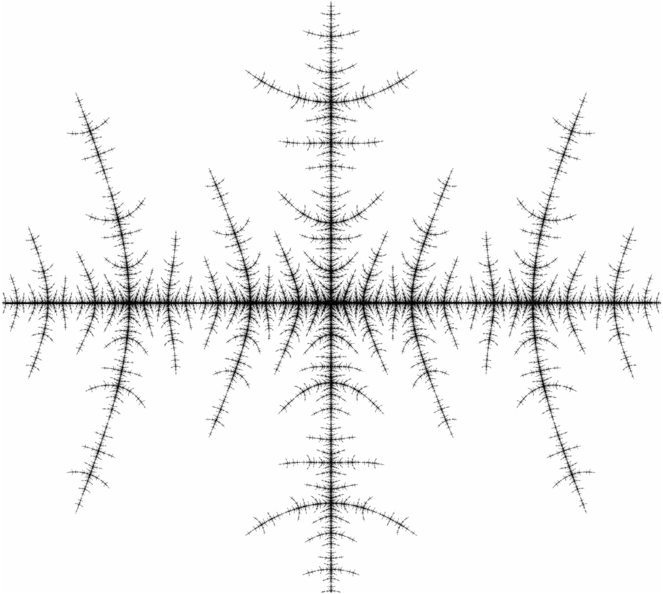


Feigenbaum Julia set $J(f_*)$, zoomed



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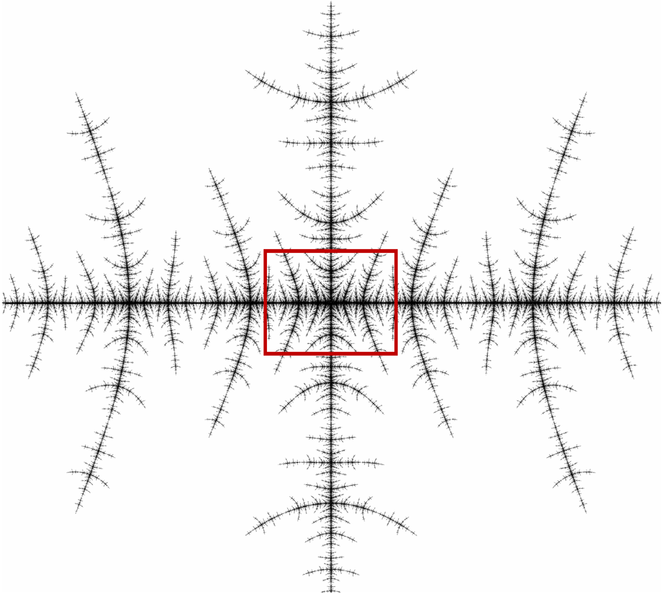
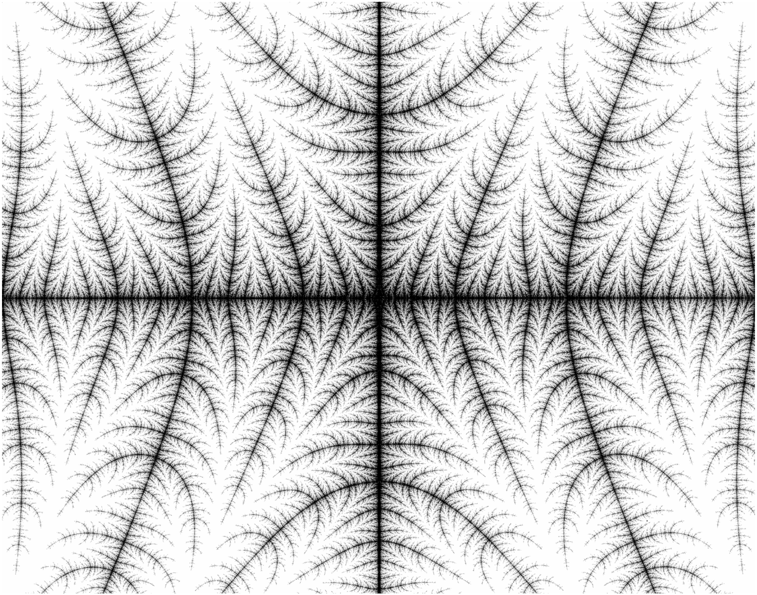


Feigenbaum Julia set $J(f_*)$, zoomed again



Positive or zero measure?

Successive zooms lead to a Julia set which grows more and more hairs.
(Similarly, the Mandelbrot set gains more decorations while limiting on the Feigenbaum point.)

This leads to the natural question:

> **Does the Julia set of the Feigenbaum quadratic polynomial have positive or zero measure?**

If zero, is its Hausdorff dimension less than 2?.

The **Sullivan dictionary** of analogies between Kleinian groups and rational maps also indicates that $HD(J_{f_*}) = 2$ should hold.

McMullen suggested a correspondence between maps of Feigenbaum type and hyperbolic 3-manifolds which fiber over the circle, and then proved that the critical point lies "deep" inside the Julia set. This suggests that these should have the maximal Hausdorff dimension.

Levin and Świątek have shown that the Julia sets for the corresponding maps of criticality d have Hausdorff dimension which tends to 2 as $d \to \infty$, while the Lebesgue measure tends to zero.

The question of measure and dimension of J_{f_*} has been open for some time.

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Quadratic-like maps

A **quadratic-like map** $q : U \to V$ is a ramified covering of degree 2 between two topological disks U and V, with U relatively compact in V.

The **Julia set J_q** and the corresponding **filled Julia set K_q** are defined as

$$K_q = \{\, z \in U \mid q^n(z) \in U \text{ for all } n \in \mathbb{N} \,\},$$
$$J_q = \partial K_q$$

Above is the Julia set for $f(z) = z^2 - 1.75486\cdots$, the "airplane".
The map $f^3 \colon U \to V$ is quadratic-like, and $f^3|_U$ is conjugate to $z \mapsto z^2$.

