Emergent space time from Path Dependent Observables

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Introduction

There exists a renewed interest in the description in terms of observables of gauge theories and gravity.

Recently, Giddings and Donnelly proposed explicit constructions that extend the observables of gauge theories in order to include diffeomorphism invariance in gravitation with weak fields.
They noted that an important feature of the resulting formulation of gravity is that the algebra of observables becomes non-local.

The most ambitious attempt to describe gravity intrinsically without coordinates was proposed by Mandelstam in the 1960's.

In this approach, points in space-time are not specified by coordinates but by paths stemming from a reference point. The paths are constructed intrinsically, without setting a coordinate system in the manifold.

All physical variables become path-dependent and are Dirac observables, they are invariant under space-time diffeomorphisms.

This could be the basis to a new approach to loop based quantizations, leading to a potentially very different Hilbert space than the ones considered up to now.

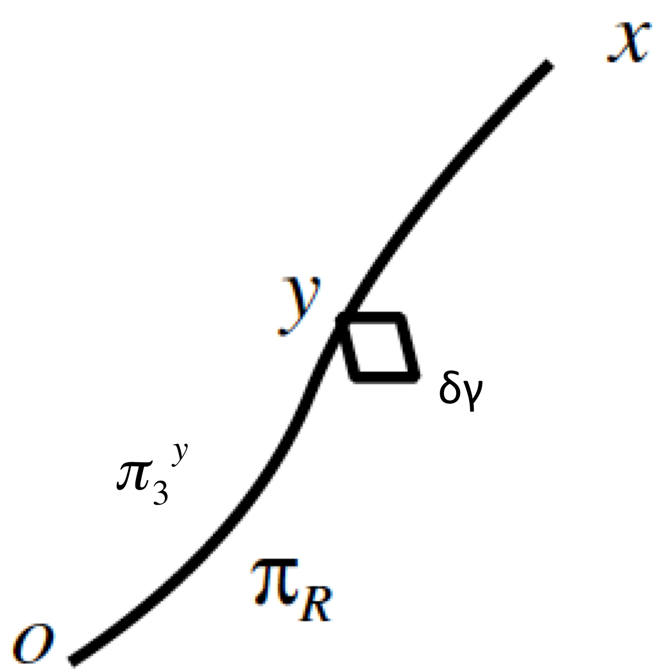
Mandelstam's approach ran into technical difficulties. It was complicated in practice to determine when two paths end up at the same point in this approach. Teitelboim made some limited progress in 1993 for infinitesimally nearby paths.

Here we would like to point out that the use of the techniques of the group of loops, that we originally developed for gauge theories allows to make much more progress and in particular the formulation of a theory of gravity as the one Mandelstam envisioned.

Here is how Mandelstam proposes to specify a path intrinsically, starting from a reference point:

> we do by giving a prescription for constructing a path from the reference point to the field point. As we construct the path, we take the coordinate basis with us by parallel displacement, and we use this basis to fix the direction of the next path element. To illustrate what we mean, a specimen prescription may run as follows: Move a distance d_1 from the reference point in the x-direction, to the point P_1 taking the coordinate basis along by parallel displacement. Now move a further distance d_2 in the y-direction (defining the y-direction with respect to the coordinate basis which has been taken to P_1). This takes us to a point P_2. Move the coordinate system by parallel displacement to P_2 as before. Finally, move a distance d_3 from P_2 in the x-direction—defined with respect to the coordinate basis at P_2. Perform the required measurement at the point just reached.

The path is given by a set of instructions similar to the ones given by the G.P.S. to a driver: Advance 200 meters then turn right and advance 500 meters, and so on… All the instructions are given with respect to the car that plays the role of local frame of reference…but in more dimensions.



To study how a physical quantity would behave along a path Mandelstam discusses the example of two paths that coincide up to a point *y* and there they differ by a small element of area and continue up to a point *x*.
A vector *a* transported along the paths π_3 and π_3 followed by δγ would change by

$$da^\mu = \frac{1}{2} R^\mu{}_{\nu\kappa\lambda}(\pi_3) a^\nu \sigma^{\kappa\lambda}$$

But the addition of the area also changes the intrinsically specified path, as the basis dragged along also gets rotated. Mandelstam computes the total change in the vector as,

$$\delta_z A_\mu(x, P_1) = \tfrac{1}{2} R_{\mu\nu\kappa\lambda}(z, P_3) A_\nu(x, P_1)\sigma_{\kappa\lambda}$$

$$- \tfrac{1}{2} R_{\iota\nu\kappa\lambda}(z, P_3)(x - z)_\iota \frac{\partial A_\mu(x, P_1)}{\partial x_\nu} \sigma_{\kappa\lambda} .$$

This formula already exhibits the challenges of the approach. It is only valid for weak fields and we do not have an analogous expression for arbitrary gravitational fields.

Gauge theories and the group of loops: a brief review

Many years ago we showed that gauge theories arise as representations of the group of loops in certain Lie groups. The complete geometric structure of gauge theories can be recovered from identities obeyed by the infinitesimal generators of the group of loops. The description may be given completely in terms of observable gauge invariants objects.

The possibility of extending this approach to the gravitational case and its intrinsic coordinate independent description did not appear possible at that time due, in particular, to the issue we mentioned with Mandelstam's approach.

Here, we would like to extend the group of loops techniques valid for gauge theories to the gravitational case. In the loop representation the objects constructed are gauge invariant whereas in the present construction the objects will be both gauge invariant and space-time diffeomorphism invariant. That is, the objects will be Dirac observables.

Brief reminder of the group of loops

R.G., Trias PRD (1981, 1983)
R.G, Pullin, CUP book (1996)
J. Lewandowski, Class. Quan. Grav. 10, 879 (1993);

We consider piecewise continuous closed curves *l, m* with origin *o* and their holonomies *H(l),* and we call loops α, β, the equivalence classes of curves that yield the same holonomies for any gauge group.