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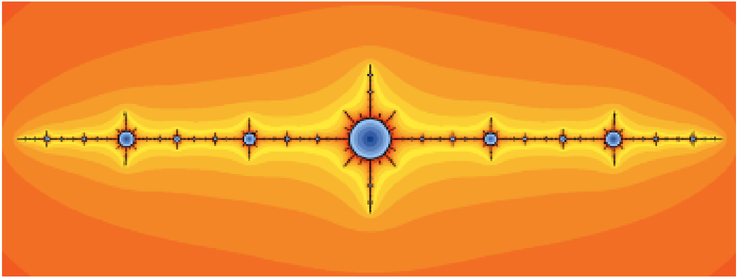

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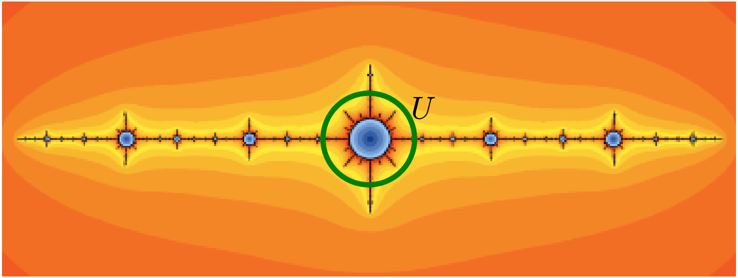

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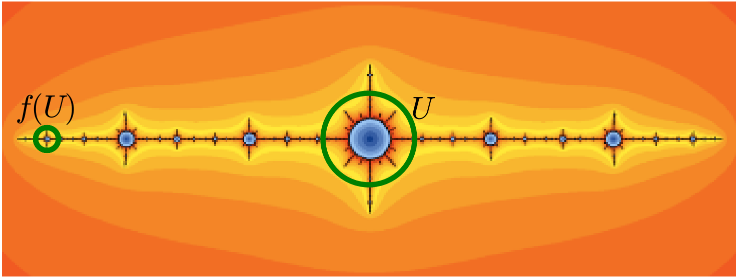

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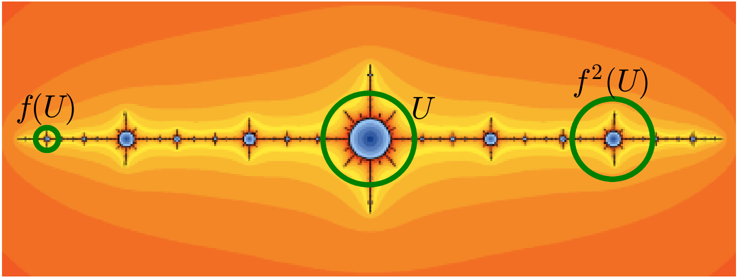

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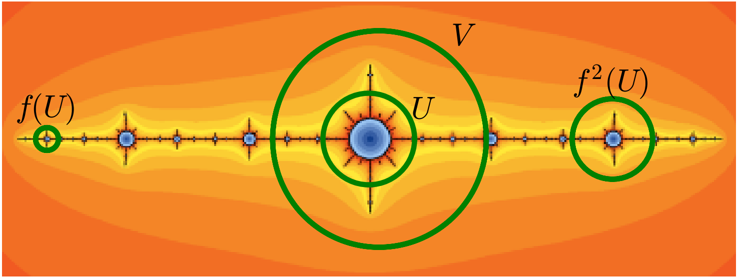

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Renormalization

Let $f\colon U \to V$ be a quadratic-like map with connected Julia set, and let c be its critical point. Then f is **renormalizable with period n** if there is an integer $n > 1$ and open disks $U' \Subset V'$ containing c such that

1. the map $f^n\colon U' \to V'$ is a quadratic-like map,
2. its Julia set J_0 is connected, and
3. the small Julia sets $J_i = f^i(J_0)$, with $i = 1, \ldots, n-1$, are either disjoint from J_0 or intersect it only at its β-fixed point.

The quadratic-like map $f^n\colon U' \to V'$ is called a **pre-renormalization of f**.

A pre-renormalization taken up to affine conjugacy is a **renormalization of f**. The renormalization with minimal possible period n is denoted $\mathcal{R}f$.

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Renormalization Fixed Point

The conjecture of **Coullet-Tresser-Feigenbaum** was that successive renormalizations of maps with a periodic critical orbit of period $2, 4, 8, 16, \ldots$ would converge to a function F which is a **fixed point of renormalization**, i.e.

$$F = \mathcal{R}F.$$

That is, there should be an analytic F such that the **Cvitanović-Feigenbaum functional equation** holds:

$$\begin{cases} F(0) & = 1 \\ \lambda & = -F(1) \\ \mathbf{F(z)} & = -\frac{1}{\lambda}\mathbf{F^2}(\lambda \mathbf{z}) \\ F(z) & = H(z^2), \text{with } H^{-1}(z) \text{ univalent on the unit disk} \end{cases}$$

The above relations imply nice scaling properties, such as

$$F^{2^m}(z) = (-\lambda)^m F(z/\lambda^m)$$

whenever both sides are defined.

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An approximation of F

Computation gives

$$F(z) \approx 1 - 1.5276z^2 + 0.1048z^4 + 0.0267z^6 - 0.00352z^8 + 0.00008z^{10} + \dots$$
$$\lambda \approx .39953528$$

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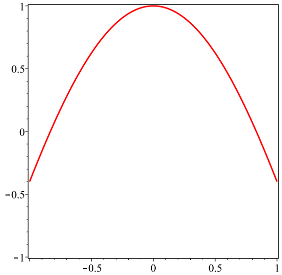

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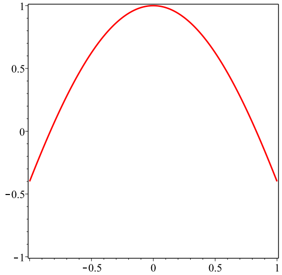

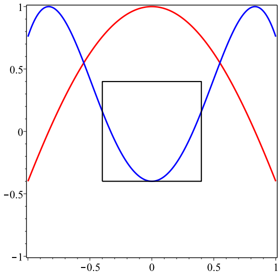

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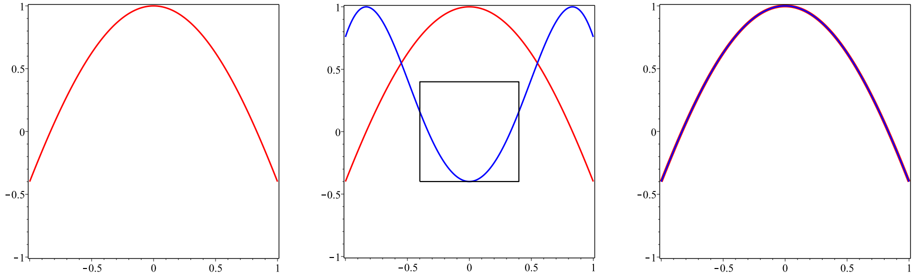

Proof that the Renormalization Fixed Point Exists

The existence of an analytic map F with $F = \mathcal{R}F$ was established independently in the early 1980s by **Lanford** and **Campanino, H. Epstein, and Ruelle**; Lanford's proof also showed that F was a hyperbolic fixed point of $\mathcal{R}$ with one expanding direction and used computer-assisted methods.

Theorem (Lanford '82; Campanino/Epstein/Ruelle '82)

- *There exists an even function F, analytic on $|z| < \sqrt{2}$, for which $F = \mathcal{R}F$ on $|z| \leqslant 1$.*
- *$\mathcal{R}$ is a smooth mapping in a neighborhood of F containing f_*; the derivative $D\mathcal{R}|_F$ is hyperbolic with one expanding direction.*
- *The expanding eigenvalue of $D\mathcal{R}|_F$ is approximately* 4.669.

Developing the above result without computer assistance (and in greater generality) continued for another 20 years with the work of **Sullivan**, **McMullen**, **Lyubich** and others.

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Computation of F

To obtain an approximation of F, we can begin with an approximation of the quadratic Feigenbaum map $f_*(z)$.

Attempting to compute F by successive renormalizations using

$$F(z) \approx (-\lambda)^m \, f_*^{2^m}(z/\lambda^m)$$

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Computation of F (continued)

Instead, Lanford's method used a modified version of Newton's method to solve $\mathcal{R}\mathbf{F} = \mathbf{F}$. First, he began with the initial estimate f_* and renormalized a few times to get a higher degree initial approximation to F.

Then, rather than using Newton's method

$$N_{\mathcal{R}}(f) = f - \big(D\mathcal{R}(f) - \mathbb{1}\big)^{-1}\big(\mathcal{R}f - f\big),$$

Lanford used

$$M(f) = f - A\big(\mathcal{R}f - f\big), \quad \text{where} \quad A = \begin{pmatrix} 1/3.669 & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix}.$$

(with f in suitable coordinates).

For f in appropriate neighborhood, M is a contraction with the same fixed point as $\mathcal{R}$. Estimates on the on the errors of succesive approximation can be obtained (and are given explicitly by Lanford for his degree 80 approximation).

One can use a basis of Chebyshev polynomials rather than the standard polynomial basis, which gives nicer truncation properties. The matrix A isn't as nice in these coordinates, but the idea is the same.

Or one can just use Lanford's polynomial, which is good to about 10^{-22} for $|z^2| < 1.5$.

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Is $HD(J) < 2$?

Define a **Feigenbaum polynomial** to be any infinitely renormalizable map of bounded combinatorial type with *a priori* bounds. That is, the periods n of renormalization are bounded, and at each renormalization, the modulus of the annuli $V \setminus U$ is bounded.

Avila and Lyubich showed in 2008 that there are Feigenbaum polynomials with $HD(J) < 2$. Indeed, the Hausdorff dimension can be arbitrarily close to 1.

There are three possibilities

Lean: $HD(J) < 2$

Balanced: $HD(J) = 2$ but $area(J) = 0$

Black Hole: $area(J) > 0$

In 2012, they gave a proof that all three cases occur for Feigenbaum polynomials.

But the question for period-doubling combinatorics in degree 2 remained open.

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Balanced: $HD(J) = 2$ but $area(J) = 0$

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In 2012, they gave a proof that all three cases occur for Feigenbaum polynomials.

But the question for period-doubling combinatorics in degree 2 remained open.

Is $HD(J) < 2$?

Define a **Feigenbaum polynomial** to be any infinitely renormalizable map of bounded combinatorial type with *a priori* bounds. That is, the periods n of renormalization are bounded, and at each renormalization, the modulus of the annuli $V \setminus U$ is bounded.

Avila and Lyubich showed in 2008 that there are Feigenbaum polynomials with $HD(J) < 2$. Indeed, the Hausdorff dimension can be arbitrarily close to 1.

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Avila-Lyubich Trichotomy

Avila and Lyubich introduced **nice domains** $U^n \subset V^n$ for which

- $f_n(U^n) = V^n$
- $J_n \cap \mathcal{O}(f) \subset U^n$
- $V^{n+1} \subset U^n$
- $f^k(\partial V^n) \cap V^n = \emptyset$ for all n and k
- $A^n = V^n \setminus U^n$ is "far" from $\mathcal{O}(f)$
- $\text{area}(A^n) \asymp \text{area}(U^n) \asymp \text{diam}(U^n)^2 \asymp \text{diam}(V^n)^2$

$X_n = \{ z \in U^0 \mid f^k(z) \in V^n \text{ for some } k \}$ $\quad \eta_n = \text{area}(X_n)/\text{area}(U^0)$

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Theorem (Avila-Lyubich 2008)

Let f be a Feigenbaum map which is a periodic point of renormalization, i.e. there is a p so that $\mathcal{R}^p f = f$. Then **exactly one** *of the following is true:*

- Lean case: η_n *converges to* 0 *exponentially fast,* $\inf \xi_n > 0$*, and* $HD(J_f) < 2$;
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The proof shows that if there is a constant C so that for some m divisible by p, we have $\eta_m < \xi_m / C$, then $\eta_n \to 0$ exponentially fast and thus f is in the lean case.

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Nice domains aren't so nice

The "nice domains" of Avila-Lyubich are quite challenging to work with computationally.

- The sets U^n and V^n are defined by cutting neighborhoods of zero by equipotentials and external rays of f_n, and taking preimages under long chains of iterates of f_n. Consequently, it is difficult to obtain rigorous approximations of them.
- The geometry of U^n and V^n is complicated, and U^n is not compactly contained in V^n; the resulting geometric bounds become very rough.
- The definition of the constant C with $\eta_m < \xi_m / C$ is given implicitly; estimates show that it can be on the order of 10^{10}.

Instead, we work with an alternative set of domains on which F is quadratic-like, and which can be easily approximated with good geometric bounds. These are defined via **Buff tiles**, which have a scale-invariant structure, enabling us to construct explicit recursive estimates for a quantity analagous to η_n directly, without relying on the Poincaré series.

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As noted earlier, it was shown in the 1980s that F (the fixed point of period-doubling renormalization) is analytic and defined on the disk of radius $\sqrt{2}$. But more can be said.

Theorem (H. Epstein 1999)

The map F has a maximal analytic extension $\widehat{F}\colon \widehat{W} \to \mathbb{C}$, where $\widehat{W}$ is an open, simply connected set which is dense in $\mathbb{C}$.