Loops are equivalence classes of closed curves that differ by retracings of a portion.
We will say that differ by a tree:

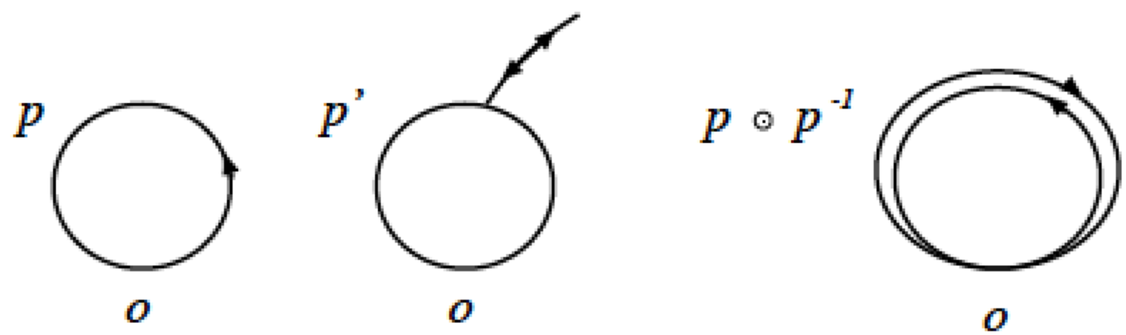


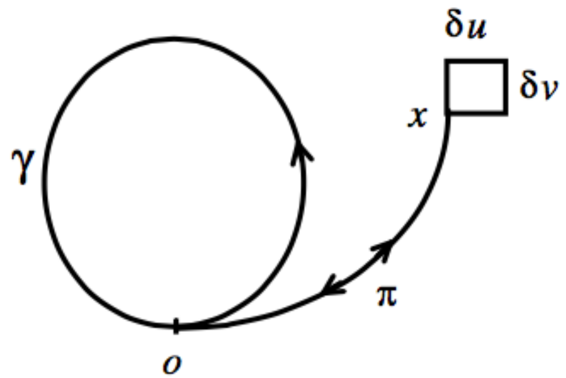
FIG. 1: Curves p and p' differ by a tree. The composition of a curve and its inverse is a tree.

They form a group under composition. The product is $\alpha \circ \beta$ and the inverse is the loop with opposite orientation.

Infinitesimal generators:

The loop derivative

Given Ψ(γ) a continuous, complex-valued function of loop, we want to consider its variation when the loop γ is changed by the addition of an infinitesimal loop δγ base-pointed at a point x connected by a path π to the base-point of γ, *o* as shown in the figure.

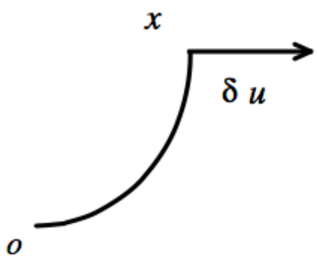


$$\Psi(\pi_o^x \circ \delta\gamma \circ \pi_x^o \circ \gamma) = (1 + \tfrac{1}{2}\sigma^{ab}(x)\Delta_{ab}(\pi_o^x))\Psi(\gamma),$$

Notice that no matter what path π one chooses, the added path is infinitesimal due to the invariance of loops under re-tracings —additions of trees— and therefore induces an infinitesimal deformation of γ.

Thus, the generators of loop deformations $\Delta_{ab}(\pi_o^x)$ are path dependent . Mathematically this derivatives are part of a differentiable calculus on the so called path bundle: an equivalence class of open paths with common origin *o* under re-tracings, on a manifold *M.* M. Reiris et al. (1998) J.N. Tavares (1994)

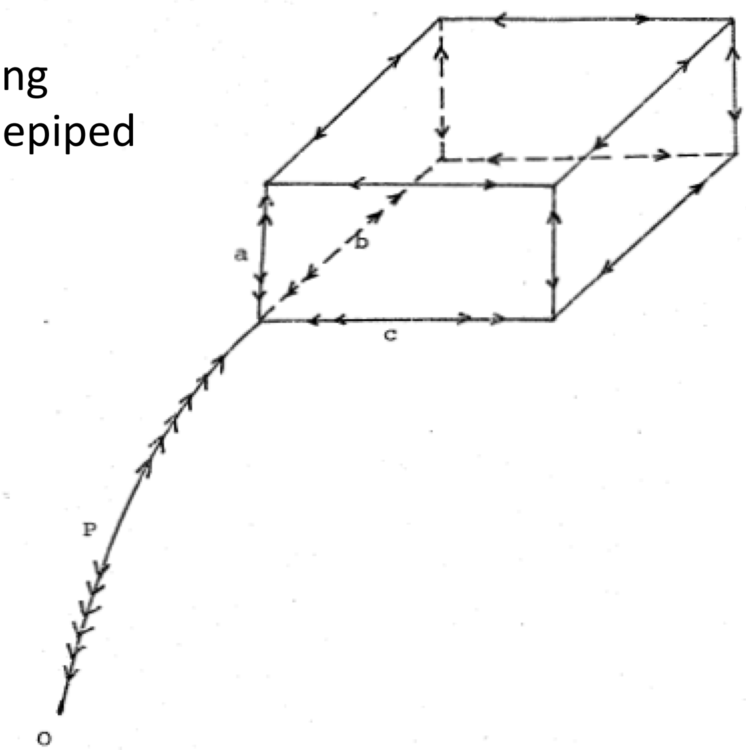
The Mandelstam covariant derivative acts on path dependent objects like the loop derivative. It introduces an extension at the end point of an open path



$$\Psi(\pi_o^x \circ \delta u) = (1 + \epsilon u^a D_a)\Psi(\pi_o^x).$$

Together with the loop derivative they satisfy the Bianchi and Ricci identities

The Bianchi identities may be obtained by considering the following tree that covers the faces of a parallelepiped