Furthermore, $\widehat{F}(\widehat{W}) = \mathbb{C} \setminus \left((-\infty, -\frac{1}{\lambda}] \cup [\frac{1}{\lambda^2}, \infty)\right)$

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All critical points of $\widehat{F}$ are simple, and the critical values of $\widehat{F}$ are all real. Moreover, for any $z \in \widehat{W}$ with $\widehat{F}(z) \notin \mathbb{R}$, there exists a bounded open set P_z containing z such that $\widehat{F}$ is one-to-one on P_z, and $\widehat{F}(P_z)$ is either $\mathbb{H}_+$ or $\mathbb{H}_-$.

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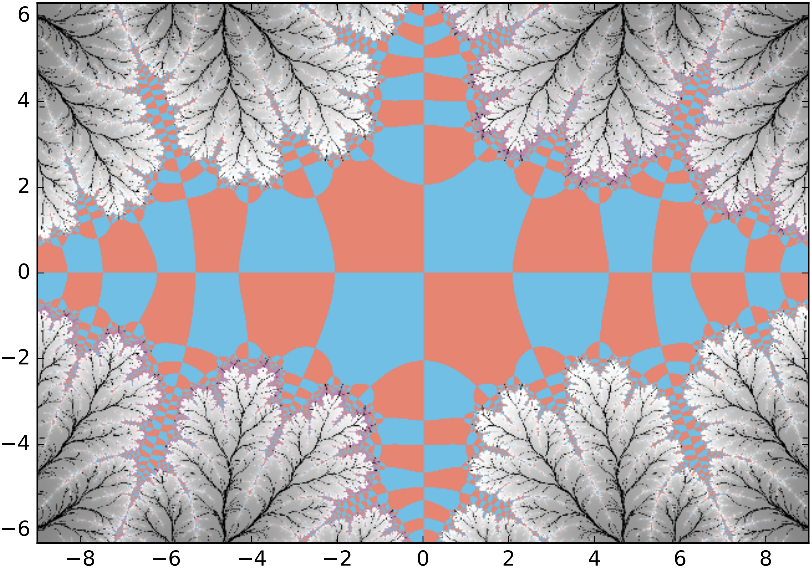
The domain of analyticity $\widehat{W}$ for $\widehat{F}$



Blue is the preimage of the upper half plane, red the preimage of the lower half plane. Shades of gray contain points of $\widehat{W}$ and its complement.

Buff tiles

Let $\mathcal{P}$ denote the set of connected components of $\widehat{F}^{-1}(\mathbb{C} \setminus \mathbb{R})$.

Let $\mathcal{P}^{(n)} = \{ \lambda^n P \mid P \in \mathcal{P} \}$ for $n \geqslant 0$.

We shall use the name **tiles** for the connected components of $F^{-k}(\mathbb{H}_\pm)$, as well as the half-planes $\mathbb{H}_+$ and $\mathbb{H}_-$.

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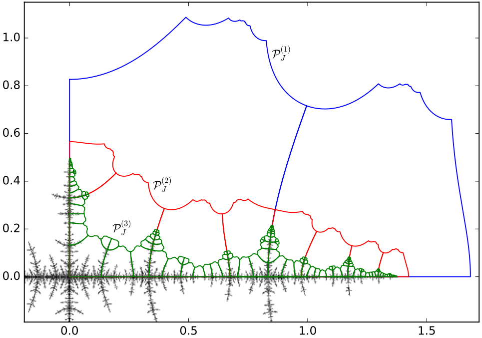

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Nesting Property

For any $P \in \mathcal{P}$, the map $\widehat{F}$ sends P one-to-one either onto $\mathbb{H}_+$ or onto $\mathbb{H}_-$. Notice that $\mathcal{P}$ has four-fold symmetry: it is invariant under multiplication by -1 and under complex conjugation.

Using the result Epstein/Buff together with the Cvitanović-Feigenbaum equation, we obtain the following.

Lemma (The Nesting Property)

Two tiles are either disjoint or one is a subset of the other.

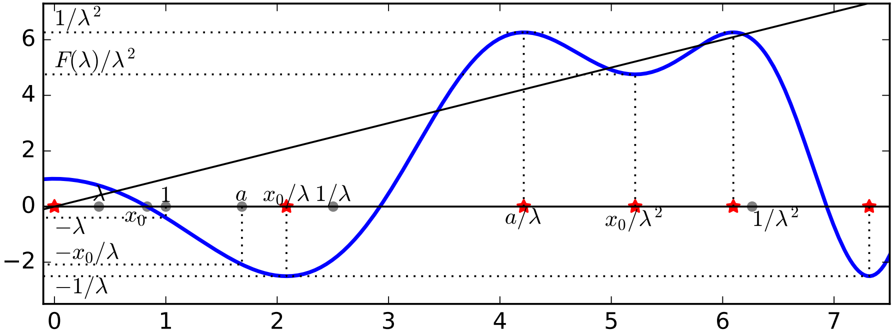
Furthermore, for any tile $P \in \mathcal{P}^{(n)}$ and any $0 \leqslant m < n$, the map $\widehat{F}^{2^n - 2^m}$ sends P bijectively onto some tile $Q \in \mathcal{P}^{(m)}$.

$\widehat{F} : \mathbb{R} \to \mathbb{R}$

Let $\boldsymbol{x}_0$ be the smallest positive solution to $\widehat{F}(x_0) = 0$.
Then $\widehat{F}(\lambda x_0) = x_0$ and x_0/λ is the first positive critical point of $\widehat{F}$.

Let $\boldsymbol{a}$ be the solution to $\widehat{F}(a) = -x_0/\lambda$ with $1 \leqslant a \leqslant x_0/\lambda$.

The first three positive critical points of $\widehat{F}$ are $\dfrac{x_0}{\lambda}, \quad \dfrac{a}{\lambda}, \quad \dfrac{x_0}{\lambda^2}$.



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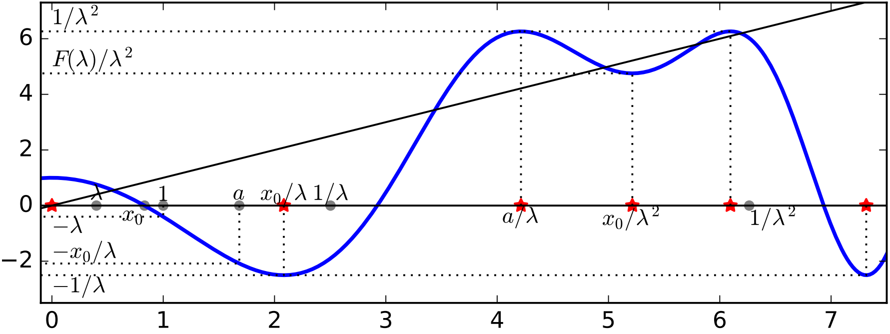
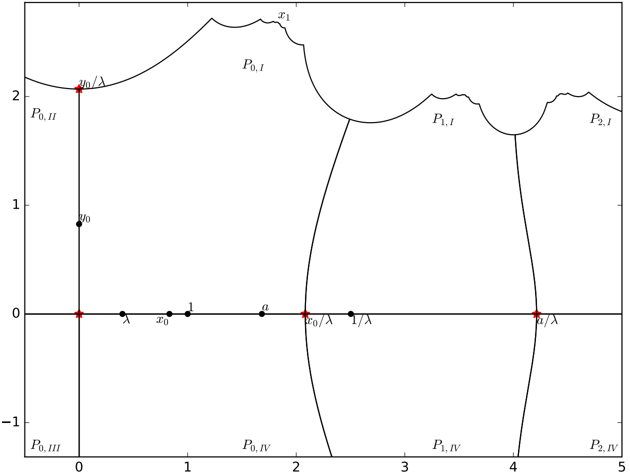


On the naming of pieces

Let c_j be the non-negative real critical points of $\widehat{F}$, with $0 = c_0 < c_1 < c_2 < \dots$, and let $\boldsymbol{P_{j,I}}$ be the tile of $\mathcal{P}$ in the first quadrant with $[c_j, c_{j+1}]$ in its boundary; let $\boldsymbol{P_{j,K}}$ be the symmetric tile in quadrant K, with $K \in \{\, I, II, III, IV \,\}$.



For any set P and any integer $n \geqslant 0$, let $\boldsymbol{P^{(n)} = \lambda^n P}$.

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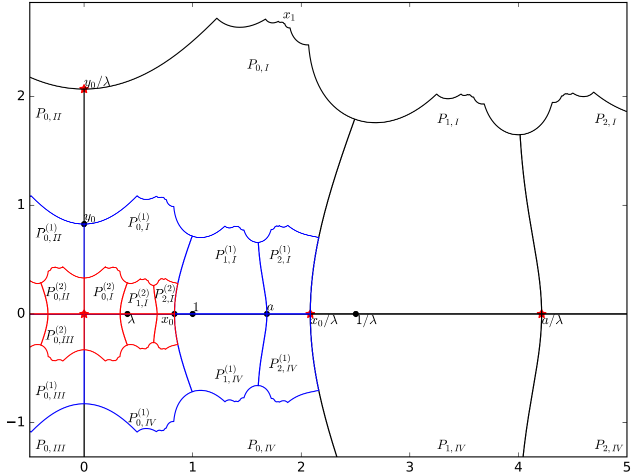

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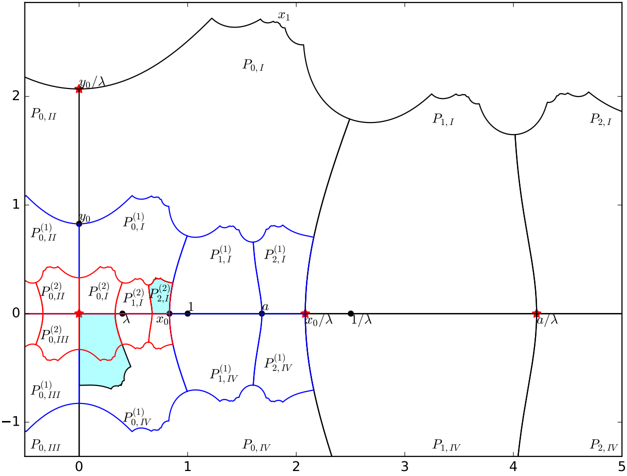

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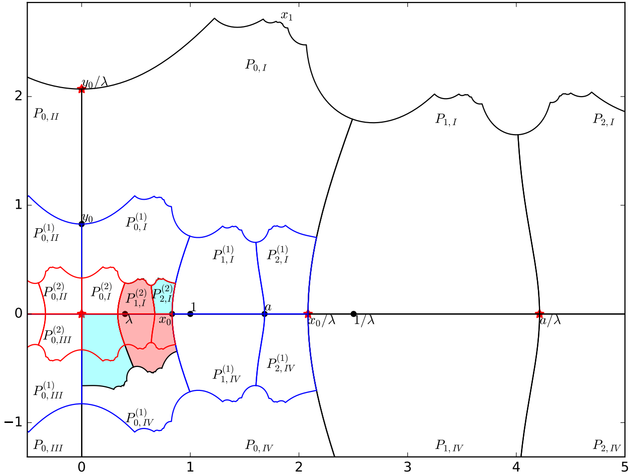

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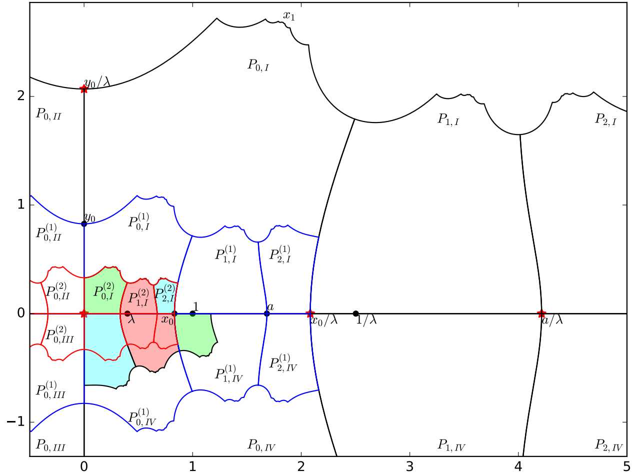

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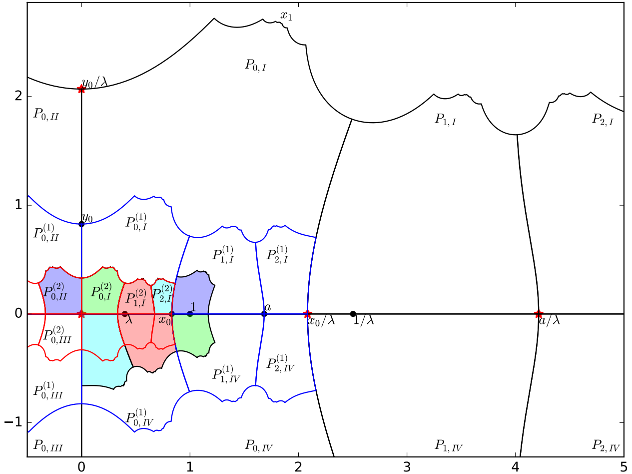