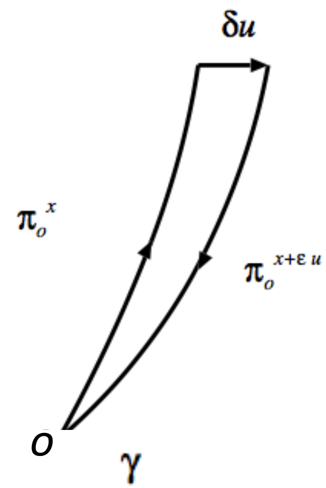
$$L_O = Pabc\overline{bcaP}\ Pac\overline{acP}\ Pcab\overline{abcP}$$
$$\times\ Pcb\overline{cbP}\ Pbca\overline{cabP}\ Pba\overline{baP},$$

$$D_a\Delta_{bc}(\pi_o^x) + D_b\Delta_{ca}(\pi_o^x) + D_c\Delta_{ab}(\pi_o^x) = 0.$$

$$[D_a, D_b]\Psi(\pi_o^x) = \Delta_{ab}(\pi_o^x)\Psi(\pi_o^x)$$

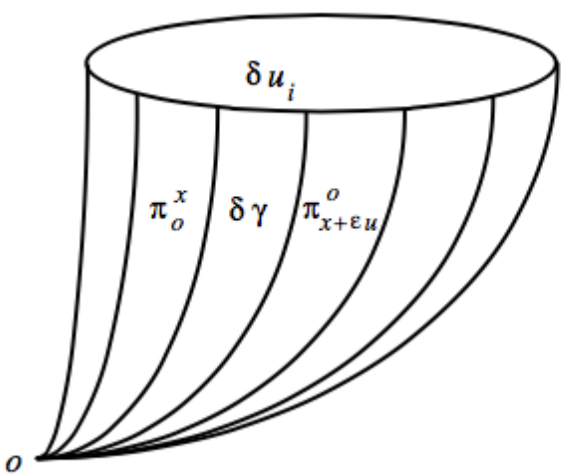
The connection derivative

One can introduce a differential operator with properties similar to those of the connection or vector potential of a gauge theory, this allows for a better contact with the usual formulation of gauge theories. We start by considering a continuous assignment or paths to each point of a given region



$$\Psi(\pi_o^x \circ \delta u \circ \pi_{x+\epsilon u}^o \circ \gamma) = (1 + \epsilon u^a \delta_a(x))\Psi(\gamma),$$

$$\Delta_{ab}(\pi_o^x) = \partial_a \delta_b(x) - \partial_b \delta_a(x) + [\delta_a(x), \delta_b(x)],$$



One can generate a finite loop operator from the connection derivative.

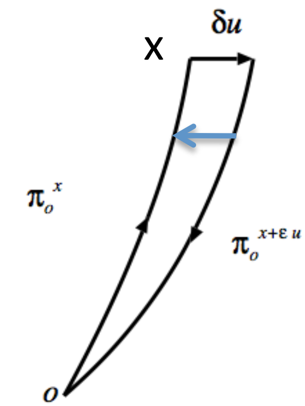
$$U(\gamma) = \mathrm{P}\exp\left(\int_\gamma dy^a \delta_a(y)\right)$$

The connection derivative may be computed by integration of the loop derivative as follows.

Consider a deformation of a path π to π' with displacement field

$x'^{\alpha}(\lambda) = x^{\alpha}(\lambda) + \epsilon^{\beta} w^{\alpha}_{\beta}(\lambda)$

$$\delta_{\mu}(\pi^{x}) = \int_{0}^{\lambda_f} \Delta_{\alpha\beta}(\pi^{x}(\lambda))\dot{x}^{\alpha}(\lambda) w^{\beta}_{\mu}(\lambda) d\lambda.$$



The kinematics of Yang-Mills theories can be understood as representations of the group of loops onto a Lie group *G*.

$$\gamma \Rightarrow H(\gamma) \qquad \text{such that} \qquad H(\gamma_1 \circ \gamma_2) = H(\gamma_1)H(\gamma_2)$$

For each assignement of paths π^x correspond a gauge fixing, and the connection and loop derivatives determine the gauge connections and fields of the Yang Mills theory.

$$\delta_a(x)H(\gamma) = iA_a(x)H(\gamma) \qquad \Delta_{ab}(x)H(\gamma) = iF_{ab}(x)H(\gamma)$$

THE GRAVITATONAL CASE: The intrinsic description

Let us recall the definition of intrinsic paths on a manifold with a given geometry.
We assume that all the paths start at some point *o* of M. If the manifold is asymptotically flat we choose *o* at infinity. We will describe paths intrinsically in terms of a Lorentz reference frame *F* in *o*. Paths are described as follows: starting from *o* we parallel transport an invariant distance *ds* the reference frame *F* with "velocity" $v^{\alpha}(0)$ to a new point $d_1 x^{\alpha} = v^{\alpha}(0)ds$.

Starting at this point we proceed to a new point moving further the reference frame with intrinsic velocity $v^{\alpha}(ds)$ and displacement $d_2 x^{\alpha} = v^{\alpha}(ds)ds$.

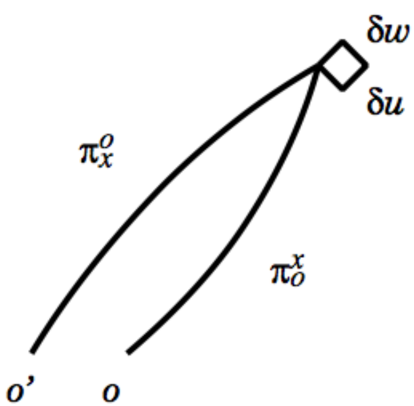
The intrinsic description of a path may be therefore described by

$$x^{\alpha}(s) \qquad \text{such that} \qquad v^{\alpha}(s) = \frac{dx^{\alpha}(s)}{ds} \qquad \text{and} \qquad x^{\alpha} = x^{\alpha}(s_f) \qquad \text{is the end point}$$

The construction is such that retraced portions of the path are excluded of the final description of the path π given in the frame *F* by $x^{\alpha}(s)$

The group of loops with intrinsic paths

We have already noticed that in the Mandelstam construction paths ending at the same physical point cannot be easily recognized. They may be identified only indirectly by noticing that all the physical fields defined at the end of two paths π_1 and π_2 are related by Lorentz and internal transformations. Furthermore this difficulty implies that closed loops in physical space will appear as open in intrinsic notation. In order to go back to the origin we need to keep track of the transformation suffered by the Lorentz frame



$$H(\pi_0^x \circ \delta\gamma \circ \Lambda(\delta\gamma)\pi_x^0)^\alpha{}_\beta = \delta^\alpha{}_\beta + \delta u^\rho \delta w^\sigma R_{\rho\sigma}{}^\alpha{}_\beta(\pi_0^x),$$

Keeping track of the transformations of the frame, the product of Infinitesimal generators is well defined:

$$\pi_1 \circ \delta\gamma_1 \circ \Lambda(\delta\gamma_1)\,\overline{\pi_1} \circ \Lambda(\delta\gamma_1)\,\pi_2 \circ \Lambda(\delta\gamma_1)\,\delta\gamma_2 \circ \Lambda(\delta\gamma_2)\,\Lambda(\delta\gamma_1)\,\overline{\pi_2}$$

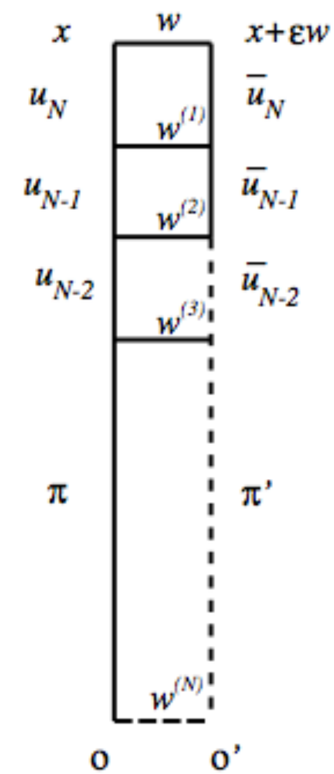
We can now apply the composition rule to construct the loop generating the connection derivative on a manifold with a given geometry:

The parallel connection derivative

$$\left(\delta_\gamma^\eta + \epsilon^2 u_N^\alpha w^\beta R_{\alpha\beta\gamma}{}^\eta \left(\pi_o^{x-\epsilon u_N}\right)\right) \left[\delta_\eta^\rho + \epsilon^2 \left(\delta_\sigma^\nu - u_N^\kappa w^\beta \epsilon^2 R_{\kappa\beta\sigma}{}^\nu \left(\pi_o^{x-\epsilon u_N}\right)\right) \times \right.$$
$$\left. \times \; u_{N-1}^\sigma \left(\delta_\mu^\lambda - \epsilon^2 u_N^\chi w^\tau R_{\chi\tau\mu}{}^\lambda \left(\pi_o^{x-\epsilon u_N}\right)\right) w^{(1)\mu} R_{\nu\lambda\eta}{}^\rho \left(\pi_o^{x-\epsilon u_N - \epsilon u_{N-1}}\right)\right] \ldots,$$

$$w^{(1)\mu} = w^\mu + \frac{1}{6} R^\mu_{\alpha\rho\gamma} w^\alpha u^\rho w^\gamma \epsilon^2 + \frac{1}{2} R^\mu{}_{\alpha\rho\gamma} u^\alpha u^\rho w^\gamma \epsilon^2.$$

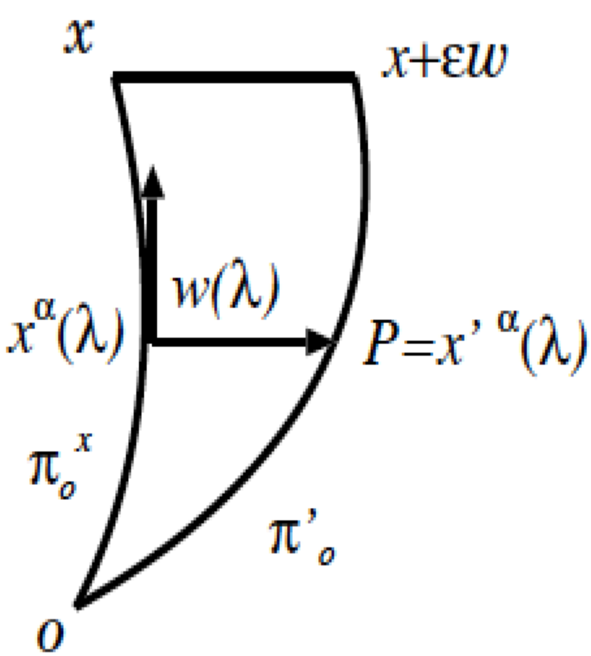
$$H_\gamma{}^\nu \left(\epsilon u_1 \circ \ldots u_N \circ \epsilon w \circ \epsilon \overline{u_N} \circ \ldots \circ \epsilon \overline{w^{(N)}}\right) = \delta_\gamma^\nu + \epsilon w^\rho A_{\rho\gamma}{}^\nu \left(F, \pi_o^x\right),$$



$$A_{\rho\gamma}{}^\nu \left(F, \pi_o^x\right) = \int_0^{s_f} ds \dot{y}^\alpha (s) R_{\alpha\rho\gamma}{}^\nu \left(\pi_o^{y(s)}\right)$$
$$+ \frac{1}{6} \int_0^{s_f} ds" \int_{s_f}^{s"} ds' \int_{s_f}^{s'} ds R_{\rho(\beta\alpha)}{}^\mu \left(\pi_o^{y(s)}\right) \dot{y}^\beta (s) \dot{y}^\alpha (s') R_{\mu\delta\gamma}{}^\nu \left(\pi_o^{y(s")}\right) \dot{y}^\delta (s")$$

The general connection.

The previously defined connection is a particular example of connections relating two "parallel" neighboring paths. But more generally, one can define a connection derivative for each tangent vector in the path manifold. If a path π^x is defined by $u^\alpha(\lambda) = \frac{dx^\alpha(\lambda)}{d\lambda}$ in the intrinsic frame parallel transported to the point $x^\alpha(\lambda)$, the tangent at the element π^x of the manifold of intrinsic paths may be described by the vector field $w^\alpha(\lambda)$ as shown in the figure. One can determine from this information the intrinsic description of a path π' infinitesimally closed to π^x



$$u'^\alpha = \frac{dx'^\alpha(\lambda)}{d\lambda}$$

$$\Lambda^\alpha{}_\beta = \delta^\alpha{}_\beta + \Omega^\alpha{}_\beta(\lambda),$$

$$\Omega^\alpha{}_\beta(\lambda) = \int_o^\lambda \epsilon R_{\gamma\delta}{}^\alpha{}_\beta(\lambda')\, u^\gamma(\lambda') w^\delta(\lambda') d\lambda'.$$