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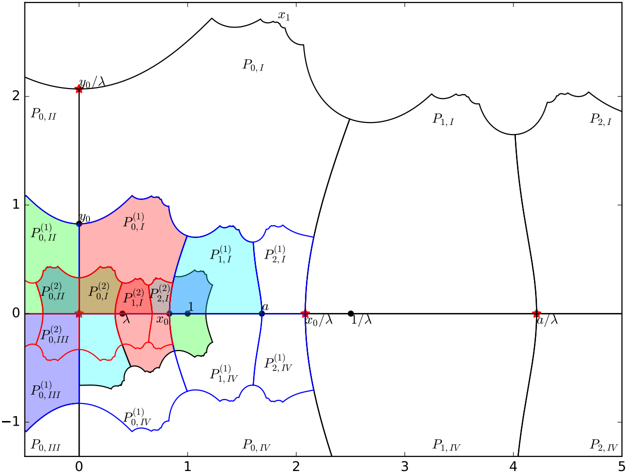

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F as a quadratic-like map

Let $\boldsymbol{W} = \operatorname{Int}\left(\overline{P_{0,I} \cup P_{0,II} \cup P_{0,III} \cup P_{0,IV}}\right)$
and let $\boldsymbol{F}$ denote $\widehat{F}$ restricted to W.

Then $F : W \to \mathbb{C} \smallsetminus \left((-\infty, -\frac{1}{\lambda}] \cup (\frac{1}{\lambda^2}, \infty)\right)$ is a quadratic-like map.

If we have some finite orbit $z_0, z_1 = F(z_0), \ldots, z_k = F(z_{k-1})$ for which z_k lies in the closure of some tile T and $DF^k(z_0) \neq 0$, then we can pull back T univalently along the orbit in a unique way.

In particular, this holds as long as the orbit remains in W and $z_i \neq 0$ for $i = 0, \ldots, k$.

Note that for all $n \geqslant 1$ and $1 \leqslant j \leqslant 2^{n-1}$, $F^j(W^{(n)})$ is disjoint from $W^{(n)}$.

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Copies of tiles

A tile Q will be called a **copy of the tile P under F^k** if there is a non-negative integer k so that $F^k(Q) = P$.

A copy Q of ${P_0}^{(m)}$ under F^k is called **primitive** if $F^j(Q)$ is disjoint from $W^{(m)}$ for all $0 \leqslant j < k$.

A copy T of ${P_0}^{(m)}$ under F^k is called **separated** if for some $0 \leqslant j < k$, $F^j(P) \subset W^{(m)}$ and $F^j(P)$ is disjoint from ${J_F}^{(m-1)}$ for the maximal such j.

1. If Q is a copy of a separated copy T, then Q is separated.
2. Let T be a separated copy of ${P_0}^{(m)}$ under F^k. Then for each $j \leq k$, $F^j(T)$ is either a primitive or a separated copy of ${P_0}^{(m)}$.
   In particular, the set ${P_0}^{(m)}$ is a primitive copy of itself.
3. Copies of primitive copies need not be primitive or separated.
   For example, let $T = F({P_{0,I}}^{(2)})$. $F^2(T) = {P_{0,II}}^{(1)}$ and $T \subset {P_{1,IV}}^{(1)} \not\subset W^{(1)}$; T is a primitive copy of ${P_{0,II}}^{(1)}$.
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Computational scheme

Our goal is to prove that the Hausdorff dimension of J_F is less than 2, (that is, that F **is lean**, via rigorous computations.

Rather than using the nice domains of Avila-Lyubich, we can use the partitions $\mathcal{P}^{(n)}$ consisting of the Buff tiles to estimate $\widetilde{\eta}_n$, the analogue of the landing parameter η_n.

If $\widetilde{\eta}_n \to 0$ exponentially, then $\eta_n \to 0$ exponentially as well.

Recall that $W^{(n)}$ is the four central tiles of the partition $\mathcal{P}^{(n)}$ at level n, filled in to form an open disk.

In general for $n \geqslant 1$, we have $F_n \colon W^{(n)} \to W^{(0)}$.

Define the following

$$\boldsymbol{\widetilde{X}_{n,m}} = \left\{ z \in W^{(n)} \;\middle|\; F_{n-1}^k \in W^{(n+m)} \text{ for some } k \geqslant 0 \right\}$$

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Distortion Bounds

Let $\mathbb{C}_\lambda = \mathbb{C} \setminus \left((-\infty, -\frac{1}{\lambda}] \cup [\frac{F(\lambda)}{\lambda^2}, \infty)\right)$ and let $\phi : \mathbb{C}_\lambda \to \mathbb{C}$ be univalent. Then the **Koebe Distortion Theorem** tells us that

$$C(z,w) = \sup \left\{ \frac{|\phi'(z)|}{|\phi'(w)|} \;\middle|\; \phi : \mathbb{C}_\lambda \to \mathbb{C} \text{ is univalent} \right\}$$

is nonzero and finite for all z and w, where the bounds depend on the distances of z and w to the slits.

Integrating this gives us the following

Corollary

Let A, B be two measurable subsets of P_0 of positive measure and let T be a primitive or a separated copy of ${P_0}^{(m)}$ under F^k for some $k \geq 0$ and $m \geqslant 2$. Then

$$\frac{\text{area}(F^{-k}(B^{(m)}) \cap T)}{\text{area}(F^{-k}(A^{(m)}) \cap T)} \leqslant M(A)\,\text{area}(B).$$

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Let $\boldsymbol{M_{n,m}} = M\big((\lambda^{-n}\Sigma_{n,m}) \cap P_{0,I}\big)$.

Theorem

For every $n \geqslant 2$ and $m \geqslant 1$, one has

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Consequently, the Hausdorff dimension of J_F is strictly less than 2, (and has measure zero).

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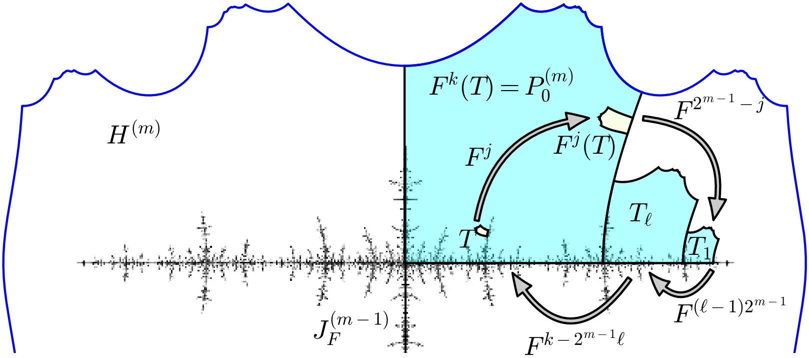
Separated copies avoid the little Julia set

Lemma

Let T be a separated copy of ${P_0}^{(m)}$. Then $T \cap {J_F}^{(m-1)} = \emptyset$.



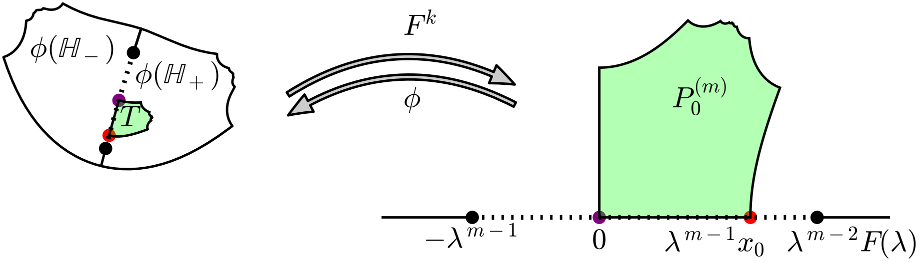
The above cannot occur.

Separated copies have Koebe space

Lemma

Let T be a separated or primitive copy of ${P_0}^{(m)}$ with $m \geqslant 2$ and $F^k(T) = {P_0}^{(m)}$. Then the inverse branch $\phi : {P_0}^{(m)} \to T$ of F^k analytically continues to a univalent map on $\operatorname{sign}({P_0}^{(m)})\, \lambda^m \mathbb{C}_\lambda$*, where*

$$\mathbb{C}_\lambda = \mathbb{C} \setminus \left((-\infty, -\tfrac{1}{\lambda}] \cup [\tfrac{F(\lambda)}{\lambda^2}, \infty)\right).$$



$H^{(1)}$, V_2, and $\widetilde{W}_n$

Let $\boldsymbol{H} = \mathrm{Int}\left(\overline{W \cup P_{1,I} \cup P_{1,II} \cup P_{1,III} \cup P_{1,IV}}\right)$
Let $\boldsymbol{V_2}$ be the interior of $\bigcup F^{-3}(\overline{W})$; observe $J_F \subset V_2 \subset H^{(1)} \subset W$.

For $n \geqslant 3$, let $\widetilde{\boldsymbol{W}}_{\boldsymbol{n}}$ be the interior of the closure of the copies P of $\mathbb{H}_\pm$ under F^{2^n-6} with $0 \in \overline{P}$.

For all $n \geqslant 3$, $F^{2^n-6}(\partial \widetilde{W}_n) = \mathbb{R}$ $\qquad$ $\widetilde{W}_3 = W^{(1)}$

For all $n \geqslant 3$, $\widetilde{W}_{n+1} \subset \lambda \widetilde{W}_n$ $\qquad$ For all $n \geqslant 4$, $W^{(n)} \subset \widetilde{W}_n \subset W^{(n-1)}$

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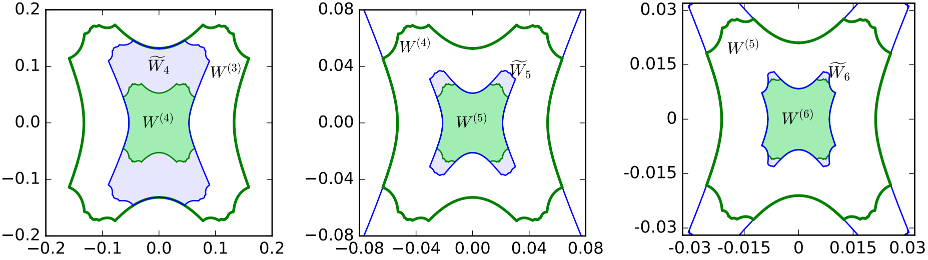

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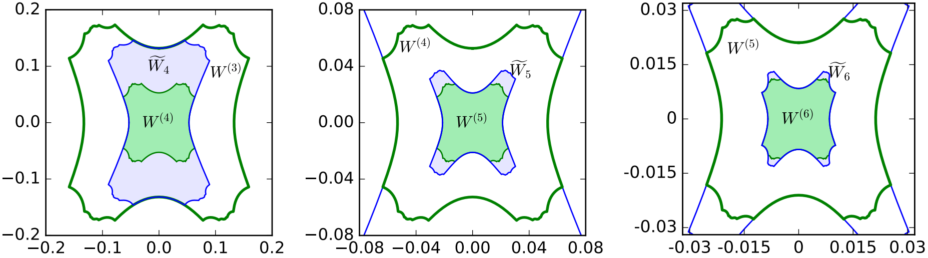

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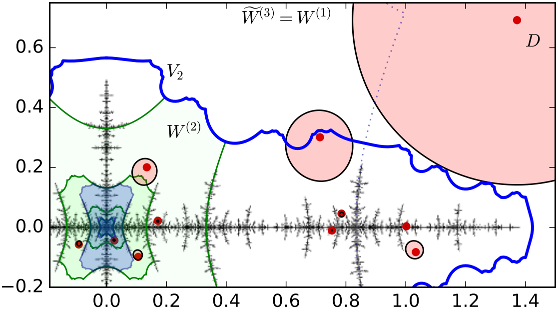
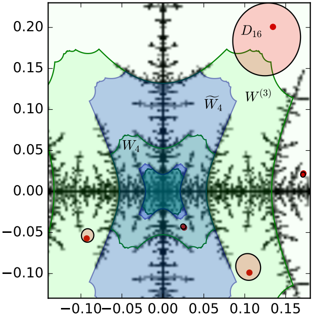
Escaping disks

Lemma

Let D be a disk in the complement of $(-\infty, -\frac{1}{\lambda}] \cup V_2 \cup [\frac{1}{\lambda^2}, \infty)$, and let D_0 be a connected component of $F^{-k}(D)$ for any $k \geqslant 0$.

Then for $n \geqslant 3$, either $\quad D_0 \cap W^{(n)} = \emptyset \quad$ *or* $\quad D_0 \subset \widetilde{W}_n$.