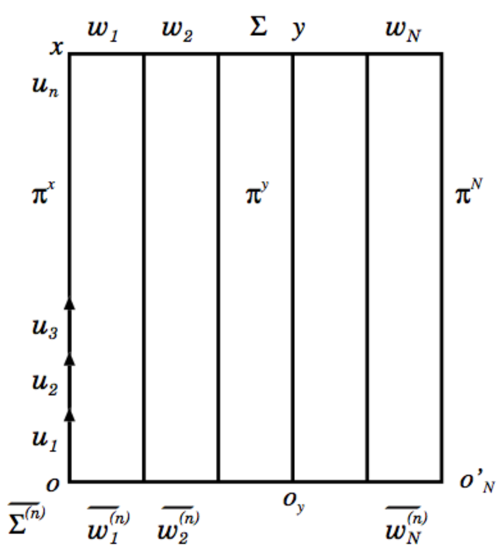
$$u'^\alpha = \Lambda^\alpha{}_\beta u^\beta(\lambda) + \epsilon \frac{dw^\alpha}{d\lambda},$$

$$A_\alpha{}^\beta{}_\sigma\left(\pi^x_o\right) = \int_o^{\lambda_f} R_{\gamma\delta}{}^\beta{}_\sigma\left(\lambda'\right) u^\gamma\left(\lambda'\right) E^\delta_\alpha\left(\lambda'\right) d\lambda' = \int_o^{s_f} R_{\gamma\delta}{}^\beta{}_\sigma\left(y\right) E^\delta_\alpha\left(y\right) dy^\gamma,$$

where the tangent vector to the path π^x is given by

$$w^\alpha(\lambda) = w^\beta(\lambda_f) E^\alpha_\beta(\lambda) \qquad \text{with} \qquad E^\alpha_\beta(\lambda_f) = \delta^\alpha_\beta$$

Recalling how finite loops are generated by the connection derivative we get the Holonomy of a closed finite intrinsic paths.

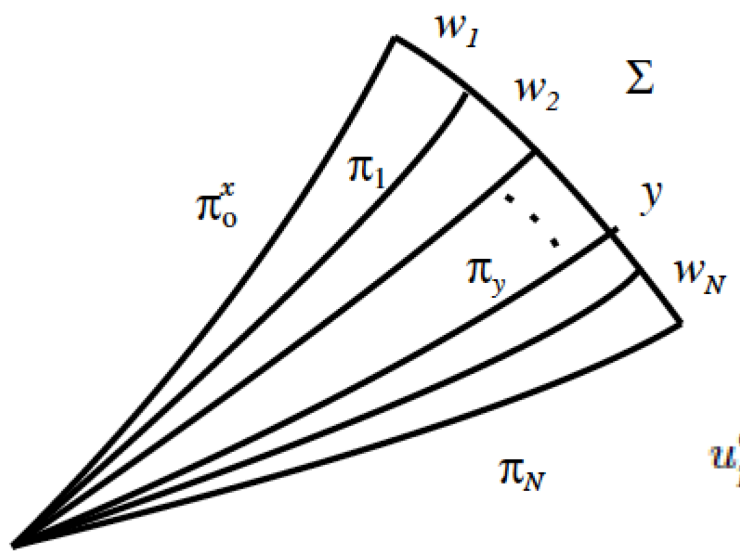


$$H(\gamma) = \mathrm{P}\exp\left(i\int_\Sigma dy^\alpha A_\alpha\left(F_y, \Sigma_{oo_y}\pi^y_{o_y}\right)\right)$$

$$\Sigma_{oo_p} = \Lambda_p\Lambda_{p-1}\cdots\Lambda_2 w_1^{(n)} \circ \Lambda_p \ldots \Lambda_3 w_2^{(n)} \circ \cdots \Lambda_p w_{p-1}^{(n)} \circ w_p^{(n)}$$

This is the intrinsic version of the non-Abelian Stokes' theorem for parallel connection derivatives

A finite loop based on general connections



The path $\gamma = \pi^x \circ \Sigma \circ \overline{\pi^N}$ used in the construction of the holonomy associated to closed finite path.

We consider a closed path that connects

$$\pi_o^x \circ \Sigma \circ \overline{\pi}_N$$

$$\gamma = \left(\pi_o^x \epsilon w_1 \overline{\pi}_1^{y_1}\right)\left(\pi_1^{y_1} \epsilon w_2 \overline{\pi}_2^{y_2}\right)\ldots\left(\pi_p^{y_p} \epsilon w_p \overline{\pi}_{p+1}^{y_{p+1}}\right)\ldots$$

$$u_p^\alpha(\lambda) = \left(\delta^\alpha{}_\beta + \Omega_p{}^\alpha{}_\beta(\lambda)\right) u_{p-1}^\beta(\lambda) + \epsilon \frac{dw_p^\alpha}{d\lambda},$$

$$H^\alpha{}_\beta = \left(\delta^\alpha{}_{\beta_1} + \delta w_1^\rho A_\rho{}^\alpha{}_{\beta_1}(\pi_o^x)\right)\left(\delta^{\beta_1}{}_{\beta_2} + \delta w_2^\rho A_\rho{}^{\beta_1}{}_{\beta_2}(\pi_1^{y_1})\right)\ldots\left(\delta^{\beta_p}{}_\beta + \delta w_p^\rho A_\rho{}^{\beta_p}{}_\beta(\pi_{p-1}^{y_{p-1}})\right)\ldots$$

$$H(\gamma) = \mathrm{P}\exp\left(i\int_\Sigma dy^\alpha A_\alpha(\pi^y)\right),$$

$$\Sigma = \delta w_1 \circ \Lambda_1 \delta w_2 \circ \ldots \circ \Lambda_1 \Lambda_2 \ldots \Lambda_{N-1} \delta w_N$$

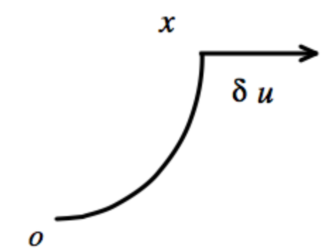
The key point is that we now have a well defined relation among path-dependent fields at a given point.

A field evaluated at two different path ending at the same point satisfy

$$A^{\alpha}{}_{I}(\pi') = H(\gamma)^{\alpha}{}_{\beta} H(\gamma)_{I}{}^{J} A^{\beta}{}_{J}(\pi), \qquad \pi' = \gamma \circ \Lambda(\gamma)\pi$$

Physical points are defined as equivalence classes of paths. The main difference with the standard formulation fot gauge fields is that intrinsic paths arriving to the same point depend on the geometry.

We can now introduce the **Mandelstam covariant derivative,**



$$(1 + \epsilon u^{\beta} D_{\beta}) A^{\alpha}{}_{I}(\pi^{x}) = A^{\alpha}{}_{I}(\pi_{E}^{x+\epsilon u})$$

It compares the field parallel transported from *x+εu* to *x* with the field at x and therefore gives us the component of the space time covariant derivative with respect to the intrinsic basis parallel transported along π.

From this definition it is easy to prove identities for the intrinsic Riemann tensor

$$[D_\beta[D_\gamma, D_\delta]]A_\alpha(\pi) = D_{[\beta}R_{\gamma\delta]\alpha}{}^{\epsilon}(\pi)\, A_\epsilon(\pi) = 0$$

$$[[D_\alpha, D_\beta]D_\gamma]\phi(\pi) = R_{[\alpha\beta\gamma]}{}^{\delta} D_\delta\phi(\pi) = 0.$$

And write the intrinsic Einstein equations, say, coupled to a scalar field,

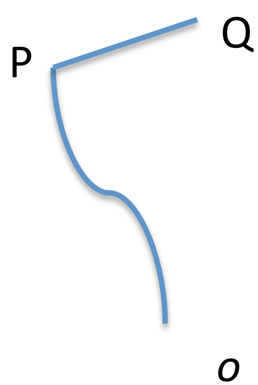
$$\begin{aligned} \left(\eta^{\alpha\beta} D_\alpha D_\beta - m^2\right)\phi(\pi) &= 0 \\ R_{\alpha\lambda\beta}{}^{\lambda}(\pi) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\rho}R_{\gamma\lambda\rho}{}^{\lambda}(\pi) &= \kappa T_{\alpha\beta}(\pi), \end{aligned}$$

$$T_{\alpha\beta}(\pi) = D_\alpha\phi(\pi)\, D_\beta\phi(\pi) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}D_\mu\phi(\pi)\, D_\nu\phi(\pi) - m^2\phi^2(\pi)\,\eta_{\alpha\beta};$$

Recovering the usual formulation:

The intrinsic construction that we have presented up to now assigns a frame and tetrad vectors to each point along the path. Let us introduce a normal system of coordinates at a region *U* around a point P to which we have arrived following a geodesic starting at the origin *o.* Then P is intrinsically defined by the geodesic $x^\alpha(s) = su^\alpha$, and the point *P* corresponds to $s=s_P$.

A point Q is given by an intrinsic path π_R^Q given by



$$x^\alpha(s) = s_P u^\alpha + s v^\alpha \qquad \text{and} \qquad x^\alpha(Q) = s_P u^\alpha + s_Q v^\alpha$$

The quantities $z^a(Q) = s_Q v^a$ define Riemann normal coordinates around P. The intrinsic construction allows to assign to each Q the coordinates of the local frame parallely transported from *o*

$$e_\alpha^a(\pi_R^Q)$$

Recovering the usual formulation:

Under changes of the path arriving to *Q*, the frames transform as,

$$e^a_\beta\left(\pi'^Q\right) = H_\beta{}^\alpha\left(\gamma\right) e^a_\alpha\left(\pi^Q_R\right) = H_\beta{}^\alpha\left(\gamma\right) e^a_\alpha\left(x(Q)\right)$$

One can use them to construct the metric components in the normal system:

$$g^{ab}\left(x\right) = \eta^{\alpha\beta} e^a_\alpha\left(\pi^x_R\right) e^b_\beta\left(\pi^x_R\right),$$

And by construction the Mandelstam covariant derivative annihilates them

$$D_\alpha e^b_\beta = 0$$

In terms of the Mandelstam derivative one can construct the usual covariant derivative that also annihilates the tetrad and metric components, making the connection metric compatible

$$\nabla_a e^b_\beta(z) \equiv e^\alpha_a D_\alpha e^b_\beta(\pi^Q_R) = 0$$

and one can show it is also torsion free:

$$D_{[\alpha} D_{\beta]} \phi(\pi) = \frac{1}{2} \Delta_{\alpha\beta}(\pi)\phi(\pi) = 0$$

The Riemann tensor in normal coordinates is simply given by:

$$R^{abcd}(x) = R^{\alpha\beta\gamma\delta}(\pi^x_R)\, e^a_\alpha(\pi^x_R)\, e^b_\beta(\pi^x_R)\, e^c_\gamma(\pi^x_R)\, e^d_\delta(\pi^x_R)\,.$$

Classically, the intrinsic and coordinate descriptions are equivalent, quantum mechanically they will be inequivalent. The notion of point as an equivalence class of intrinsic path only makes sence classically.

We would like to relate the paths described in coordinate systems with intrinsic paths and identify the local frames at an arbitrary point of the path, in terms of coordinate or intrinsic descriptions of the paths. Let $\gamma_a(\lambda)$ be a curve in an arbitrary coordinate system, then the local frame parallel transported along γ is

$$e_\alpha{}^c(\lambda) = \mathrm{P}\left(\exp\left(-\int_0^\lambda d\lambda' \dot\gamma^a(\lambda')\Gamma_a\right)\right)_\alpha^{\;c},$$

$$y^\alpha(\lambda) = \int_0^\lambda \dot\gamma^c(\lambda') e^\alpha{}_c(\lambda') d\lambda'.$$

$$\begin{aligned} e_\alpha{}^a(\lambda) &\equiv e_\alpha{}^a\left(\left[y^\beta\right],\lambda\right),\\ \dot\gamma^a &= \dot y^\alpha e_\alpha{}^a(\lambda) = \dot y^\alpha e_\alpha{}^a([y],\lambda),\\ \gamma^a(\lambda) &= \int_0^\lambda d\lambda' \dot y^\alpha e_\alpha{}^a([y],\lambda) + x^a_o, \end{aligned}$$

The action for path dependent fields

Teitelboim proposed in 1993 C. Teitelboim, Nucl. Phys. B 396, 303 (1993)
that the action for usual fields could be written in terms of fields that depend on intrinsic paths by gauge fixing them and using the Fadeev-Popov technique.

$$\mathcal{L}\left(R_{\alpha\beta\gamma}{}^{\rho}\left(\pi\right),\phi\left(\pi\right),\psi\left(\pi\right)\right) \qquad\qquad S=\int\mathcal{L}\mathcal{D}x$$

$$\mathcal{D}x=\Pi_{\lambda,\alpha}dx^{\alpha}(\lambda)\delta(\pi-\pi'_{R})\Delta_{FP}(\pi_{R})$$

He did not offer proof. However, using the developed framework we were able to show that one indeed recovers the usual action.