Corollary

Let $z \in H^{(1)} \setminus J_F$ and let k be such that $F^k(z) \notin V_2$. Suppose that $F^j(z) \notin \widetilde{W}_n$ for $0 \leqslant j \leqslant k$.
Let D_0 be the connected component of $F^{-k}\big(\mathbb{D}_R(F^k(z))\big)$ that contains z, where $R = \mathrm{dist}(F^k(z), V_2)$. Then $D_0 \cap X_n = \emptyset$. In particular,

$$\mathbb{D}_r(z) \cap X_n = \emptyset, \ \text{ where } \ r = \frac{R}{4|DF^k(z)|}.$$

Let $\boldsymbol{Y_n}$ be all $z \in W^{(n)}$ with $F^k(z) \notin W^{(n)}$ for $k \geqslant 1$, and let $\boldsymbol{\Sigma_n} = \bigcup_{m>0} \Sigma_{n,m}$

Corollary

Let $z \in W^{(n)}$ be such that $w = F^s_{n-1}(z) \in Y_n$ for some s, with $n \geqslant 3$. Let ℓ be such that $F^\ell(w) \notin V_2$. Suppose also that $F^j(w) \notin \widetilde{W}_n$ for all $0 \leqslant j \leqslant \ell$.
Set $R = \mathrm{dist}(F^\ell(w), V_2)$ and let D_0 be the connected component of $F^{-s}_{n-1}\Big(F^{-\ell}\left(\mathbb{D}_R(F^\ell(w))\right)\Big)$ that contains z. Then $D_0 \subset \Sigma_n$. In particular,

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Estimating X_n and Σ_n

The previous two corollaries give us explicit and verifiable conditions to calculate explicit, rigorous bounds on the measure of the sets X_n and $Y_n \subset \Sigma_n$.

Using Lanford's degree 80 polynomial F_0, we have explicit bounds on the errors in the approximation. For $|z^2| < 1.5$, we have

$$|F(z) - F_0(z)| < 1.5 \times 10^{-23} \quad \text{and} \quad |F'(z) - F_0'(z)| < 1.5 \times 10^{-22}.$$

We can accelerate long chains of iterates using the fact that

$$F^{2^m}(z) = (-\lambda)^m F(\tfrac{z}{\lambda^m}) \quad \text{for } z \in W^{(n)}.$$

Not only does this speed calculations, the factor of λ^m helps control numerical errors explicitly.

Using the Koebe estimates and the previous corollaries, we obtain an outer cover of X_6 and an explicit subset of Σ_6. Hence we have rigorous upper bounds for $\widetilde{\eta}_6$, $M_{6,6}$, and $|P_{0,I}|$. These are constructed in such a way that they hold for all approximations F_0 sufficiently close to F.

The computations show that $\widetilde{\eta}_6 \; M_{6,6} \; |P_{0,I}| < .846$, so $HD(J_F) < 2$, and the **Lebesgue measure of J_F is zero**.

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$$F^{2^m}(z) = (-\lambda)^m F(\tfrac{z}{\lambda^m}) \quad \text{for } z \in W^{(n)}.$$

Not only does this speed calculations, the factor of λ^m helps control numerical errors explicitly.

Using the Koebe estimates and the previous corollaries, we obtain an outer cover of X_6 and an explicit subset of Σ_6. Hence we have rigorous upper bounds for $\widetilde{\eta}_6$, $M_{6,6}$, and $|P_{0,\mathcal{I}}|$. These are constructed in such a way that they hold for all approximations F_0 sufficiently close to F.

The computations show that $\widetilde{\eta}_6 \; M_{6,6} \, |P_{0,I}| < .846$, so $HD(J_F) < 2$, and the **Lebesgue measure of J_F is zero**.

Error bounds on numerical approximations

If $\check{z}$ is a double-precision approximation of z and $\check{F}$ is Lanford's polynomial approximation of F evaluated in double-precision, then we have the following upper bounds on the error.

$\lvert F(z) - \check{F}(\check{z})\rvert$	
5.2884×10^{-14}	for $\lvert z\rvert < 1$
6.4430×10^{-13}	for $\lvert z\rvert < 1.414$
5.0001×10^{-7}	for $\lvert z\rvert < 2.449$
1.7001×10^{-2}	for $\lvert z\rvert < 2.828$

$\lvert F'(z) - \check{F}'(\check{z})\rvert$	
7.2771×10^{-14}	for $\lvert z\rvert < 1$
5.5875×10^{-12}	for $\lvert z\rvert < 1.31$
5.0001×10^{-5}	for $\lvert z\rvert < 2.34$
1.7001×10^{-1}	for $\lvert z\rvert < 2.72$

Observe that for $|z| < 1$, the error in using $\check{F}(\check{z})$ is dominated by the accumulated round-offs (since F is approximated by $\check{F}$ to better than machine precision for $|z| < \sqrt{6}$); for $|z| > \sqrt{2}$, the error is dominated by the approximation of F by $\check{F}$.

Composition errors

In calculating an approximation of the orbit of a point z we keep a running bound on the accumulated total difference between the true orbit $F^j(z)$ and the aproximation $\check{F}^j(\check{z})$, as well as the corresponding derivatives.

We can compute an approximation of F^j as compositions of $\check{F}^{2^n}$, each of which can be computed with good accuracy by using the fact that $F^{2^n}(z) = (-\lambda)^n F(\frac{z}{\lambda^n})$ and powers of λ and $1/\lambda$ can be readily calculated to more than 20 decimal digits.

In particular, if $\check{g}_k$ is a k-fold composition of such approximations and g_k is the same composition of Feigenbaum maps F^{2^n}, we have the following worst-case bounds on the accumulated errors in the approximations of the orbit and the derivative for $\check{z}_k \in W^{(1)}$.

k	$\lvert g_k(z) - \check{g}_k(\check{z})\rvert$	$\lvert g'_k(z) - \check{g}'_k(\check{z})\rvert$
1	6.45×10^{-13}	5.59×10^{-12}
5	2.15×10^{-10}	4.45×10^{-9}
10	2.14×10^{-7}	1.43×10^{-5}
15	2.13×10^{-4}	4.57×10^{-2}
18	1.20×10^{-2}	15.14

Thank you!

And now, a brief commercial message

Analytic Low-Dimensional Dynamics: a celebration of Misha Lyubich's 60th birthday

will take place at the **Fields Institute** on

May 27 – June 7, 2019

Mark your calendar and plan to join us in Toronto on this festive occasion!

Organizers:
Anna Miriam Benini, Tanya Firsova, Peter Makienko,
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