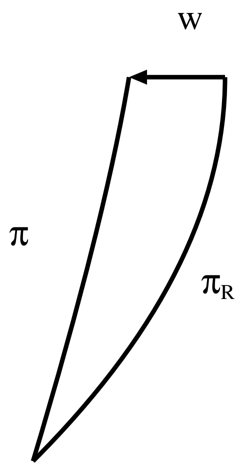
Sketch of the proof:

Pick a geodesic reference path for simplicity: $x^\alpha = u^\alpha \lambda.$

This choice allows to compare the intrinsic formularion with a Lagrangian formulation in Riemann normal coordinates

We have that:

$$u'^\alpha = \Lambda^\alpha{}_\beta u^\beta(\lambda) + \epsilon \frac{dw^\alpha}{d\lambda},$$



$$\Lambda^\alpha{}_\beta(\lambda) = \delta^\alpha_\beta + \Omega^\alpha_\beta(\lambda) = \delta^\alpha_\beta + \int_0^\lambda d\lambda' R_{\gamma\delta}{}^\alpha{}_\beta(\lambda') w^\delta(\lambda') u^\gamma(\lambda').$$

$$\delta\left(x^\alpha_\pi(\lambda) - x^\alpha_R(\lambda)\right) = \delta\left(\int_0^\lambda \Omega^\alpha{}_\beta(\lambda') u^\beta \lambda' d\lambda' + w^\alpha(\lambda)\right)$$

$$\delta\left(\pi - \pi_R\right) = \Pi_{\lambda,\alpha} \delta\left(x^\alpha_\pi(\lambda) - \lambda u^\alpha\right),$$

$$\delta\left(x^\alpha_\pi(\lambda) - x^\alpha_R(\lambda)\right) = \delta\left(M^{\alpha(\lambda)}{}_{\delta(\mu)} w^\delta(\mu)\right)$$

The determinant of M gives the Fadeev-Popov term and allows to recover the standard form of the volume element. And therefore:

$$\begin{aligned}\sqrt{-g} &= \sqrt{1 - \frac{1}{3}\eta^{mn} R_{manb} u^a u^b \lambda_f^2} \\ &= 1 - \frac{1}{6}\eta^{mn} R_{manb} z^a z^b.\end{aligned}$$

$$\mathcal{D}x = \Pi_{\alpha,\lambda} dx^\alpha_\pi(\lambda) \delta(\pi - \pi_R) \Delta_{FP} = \Pi_a dz^a \sqrt{-g},$$

By studying how it behaves under changes of reference reference paths, one can show that the action is diffeomorphism invariant.

This opens the possibility of studying canonical formulations.

The Poisson bracket of path dependent Riemann tensor components takes the form

$$\{R_{0abc}(\pi_1),R_{ijkl}(\pi_2)\} = -\frac{\kappa}{2}D^1_{[_1 b|}D^2_{[_3 k|}D^2_{[_2 i|}(\delta_{a|j]_2}\delta_{|c]_1|l]_3} + \delta_{|c]_1|j]_2}\delta_{a|l]_3} - \delta_{a|c]_1}\delta_{|j]_2|l]_3})\delta(\pi_1,\pi_2))$$

$$+\frac{\kappa}{2}R_{ij[_2 k|m}(\pi_2)D^1_{[_1 b|}(\delta_{am}\delta_{|c]_1|l]_2} + \delta_{|c]_1 m}\delta_{a|l]_2} - \delta_{a|c]_1}\delta_{m|l]_2})\delta(\pi_1,\pi_2)$$

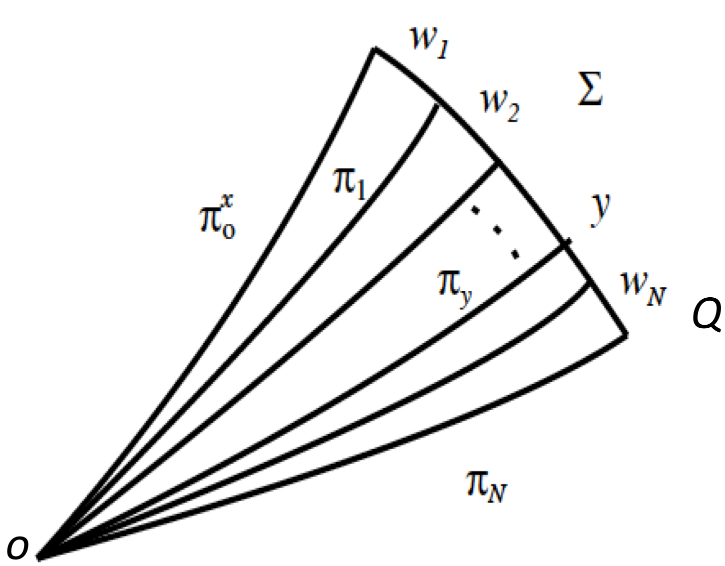
The square brackets with a subindex allow to identify couple of indices which are antisymetrized.

The quantization is still under construction, but some partial insights about the quantum structure of space time can be derived.

Fuzzy points

In intrinsic gravity points are equivalence classes of paths that describe the same classical fields in all possible Lorentz frames. In a quantum geometry the notion of point becomes fuzzy.

Even though the quantization has not been completely accomplished we are going to introduce an heuristic procedure for the study of the fuzziness of the quantum space-time points.
We start by introducing geodesic Riemann normal coordinates in an open set containing the point *Q*:



$$x_Q^a = s_Q \hat{u}^a \qquad \text{with} \qquad \eta_{ab}\hat{u}^a\hat{u}^b = 1$$

The geodesic path π_N going from *o* to *Q* is Intrinsically described by

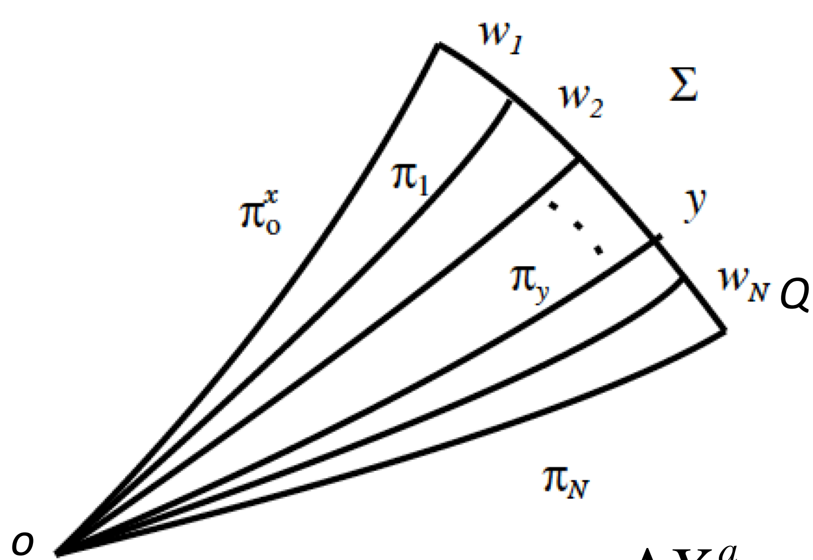
$$(s) = s\hat{u}^a \qquad \text{or} \qquad y^a(\lambda) = \lambda u^a \qquad \text{with} \qquad \lambda \in [0,1]$$

Let us now consider that our quantum space time is given by a atate centered arround flat space $|\psi_0>$ such that

$$<\psi_0 | R_{\alpha\beta\gamma\delta}(\pi) | \psi_0 >= 0$$

We are interested in the fluctuations of the coordinates assigned by different Intrinsic paths. To do that we consider the operator

$$\hat{X}^a_{12} = \hat{X}^a(\pi_1) - \hat{X}^a(\pi_2) = \hat{X}^a(\pi \circ \Sigma) - \hat{X}^a(\pi_N)$$

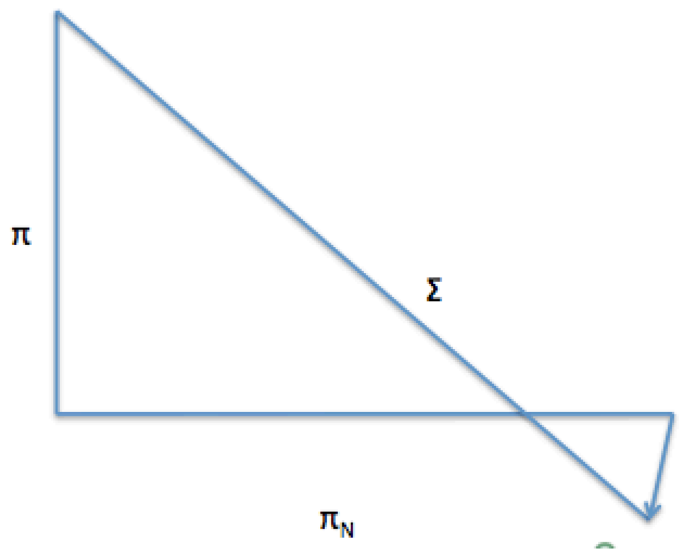


Let us chose π and Σ such that for flat space π•Σ and π_N arrive to te same point

$$<\psi_0 | \hat{X}^a_{12} | \psi_0 >= 0$$

By using the non Abelian Stokes theorem one can compute the standard deviations in terms of expectation values of Riemann tensor operators

$$\Delta X^a_{12} = \sqrt{<\psi_0 | \hat{X}^{a\ 2}_{12} | \psi_0 > - <\psi_0 | \hat{X}^a_{12} | \psi_0 >^2}$$



$$\Delta X^{a}_{12} = \sqrt{<\psi_0 \,|\, (\delta\hat{w}^{a})^2 \,|\, \psi_0>}$$

$$\delta\hat{w}^{a} = \int_0^1 d\mu\,\delta\hat{w}^{\alpha}(\lambda,\mu)$$

$$\delta\hat{w}^{\alpha}(\lambda,\mu) = \int_0^{\lambda} d\lambda' \int_0^{\lambda'} d\lambda''\{R^{\alpha}_{212}((1-\mu)\lambda'' u\delta_2^{\rho} + \mu v\lambda''\delta_1^{\rho})u^2 v(1-\mu)\mu$$

And using the commutation relations of the Riemann components and assuming that the quantum theory admits a positive definite inner product one gets:

$$\Delta X^{1}_{12}\Delta X^{0}_{12} \geq l_{Planck}\sqrt{A(\pi \circ \Sigma, \pi_2)}$$

$$\Delta X^{1}_{12}\Delta X^{2}_{12} \geq 0$$

Summarizing, points or the causal structure of space time could become fuzzy, however the second inquality suggests that microcausality would be preserved..

This uncertainty in the position of the end points for different intrinsic paths induces a non-locality in the algebra of path dependent fields. For instance if ϕ(π) is a massless scalar field the non-locality induced by quantum gravity leads to commutation relations

$$[\hat{\phi}(\pi \circ \Sigma), \hat{\phi}(\pi_N)] = \Delta(x^\alpha(\pi \circ \Sigma), x^\alpha(\pi_N))$$

where Δ is an extension of the odd propagator for the scalar field that vanish outside the light cone and is nonvanishing inside. This commutation relations are the manifestation of the size of the field excitations in the quantum background. They behave as if they have acquired mass.

This kind of non-locality is also predicted by causal sets

L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, Physical Review Letters 59, 521 (1987).
[2] R. D. Sorkin, ArXiv General Relativity and Quantum Cosmology e-prints (2007), gr-qc/0703099

and has been recently taken as an explanation of the origin of *dark matter* that is not based on another new particle in nature, but that it is a remnant of quantum gravitational effects on known fields.

Mehdi Saravani, Siavash Aslanbeigi Phys Rev D 92. 103504 (2015)

Summary

- By using the group of loops techniques, the idea of Mandelstam of formulating field theories and gravity in terms of path dependent variables becomes possible.
- The formulation is entirely in terms of Dirac observables.
- Diffeomorphisms can be recovered, but they are not a fundamental symmetry of the formulation.
- We have an action suitable for canonical formulation in terms of the path dependent variables.
- The resulting quantum theory will be loop-based but will likely have a very different kinematics and dynamics than ordinary loop quantum gravity.
- More details arXiv:1802.02661.